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THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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SOME APPLICATIONS OF GROUPS TO GEOMETRY[1]

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Introduction

The subject of groups is of wide appeal, both for its intrinsic interest and the range and importance of its applications—applications to geometry, to algebra,—notably in the theory of equations and the invariant theory,—to mathematical physics, to differential equations, to crystal structure, to the principles of design. Yet it is worthy of remark that one can start from elementary notions and reach the frontier in a comparatively short time without the aid of other branches of analysis. And the postulates are few—Dickson's set consists of four, while for finite groups of collineations, a single postulate suffices.

In the brief time allotted to this paper, it is possible to sketch but one phase of the applications even to geometry. I only mention in passing the great field of the Lie theory where continuous groups are so effectively brought into play. Again the Galois theory of equations has its geometric counterpart in such problems as those associated with the inflexions of the cubic curve, the bitangents of the plane quartic and the twenty-seven lines on the cubic surface. There is the large and increasing domain of Cremona geometry and Cremona groups. Klein has made the theory of groups of linear transformations the basis of the classification of the various geometries which occur as special cases of general projective geometry. Indeed the theory of linear transformations is the very soul of projective geometry and its algebraic dual, the classical invariant theory. In spite of this amazing richness of application, many scholars are content to explore the abstract theory of the groups themselves, indifferent to the applications or perhaps serene in the delusion that some of their results are immune to application.

I cannot refrain from giving voice here to my own and other geometers' keen disappointment that in the recent invaluable *Report on Algebraic Geometry*,[2] space could not have been found for the important subject of collineation groups, more especially since the keynote of the Report is transformation.

§1. *The Problem of Symmetry*

I propose to consider briefly the application of collineation groups, binary and ternary, to the geometry of plane curves, beginning with the metrical question of symmetry. We recognize in the plane two elementary types of symme-

[1] Read at the summer meeting of the Mathematical Association of America at Boulder, Colorado, August 27, 1929.

[2] National Research Council Bulletin No. 63.

try (*a*) with respect to a line (*b*) with respect to a point. These are illustrated respectively by the ordinary parabola $y^2 = x$, and the cubical parabola $y = x^3$. In the latter case, the center of symmetry is defined to be the center of the curve. Their utility for tracing the curves is obvious, for in either case the points of the curve are paired in the symmetry—paired in a mutual way so that either point of a pair determines the other.

It is desirable now to have a mechanism that will enable us, given one point of a pair, to construct its partner. Metrically stated, we are given one end and the middle point of a line segment and are required to construct the other end. But in projective language the middle point of a segment and the infinite point of the segment (produced) divide the end-points harmonically and the problem becomes: Given three points of a line, to construct the harmonic conjugate of one with respect to the other two. The projective view of this operation, which is called *reflexion*, is illustrated by a conic and a polar point and line which are not incident: Any line through the pole cuts the conic in a pair of points which are harmonically separated from the pole by the point in which the line cuts the polar line. We say that the conic is reflected into itself, calling the pole the center and the polar line the axis of the reflexion.

It appears now that from the projective point of view the two elementary types of symmetry are identical: In the first the axis of symmetry is the axis of reflexion while the center of reflexion is a point at infinity in the direction of a perpendicular to the axis; in the other the axis of reflexion is the line at infinity while the center of reflexion coincides with the center of symmetry. Thus we may summarize as follows:

If a curve is symmetrical with respect to a line, it is invariant under a reflexion of which the line is the axis. And if a curve is symmetrical with respect to a point, it is invariant under a reflexion of which the point is the center. Conversely, if a curve is invariant under a reflexion, it may be projected indifferently into one having (*a*) *an axis of symmetry or* (*b*) *a center of symmetry.*

The geometrical apparatus for constructing a pair of points interchanged by a reflexion is the complete quadrangle in the general projective case; in the metrical case when the points are symmetrical with respect to a line or a point, the familiar Euclidean constructions are available. As an algebraic means of determining such a pair of points, we resort to a *transformation*. Thus in the metrical case of the parabola, if the reflexion carries the point (x, y) into the point (x', y'), the relation between the coordinates of the two points is

$$R_1: \quad x' = x, \quad y' = -y.$$

These are the equations of a reflexion which we regard as the operation which carries the point (x, y) into the point (x', y'). A second application of the reflexion restores the point to its original position. Thus a reflexion is of period two and in fact any linear transformation of period two in the plane is a reflexion. Under a reflexion the center and all points of the axis and hence the axis itself and all lines through the center are fixed, while all other points and lines of the

plane are paired. Consider now a second reflexion, in the y-axis, whose defining equations are

$$R_2: \; x' = -x, \quad y' = y.$$

If R_1 carries a point P into a point P' and R_2 carries P' into P'', then P'' can be obtained from P by means of the transformation

$$R_3: \; x' = -x, \quad y' = -y.$$

R_3, which is the resultant or *product* of R_1 and R_2, is obviously again a reflexion —in the origin. Or we may regard R_3 as a rotation about the origin through the angle 180°. Finally we associate with the reflexions the identical transformation, or simply the *identity*

$$1: \; x' = x, \quad y' = y,$$

which leaves every point fixed. The combined effect of these reflexions when applied to an arbitrary point of the plane is to carry the point into four positions, including the original. Such a set of linear transformations, finite in number, all possible combinations of which carry a point into different positions, ultimately restoring it to the original one, are said to form a collineation group. Or, somewhat more conventionally,

A set of collineations, finite in number, the product of any two of which produces the same effect as some one collineation of the set, constitute a finite collineation group. It is understood that the several powers of each collineation (successive products of the collineation by itself) are included in the definition. Thus in the group above, we have the relations

$$R_i^2 = 1, R_iR_j = R_k, \qquad i,j,k, = 1,2,3, i \neq j \neq k.$$

The individual collineations are called *elements* and the *order* of a group is the number of elements it contains. A group of order n we shall uniformly denote by G_n. Two points which can be transformed into each other by the elements of the group are said to be *conjugate*. In the sequel "group" shall be understood to mean finite collineation group.

As immediate consequences of the definition we have

1°. *Every collineation S_i of a finite group is periodic, i.e. for some integer p,* $S_i^p = 1$, otherwise the order of the group would be infinite. If p is the least integer for which this relation holds, S_i is of period p.

2°. *The identity is included in every group*—a corollary of 1°.

3°. *The inverse S_i^{-1} of every element S_i belongs to the group.* For if p be the period of S_i, $S_i^{p-1} = S_i^{-1}$.

We return now to the question of symmetry. Suppose a curve has two axes of symmetry, inclined at an angle θ. The curve is then invariant under two reflexions, say T and T'. The product of the two is obviously a rotation, say S, about the origin through an angle 2θ. Hence the curve is also invariant under S, which simply means that under repeated applications of S a point of the curve always goes again into a point of the curve. If now $2n\theta = 360°$, S repeated will

carry an arbitrary point of the curve into n positions, including the original. In other words $S, S^2, \cdots, S^n \equiv 1$, will constitute a group of order n which transforms the curve into itself. S and T in combination generate a group of order $2n$, the ternary dihedral G_{2n}, which leaves the curve invariant. The group contains n reflexions whose axes are equispaced about the origin, i.e. the curve has n equispaced axes of symmetry. Evidently the dihedral group, in the metrical setting here described, leaves unaltered each circle of the system having the origin for common center. An arbitrary point of any invariant curve is one of a set of $2n$ conjugate points cut from the curve by a circle of this concentric system.

In the metrical representation of the ternary dihedral group now employed, the centers of the n reflexions lie on the line at infinity, which is a fixed line as the origin is a fixed point. In a projective representation, the dihedral group has likewise a fixed line, the line of centers, and a fixed point, the intersection of the n axes of reflexion. It also leaves invariant a pencil of double-contact conics, with respect to every (proper) member of which the fixed point and the line are pole and polar.

It is plain now that there is an intimate connection between multiple symmetry and dihedral groups, for a dihedral group can always be given the metrical representation described above. Thus

If a curve is invariant under a dihedral G_{2n}, it may be projected into one having n axes of symmetry, equispaced about a point. Conversely, if a curve is symmetrical with respect to n equispaced axes, it admits a dihedral G_{2n}.

Again, *a curve of order n, a conic alone excepted, may not admit a dihedral group of order greater than $2n$, i.e the maximum number of lines with respect to which an algebraic curve may be symmetrical is $2n$. When n is even however, the intersection of two or more axes of symmetry is a center of the curve.* For if the curve is invariant under a dihedral G_{2n}, any general set of conjugate points on the curve is cut out by one of the invariant conics (circles) which cuts the curve in a maximum of $2n$ points.

There is however at least one algebraic curve for each value of n possessing this maximum symmetry. Indeed there is at least one rational curve—one cubic (the general rational cubic) one quartic (the projective lemniscate) two quintics, one sextic and three septimics. The general formula for rational curves is[1]

$$\begin{aligned} x_1 &= t^r & r + s &= n, \\ x_2 &= t^s & 1 &\lessgtr r \neq s, \\ x_3 &= t^n + 1, \end{aligned}$$

where r, s and n are integers and have no common factor.

We may now state a further theorem about multiple symmetry:

[1] Winger, American Journal of Mathematics vol. 36 (1914), p. 66. To plot the curves so that the symmetry appears, absolute or circular coordinates x, $\bar{x}$ must be used, writing $x = x_1/x_3$, $\bar{x} = x_2/x_3$, while to the parameter t are to be assigned complex numbers of absolute value 1.

If an algebraic curve have two axes of symmetry, it is invariant under a dihedral group. For the curve is then invariant under two reflexions, whose product is a rotation. This rotation must be periodic, else the curve would have an infinite number of axes of symmetry, whereas the maximum number is n. If the period of the rotation is m, then the two reflexions generate a dihedral group of order[1] $2m$ and the curve has m axes of symmetry.

A curve may admit a rotation into itself of period n without admitting a reflexion. Such a curve possesses a sort of "one-sided" symmetry which is admirably illustrated by a swastika or a cross section of an anemometer. For example the quartic curve

$$xy^3 - x^3y = 1$$

is invariant under the G_4 of rotations generated by $x' = -y$, $y' = x$.

A very interesting analysis of symmetry in design from the group standpoint, with references, is given by Speiser in his admirable book *Theorie der Gruppen von Endlicher Ordnung*, Springer, 1927.[2] Weyl in his *Gruppentheorie und Quantenmechanik* goes so far as to say that a substantial part of group theory is implicit in the art of ornamental design, which was brought to a high state of perfection by the Egyptians.

§2. *Applications to Configurations*

Since every element of a finite group is periodic, a useful theorem is the following:

Any periodic collineation in a space of n dimensions may, by a suitable choice of the reference frame, be reduced to the canonical form

$$\rho x_i' = a_i x_i, \quad i = 0, 1, \cdots, n, \tag{1}$$

where the x's are homogeneous projective coordinates and the a's (called *multipliers*) are roots of unity. If the collineation is of period n, one of the a's must be a primitive nth root. The theorem does not imply of course that all the collineations of a group can be thrown simultaneously into the form (1). When that is possible, the group is *monomial*. The distinguishing property of a monomial group is that it leaves invariant an $(n+1)$-point which is the reference frame when the group is in the canonical form. This gives a broad classification of collineation groups into two classes, those which are monomial and those which are not. Because of their comparative simplicity monomial groups have been almost ignored, but they are deserving of a better fate because of the interesting geometry associated with them.

The theorem restricts the types of collineations that may occur in a finite group to two main classes represented by (1) when (a) the multipliers are distinct and (b) two or more multipliers are the same. Thus of the five projectively distinct types of collineations in the plane only two may belong to a finite group while of the thirteen possible varieties in space but four may occur in a finite

[1] Miller, Blichfeldt, Dickson, *Finite Groups*, p. 61.

[2] See also the delightful *Lectures on the Principle of Symmetry* by Jaeger, Amsterdam, 1920.

group. Geometrically, type (*a*) is characterized as having a finite number of fixed points, while collineations of type (*b*) leave fixed all of the points of one or more linear spaces.[1]

Conjugate points and lines. In the absence of explicit statement to the contrary, we shall henceforth confine ourselves to the plane. The effect of operating on a point P with all the collineations of a group G_n, individually and collectively in any combination, is to carry P into a maximum of n positions, including the original. For each collineation either leaves P fixed or transforms it into a second point and by definition the effect of two elements operating in succession is the same as that of some one element. The set of points thus associated with P forms a set conjugate under the group. Paralleling the idea of conjugate points is the notion of conjugate elements in the group itself. Two conjugate collineations are projectively equivalent, or as we say abstractly identical, i.e. by a proper choice of the reference scheme each may be reduced to exactly the same form as the other. In symbols, if S and T are any two elements of a group and if $T^{-1}ST=S'$, where the multiplication is understood to be from left to right, S and S' are conjugate elements.

A cardinal principle for the geometry is the following: *Any two points in a set of conjugate points have exactly the same* (*projective*) *properties.* Thus if a curve is invariant under a group, a point of inflexion can go only into a point of inflexion, a double point into a double point, an undulation into an undulation, etc. If the curve has but one point of a species, that point must be fixed under the group. The principle applies also to the relation of a point to the group. Thus if a point is fixed under a collineation S, it can be carried by the group only into a fixed point of an element conjugate to S. The principle is illustrated by the relation of congruent figures in Euclidean geometry, where we think of congruent figures as having identical properties. Indeed congruent figures are those which can be transformed into each other by the group of motions—consisting of translations and rotations—that characterize Euclidean geometry.

The principle affords a means of classifying precisely the points of a curve. We are accustomed to speak somewhat loosely of a "generic point" of a curve, a vague expression used to distinguish it from points possessing some striking peculiarity such as the "singular points." A generic point on a plane cubic would signify any point except an inflexion. But the case is not so simple as that. There are 27 points at each of which a conic with 6-point contact can be drawn (sextactic points), 72 which are vertices of the 24 Poncelet triangles, a limited number which are vertices of Poncelet polygons of higher order, etc. All such points are in a strict sense special since other points of the curve do not share all of their properties. I submit the following criterion for the equivalence of points on an algebraic curve: *If the curve is not invariant under a collineation group, no two points of the curve are equivalent in a projective sense; while if the curve is invariant under a collineation group, only conjugate points may be regarded as equivalent.*

[1] All of the elements of a binary group, except the identity, are of type (*a*).

In the present instance the nine flexes are equivalent since they constitute a set conjugate under the G_{18} which leaves the curve invariant. The 27 sextactic points however belong in three conjugate sets of 9 points each, while the 72 vertices of the Poncelet triangles divide into four sets each containing 18 conjugate points.

Again the binomial rational curves

$$x = t^n, \quad y = t^{n-r}, \quad z = 1, \quad \text{or} \quad y^n - x^{n-r}z^r = 0$$

are invariant under infinite groups, binary and ternary, generated respectively by

$$t = at' \quad \text{and} \quad x' = a^n x, \quad y' = a^{n-r}y, \quad z' = z.$$

Each of the curves has dual singularities at $t=0$, ∞ which absorb all of the double points, flexes and double lines. Each of these points is fixed under both groups, but all other points are equivalent to each other.[1]

Special sets. Certain points of the plane may assume fewer than n positions under the operations of a G_n. Such points belong to special sets of conjugate points. Each point of a special set must be fixed under one or more collineations, hence to discover all special sets we need only examine the fixed points of the constituent elements. Suppose that a point is fixed under k (and only k) elements of G_n. Then every point conjugate to it will be fixed under k elements and the aggregate of such points will form a special set. Every point in the special set thus arises from the coincidence of k points of a general set. It follows that

The number of points in a special set is a factor of n.

Now all the elements of G_n that leave a point fixed obviously form a subgroup, since if any two elements leave a point fixed so also will their product. Hence *each point of a special set of n/k conjugate points is invariant under a subgroup G_k, of order k.* Moreover, *the set of subgroups of order k, each of which comprises all the elements that leave one point of a special set of n/k points unaltered are conjugate subgroups.* The number of conjugate subgroups in such a set may be equal to n/k or it may be less, since the subgroups need not all be distinct. The number, which in the extreme case may reduce to one, is however always a factor of n/k.

As its name indicates a collineation carries lines into lines as well as points into points. All that we have said about conjugate sets of points applies equally to conjugate sets of lines for

The configuration of fixed points and lines under the collineation (1) *is self-dual. Hence the configuration determined by every special set of conjugate lines is dual to that determined by some special set of conjugate points.*

To prove the theorem for the plane, we observe that the collineation (1), $i=1, 2, 3$, transforms the line

[1] There is an exception in the case of the conic which has no singular points. And since 0 and ∞ may be assigned as parameters to any two points of the curve, all points on a conic are equivalent according to the criterion.

$$u_1x_1 + u_2x_2 + u_3x_3 = 0$$

into

$$\rho(u_1x_1'/a_1 + u_2x_2'/a_2 + u_3x_3'/a_3) \equiv u_1'x_1' + u_2'x_2' + u_3'x_3' = 0.$$

Thus the connection between the old and new u's is

$$u_i' = \rho u_i/a_i, \tag{2}$$

which is the same collineation as (1) expressed in line form. Now the fixed points and lines are found by putting $x'=x$ and $u'=u$. The collineations reduce to

$$\rho x_i = a_i x_i \quad \text{and} \quad \rho u_i = a_i u_i$$

which are identical in form. It follows that these equations will be consistent under the same conditions, viz. $\rho = a_i$, and that the coordinates of any point which satisfy the first set will be identical with the coordinates of a line that satisfy the second. The proof is valid for n dimensions by letting i range through $n+1$ values and replacing "line" by S_{n-1}, i.e. by a linear space of $n-1$ dimensions.

If the a's are distinct the collineation (1) [or (2)] has three fixed points and three fixed lines forming a triangle. But if two of the a's are equal, the collineation—which is then called a *homology*—has a point of fixed lines (the center) and a line of fixed points (the axis). The centers and axes of the homologies [type (b)] together with the fixed points and lines of the collineations of type (a) in a group may be called the *invariant configuration* of the group. As a consequence of the theorem proved above, we have at once:

The invariant configuration of a group is self-dual.

Thus every collineation group leads us to one or more associated configurations,—the invariant configuration and others which are frequently contained in it. These configurations include some of the most interesting to be found in geometry and, when studied in relation to the invariant curves of the groups, afford an opportunity for the contemplation of the highest form of beauty, a beauty whose full appreciation is the prerogative solely of the mathematician since much of the complete figure is usually imaginary.

The converse question is, does every configuration determine a finite group? Certain well known groups may be so defined. Thus the ternary octahedral G_{24} comprises those transformations which permute in all possible ways the vertices of a complete 4-point. The ternary Hesse G_{216} may be obtained by asking for those collineations which transform into itself the Hesse configuration—the configuration associated with the nine flexes of the plane cubic. Coble has shown that the collineations which transform into themselves the seven points and seven lines in finite geometry (mod 2) constitute a group of order 168, simply isomorphic with the Klein group in the general projective plane.[1]

[1] Two groups G and G' are simply isomorphic if (a) there is a (1, 1) correspondence between their elements, (b) such that the product of any two elements of G corresponds to the product of the two corresponding elements of G'. Two groups which are simply isomorphic are said to be abstractly identical.

The major problems that any collineation group offers for study are the structure of the group itself—the relations between the elements, the nature and variety of the subgroups, its isomorphisms with other groups, etc.—and the invariant configuration and invariant curves. The structure may of course be studied purely algebraically and the knowledge gained may then be applied to the geometry. But I think enough has been said to indicate that the geometry is a useful weapon of attack on the fortress of the group itself. In particular a knowledge of special sets of conjugate points may be utilized in the discovery of the conjugate subgroups associated with them.

§3. *Self-Projective Curves*

The determination of the types of groups in any dimension, as well as the study of their structure, may be considered an algebraic or group-theoretic problem. However the first solution of this problem in one dimension, the binary realm, was effected by Klein with the aid of geometric considerations. Indeed these groups, in the abstract, are identical with the groups of rotations of a sphere which carry the regular inscribed polyhedrons into themselves. The groups associated with the cube and the octahedron are the same, since diameters of the sphere through the midpoints of the faces of the cube cut the sphere in the vertices of a regular octahedron. The two figures are in a sense space duals, the cube having 6 faces, 8 vertices and 12 edges while the octahedron has 8 faces, 6 vertices and 12 edges. Thus vertices and faces correspond in this duality while an edge is self-dual. Similarly the regular inscribed dodecahedron with 12 faces, 20 vertices and 30 edges is dual to the regular icosahedron with 20 faces, 12 vertices and 30 edges,—both admitting the same group. The regular tetrahedron is self-dual since lines joining the vertices to the center cut the sphere in the vertices of a second, the counter tetrahedron. To complete the theory, Klein devised the regular *dihedron*, a two-faced "polyhedron" consisting of two coincident regular polygons inscribed in a great circle.

The groups of rotations take their names from the polyhedrons which they leave invariant and are sometimes called the regular body groups. If a dihedron with n vertices lies in an equatorial plane it will be carried into itself by a rotation of period n about the axis passing through the poles, generating a cyclic group of order n. If to this group we adjoin the n rotations of period 2 about lines of symmetry of the dihedron, we obtain the complete group of the dihedron, a G_{2n}. For the other regular bodies there are three possible positions of the axis of rotation (a) through a vertex, (b) through the mid-point of a face and (c) through the mid-points of opposite edges. Rotations (c) are of period 2 for all of the groups; and while types (a) and (b) vary with the groups, all of either class are of the same period for any particular group. In the case of the tetrahedron, (a) and (b) coincide, both being of period 3. The number of rotations of the several periods are easily counted and the orders of the groups determined. The five regular body groups then are:

Name and Order	*Associated Polyhedron(s)*
Cyclic G_n	Dihedron
Dihedral G_{2n}	Dihedron
Tetrahedral G_{12}	Tetrahedron
Octahedral G_{24}	Octahedron, cube
Icosahedral G_{60}	Icosahedron, dodecahedron.

While the regular bodies were known to the ancients, it remained for Klein to point out the varieties of groups associated with them and to prove in particular that they are simply isomorphic with types of collineation groups in the binary domain. It seems to have been Klein's peculiar genius to excel in recognizing these analogies in mathematical theories which constantly challenge the wonder of every mathematician. I know of no more striking manifestation of that genius than his achievement in reading in the properties of the familiar —one might almost say household objects—the regular bodies, all the group theory implicit in them, a theory latent since the time of Pythagoras, eluding the searching scrutiny of the mathematical world for two and a quarter milleniums.

To my mind this geometric approach to the analysis of the groups in the binary field, because of its concreteness, simplicity and elegance, no less than for its intuitive directness is incomparably superior to any other. What other important theory perchance lies dormant in the sombre chrysalis of elementary geometry, awaiting the fullness of time and the creative touch of some future Klein to call it into a glorified being?

In the plane, as previously indicated, there are two main divisions of groups, (a) those which leave a triangle invariant (monomial groups) and (b) those which do not. In particular we have five regular body groups—ternary groups simply isomorphic with the several binary groups.[1] These are all monomial except the ternary icosahedral group. There are six varieties of groups of type (b) in the plane: The icosahedral G_{60}, a G_{36}, a G_{72}, the Hesse G_{216}, the Klein G_{168} and the Valentiner G_{360}. The G_{36} is one of three conjugate subgroups of the G_{72} which in turn is a self-conjugate or invariant subgroup of the Hesse group.

We have already mentioned one geometric problem connected with every group—that of the invariant configuration. Two others relate to the invariant curves. The first is the determination of the *complete system* of invariants, i.e. a finite number of forms in terms of which all others may be rationally and integrally expressed. This complete system for groups in n homogeneous variables comprises a minimum of $n+1$ forms of which n are algebraically independent. This problem has been solved for all of the principal classes of groups.

The second problem is that of self-projective curves. Since a collineation defines analytically a projection, a curve which is invariant under a collinea-

[1] In every dimension there is at least one regular body group simply isomorphic with each of the binary group. Ciani, Annali di Matematica (1902), has discussed the different species in space.

tion group is termed self-projective. The problem may be formulated thus: (1) What varieties of groups may a curve of given order m admit? (2) what are the projectively distinct species of curves of order m invariant under the same group? This problem for curves restricted only by the group requirements has been solved for $m=2, 3, 4, 5, 6$ and for rational curves for $m=2, 3, 4, 5, 6, 7$. A conic may admit any of the regular body groups. The general cubic is invariant under a G_{18}, a subgroup of the G_{36} above; the harmonic cubic under the G_{36}; and the equianharmonic cubic $x^3+y^3+z^3=0$ under the subgroup G_{54} comprising all of the monomial transformations of the Hesse G_{216}. The general rational cubic admits a dihedral G_6 and the cuspidal cubic an infinite group.

A self-projective quartic is necessarily special but there are numerous interesting varieties. The octahedral G_{24} has a pencil of them

$$x^4 + y^4 + z^4 + 6\lambda(y^2z^2 + z^2x^2 + x^2y^2) = 0,$$

including a repeated conic ($\lambda=1/3$), the projective lemniscate ($\lambda=\infty$), two Clebsch curves[1] ($\lambda=1,-1/2$) and their associated Lüroth curves ($\lambda=-1/3$, $7/6$, respectively), a pair of conics[2] ($\lambda=-1/6$), a Dyck curve ($\lambda=0$) and a pair of Klein quartics ($\lambda=-\frac{1}{2}\pm\frac{1}{2}\sqrt{7}$). The Dyck and Klein quartics admit groups of orders 96 and 168 respectively.

The curves of lowest orders associated with the Hesse and Valentiner groups are sextics.

Self-projective curves are always interesting because of the remarkable way in which they adjust themselves to the rather severe demands of the groups. If a curve is invariant under a G_n, the points of the curve in general distribute themselves in sets of n conjugate points, though some may belong to special sets. In particular, the singularities will be found among the conjugate sets, special or general. The number of special sets of points on an invariant curve is limited, for the number of special sets under the group is finite unless the group contains homologies. In the latter case there will be an infinite number of special sets, lying on the axes of homology, but only a finite number of these can lie on the curve.

The simplest example of a self-projective curve is one admitting a group of order two, consisting of a reflexion and the identity. Even here the curve is considerably specialized. First of all, each intersection of the axis of reflexion with the curve is either a multiple point or a contact of a tangent from the center of reflexion. All other points—except the center, if it happens to lie on the curve—must fall into pairs harmonically separated from the center by the axis. All tangents from the center whose contacts do not fall on the axis must be

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[1] See a paper by Arnold Emch, Bulletin of the American Mathematical Society vol. 35 (1929), p. 389. The first Lüroth quartic ($\lambda=-1/3$) degenerates into four lines, dual to the special set of four points conjugate under the group—see below.

[2] This pair of conics admit an enlarged group G_{48} of collineations and correlations, the correlations arising from the products of the elements of G_{24} by a polarity. See Wear, American Journal of Mathematics) vol. 42 (1920), p. 118.

multiple tangents with contacts harmonically paired. The contacts on the axis can arise only from the coincidence of an even number of points—they may be undulations, e.g., but not flexes. A multiple tangent from the center may have an odd number of contacts only if one fall on the axis. If the center lies on the curve, the least thing that can happen is that the point be an inflexion. It might be a flex of higher order, the tangent cutting out an odd number of coincident points. The center may be a biflecnode and on curves of even order it may be a tac-node. If a curve is restricted only by admitting a reflexion, the intersections of the axis will be contacts of simple tangents from the center, while other tangents from the center will be ordinary double lines. Again if the center lies on the curve, we normally expect it to be an ordinary inflexion if the curve is of odd order, a biflecnode if the curve is of even order. Now most groups contain several reflexions which comprise one or more conjugate sets. If k of these belong to such a conjugate set, the yield of geometrical fruitage is exactly k-fold, as illustrated by those curves possessing k-fold symmetry and invariant under a dihedral G_{2k}, k odd. If k is even the curve still has k-fold symmetry but the axes divide into two conjugates sets, the symmetry with respect to the two sets not being the same.

Suppose that a curve is invariant under a homology[1] of period three, e.g. $x'=\omega x$, $y'=y$, $z'=z$, $\omega^3=1$. Then lines through the center $(1, 0, 0)$ cut out triples of conjugate points. The intersections of the axis, $x=0$, thus will be flexes, cusps etc. whose tangents meet at the center—or they will be triple points, sixfold points etc. Other tangents from the center must be at least triple tangents. If the center is on the curve, it will be at least an undulation since it must arise from the coincidence of a minimum of four points. Thus the quartic curve $y(y+z)^3-x^3z=0$ admits a cyclic G_3 generated by the homology above. The center is an undulation (whose tangent is $z=0$) and the axis cuts out a triple point, and a flex whose tangent, $y=0$, passes through the center.

Another important principle is that two curves which are invariant under the same group must intersect in sets of conjugate points. Klein has utilized this principle in finding the complete system of invariants of a group.[2] It is also helpful in constructing sets of conjugate points on an invariant curve. There are ∞^2 sets of conjugate points associated with any group since any point in the plane gives rise to one set. Only a single infinity of these however will lie on an invariant curve. As an illustration, the equianharmonic cubic $x^3+y^3+z^3=0$ is invariant under a monomial G_{54}, generated by the homology of the preceding paragraph and the transformations

$$\begin{aligned} x' &= x, && z, && -x, \\ y' &= \omega y, && x, && -z, \\ z' &= \omega^2 z, && y, && -y. \end{aligned}$$

[1] The homology of course generates a cyclic group of order three which contains a second homology.

[2] See for example Fricke, *Lehrbuch der Algebra* (1926), vol. 2, p. 203.

The curve contains three special sets of points, (a) the 9 flexes, (b) the 27 sextactic points, (c) a set of 18 points, vertices of six Poncelet polygons, cut out by the invariant sextic $y^3z^3+z^3x^3+x^3y^3=0$. These as well as all general sets on the cubic are cut out by the pencil of invariant curves[1]

$$x^6y^6z^6 + \lambda(y^3z^3 + z^3x^3 + x^3y^3)^3 = 0.$$

The special sets are cut out multiply by particular curves—the flexes 6 times ($\lambda=0$), the 18 points 3 times ($\lambda=\infty$) and the sextactic points twice ($\lambda=4/27$).

A problem introduced into geometry by the analytic method is that of canonical forms, i.e. the discovery of the simplest form to which the equation of a locus may be reduced. From the point of view of pure geometry the problem is artificial since it merely concerns the optimum choice of the reference framework. If a curve is invariant under a group, the selection of the reference triangle is usually not difficult. We saw that the collineations in a group are of two kinds—those with a unique fixed triangle and the homologies with ∞^2 fixed triangles. Each of the latter has as one vertex and opposite side a center and axis of a homology. *When the curve is in canonical form the triangle of reference is some one of these tringles associated with the individual collineations*, i.e. *it is part of the invariant configuration of the group.*

Thus the equianharmonic cubic assumes the form given above when referred to the invariant triangle of the G_{54}. But there is another form entitled to consideration as a canonical form of the curve. For if we select as reference triangle any one of the six Poncelet triangles whose vertices comprise the special set of 18 conjugate points, the equation of the curve may be reduced to $x^2y+y^2z+z^2x=0$, or $xy^2+yz^2+zx^2=0$.

We shall close this section with a statement of some general theorems concerning self-projective curves.

1^0. *The dual of any self-projective curve admits the same group.*

For suppose that any collineation of the group is reduced to the form (1), then the line equation of the collineation is (2). The inverse of (2) is

$$\rho u_i' = a_i u_i, \tag{3}$$

which is identical in form with (1). Hence if the point curve $f(x_1, x_2, x_3)=0$ is invariant under (1), the dual curve $f(u_1, u_2, u_3)=0$ is invariant under (3), which is merely some power of (1) in line form. Thus, since the projective lemniscate is an invariant of the ternary octahedral G_{24}, we infer that its dual the projective astroid is likewise an invariant.

2^0. *If an invariant curve C_{2n}, of order $2n$, factors into two curves C_n and C_n', neither of which is invariant under the whole group, C_n and C_n' are projectively equivalent*, since they must be interchanged by some elements of the group.

3^0. *A C_n cannot admit a homology of period greater than n*, since conjugate points must lie in sets of n on lies through the center.

[1] The curves of this pencil are composite, each composed of three sextics.

4°. *If a C_n is invariant under a homology of period k, the first polar of C_n is composite, the axis splitting off $k-1$ times. The axis splits off from the second polar $k-2$ times, etc. If $k=n$, the first polar of an arbitrary point on the axis degenerates into $n-1$ lines passing through the center.*[1]

§4. *Self-Projective Rational Curves*

The group problem for rational curves differs from that for general algebraic curves because of the parametric representation that characterizes the former. By definition, the homogeneous coordinates of a point on a rational curve are expressible as rational, integral, algebraic functions of a single parameter,—i.e. as binary forms. Hence the geometry on a rational curve belongs to the binary domain. The following theorem is fundamental for rational curves.

1°. *If the points of a rational curve in S_{n-1} are transformed into themselves by an n-ary group, the parameters of the points are permuted by a binary group of the same order.*

The only groups admissible then are the regular body groups, binary and n-ary. Other general theorems for rational plane curves are:

2°. *If a rational curve of order m is invariant under a cyclic group of order n, $n>m$, it is invariant under a one-parameter group.*

3°. *A rational curve of order m, m odd, cannot admit the tetrahedral, octahedral nor icosahedral group, nor a dihedral G_{2k}, if k is even.*

We have already observed (§1) that the maximum order of a dihedral group is $2m$. Theorems 2° and 3° thus restrict the types of groups for curves of odd degree to cyclic groups of order n, $n=2, 3, \cdots, m$, and dihedral groups of order $2k$, $k=3, 5, \cdots, m$.

4°. *If a rational curve of order m is invariant under a cyclic group of order m (but not a dihedral G_{2m}) the group cannot contain a homology. The curve has a multiple point of order greater than two, having however but two distinct parameters.*[2]

We have in the plane then a partial parallelism between the binary and ternary groups associated with self-projective rational curves. Every element of the binary group has two fixed points (parameters). These points as well as their tangents are fixed under the ternary group, which however leaves at least one other point and line fixed. Hence there will be a difference in the special sets for the two groups. Every non-cyclic binary group has three special sets, corresponding to (a) the vertices, (b) the midpoints of the faces and (c) the midpoints of the edges. These will correspond to special sets of the same number of points under the ternary group if each parameter is attached to a single point. But if a binary set of $2k$ are parameters of k nodes, the nodes will form a special ternary set of k points although their tangents will be a special ternary set of $2k$ lines. These are the only special sets of points lying on an invariant

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[1] This theorem is proved in a paper read at the Nashville meeting of the American Mathematical Society.

[2] These theorems are proved in a series of papers by the author on self-projective rational curves in the American Journal of Mathematics, vols. 36, 38, and 47.

curve but the ternary group contains an infinite number of special sets not on the curve. A binary cyclic group leaves each of two points fixed while the corresponding ternary group has at least one additional fixed point.

Perhaps these principles will be sufficiently illustrated by the binary and ternary octahedral groups and the invariant projective astroid, which in metrical form is the familiar "hypocycloid of four cusps," a rational curve of class 4 and order 6. For convenience we shall denote ternary groups by G.

The G_{24} is the group of the complete quadrangle, permuting the vertices in all possible ways. The diagonal triangle is fixed under the whole group. The vertices and sides of this triangle are centers and axes of three reflexions, which with the identity constitute a dihedral G_4, an invariant sub-group of G_{24}. Each vertex and opposite side are likewise the fixed point and line of one of the three conjugate dihedral subgroups G_8. The sides of the quadrangle are the axes of six other reflexions, conjugate under the group. Each vertex of the quandrangle is fixed under one of the four conjugate dihedral subgroups G_6. The four cyclic subgroups G_3 of the dihedral G_6's leave each two additional points unaltered, giving rise thus to a special set of eight points. The centers of the set of 6 reflexions lie in pairs on the sides of the invariant triangle—each pair harmonically separated by a pair of vertices of the triangle. On each side of the triangle is a third pair, harmonic both to the vertices and the centers on that side. We thus obtain a second set of six conjugate points. The special sets of points of G_{24} thus comprise:

A set of 3, vertices of the invariant triangle;
A set of 4, vertices of the invariant quadrangle;
A set of 6, centers of the set of six reflexions;
A counter set of 6, described above;
A set of 8, which with the set of four are the fixed points of the four cyclic G_3's;
∞^1 sets of 12, lying in pairs on the sides of the invariant quadrangle;
∞^1 sets of 12 lying in fours on the sides of the invariant triangle.

Dual to each of these is a special set of conjugate lines.

When written in the usual Cartesian form

$$x^{2/3} + y^{2/3} = a^{2/3}, \tag{4}$$

the curve exibits symmetry with respect to the origin and four equispaced lines through the origin, i.e. one of the dihedral G_8's is real. The invariant triangle of the group is then formed by the rectangular axes and the line at infinity. Since the invariant triangle is treated symmetrically by the group, we infer that the line at infinity is a third double-cusp tangent, the cusps falling at the circular points. The six cusps thus lie on a circle, the invariant conic of the group. The flexes all go to form the cusps, while the three double-cusp tangents are the only bitangents. There are however four nodes, isolated at $(\pm a/8^{1/2}, \pm a/8^{1/2})$, which are the vertices of the invariant quadrangle.

The binary group has but three special sets of points (parameters) (a) the cusps, the counter set of 6, corresponding to the octahedron vertces, (b) the 8

nodal parameters, corresponding to the cube vertices and (c) a set of 12, the residual intersections of the sides of the quadrangle of double points. Thus (a) and (c) are special sets of the ternary group as well while (b) determine a set of 4 in the ternary group.

If we define the curve as the envelope of a line which the rectangular axes cut in a segment of constant length a, the equation in Plücker coordinates is found at once to be

$$1/u^2 + 1/v^2 = a^2$$

the point equation of which is (4). The locus of the middle point of this line segment is a circle (conic on two cusps), having four proper contacts with the curve. This is a member of the invariant pencil of double-contact conics (concentric circles) belonging to the real dihedral G_8. Since the three dihedral G_8's are conjugate, the circle is one of a set of three conics conjugate under G_{24}. Bearing in mind the projective version of middle point of a segment and the geometry of a binary cubic, we have the theorem:

If the intersections of a variable tangent line of the projective astroid with the three double-cusp tangents are represented by a binary cubic, the three cubicovariant points will trace, each a conic passing through two cusps and having four contacts with the curve. The 12 contacts are a special set under both the binary and the ternary group.

THE ELDER ĀRYABHAṬA'S VALUE OF π

By SĀRADĀKĀNTA GĀṄGULI, Ravenshaw College, Cuttack, India

The credit of discovering two remarkably approximate values of π in ancient times belongs to two Asiatic scholars of the fifth century A.D. One of them is the Chinese astronomer Tsu Ch'ung-chih (born 430 A.D.) and the other is the Indian astronomer and mathematician Āryabhaṭa (born 476 A.D.) known as the elder Āryabhaṭa as distinguished from another astronomer-mathematician of the same name who lived in the tenth century A.D. Tsu gives the value 355/113 ($=3.141592 \cdots$)[1] and Āryabhaṭa the value 62832/20000 ($=3.1416$). In point of accuracy Tsu's value is superior to Āryabhaṭa's value which, however, can claim practical advantages over the other. Tsu's value has not been the subject of much discussion and its authorship has not been assigned to a foreign writer or to a later Chinese investigator. But Āryabhaṭa's value has received an altogether different treatment. It has been the subject of a long controversy and its genuineness has been seriously questioned. Attempts have been made to attribute the elder Āryabhaṭa's value of π to an unknown Greek source or to the younger Āryabhaṭa.

[1] Mr. Kaye omits this value from his list published in *Indian Mathematics* (p. 33) but includes Tsu's inaccurate value, 22/7, in it.

The arguments urged in favour of a Greek origin are:

(i) According to Rodet "The choice of a diameter of *two myriads* or rather of the number *one myriad* for the radius is assuredly a strong argument in favour of a Greek origin."[1] For, in his opinion, the Greeks alone of all peoples made the myriad a numerical unit of the second order. Sir Thomas Heath also endorses this view.[2]

(ii) According to Alberuni, Pulisá employed a value of π equal to the elder Āryabhaṭa's value.[3] Mr. Kaye takes this Pulisá to be anterior to both the elder Āryabhaṭa and Tsu Ch'ung-chih.[4]

(iii) The value 3.1416 was known to the Alexandrian scholars whose works may well have reached India.[5]

(iv) "The Hindu mathematicians took various values of π and no writer among them seems to have been uniform in his usage."[6]

(i) The unsoundness of the first of the above arguments has been shown by the present writer[7] and Dr. Bibhūtibhūṣaṇ Datta.[8] Āryabhaṭa used the numerical unit *ayuta* which is equivalent to the Greek myriad but which has been in use in India from before the birth of the Greek civilization. I may add that Rodet's argument does not seem to have appealed to Marie who does not see to which Greek writer the elder Āryabhaṭa's value of π could be attributed but who admits, as we all admit, that there had existed, even before the fifth century A.D., intercourse between Greece and India.[9] But mere existence of intercourse can not prove India's indebtedness to Greece for a result which, so far as our present knowledge goes, the latter did not know.

(ii) The fallacy of this argument has been thoroughly exposed by Mr. Nalinbihāri Mitra[10] and Dr. Bibhūtibhūsan Datta.[11] Dr. Datta has shown that Alberuni's Pulisá is not the author of the original *Pulisá-siddhānta* or of the one known to Varāhamihira but of a recast of the same work with various changes and additions. The author of the original *Pulisá-siddhānta* was a contemporary of the elder Āryabhaṭa according to Weber while, according to Kern, the former preceded the latter by a century.[12] Alberuni's Pulisá refers to Varāhamihira

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[1] *Journal Asiatique*, vol. 13 (1879), p. 411.

[2] *History of Greek Mathematics*, vol. 1 (1921), p. 234.

[3] Alberuni's *India*, vol. 1, p. 168; vol. 2, p. 72 and p. 67, where the fraction 14/15 is a mistake for 14/25.

[4] *Indian Mathematics*, p. 33.

[5] D. E. Smith, *History of Mathematics*, vol. 2 (1925), pp. 308–309.

[6] Ibid, p. 308.

[7] Journal of the Bihar and Orissa Research Society for March, 1926, pp. 83 and 84.

[8] Journal of the Asiatic Society of Bengal, vol. 22 (1926), pp. 36–37; hereafter referred to as *JASB*.

[9] Marie, *Histoire des Sciences Mathématiques et Physiques*, vol. 2, p. 73.

[10] *Modern Review* (Calcutta), vol. 18, p. 160.

[11] *JASB*, vol. 22 (1926), pp. 27, 37, 38.

[12] *JASB*, vol. 4 (1908), p. 115.

twice[1] and not once as stated by Mr. Kaye,[2] uses the elder Āryabhaṭa's value of π, and also takes his measure (namely, 3438′) of the radius as the radius of the sphere of each planet.[3] For a Greek origin of the elder Āryabhaṭa's value of π Mr. Kaye finds it necessary to establish the priority of Alberuni's Pulisá to the elder Āryabhaṭa. Hence he wants to take away Pulisá's reference to Āryabhaṭa and Varāhamihira from the mouth of Alberuni's Pulisá and to put it into the mouth of Alberuni himself.[4] But he is silent with respect to a second reference to Varāhamihira by Alberuni's Pulisá. He then concludes that "the traditional order . . . Pulisa, Āryabhaṭa, Varāhamihira . . . may be accepted as correct." We do not object to this conclusion. Only we would extend Mr. Kaye's list as follows: Pulisá (whose $\pi=\sqrt{10}$), Āryabhaṭa, Varāhamihira, Alberuni's Pulisá (i.e., the author of Alberuni's *Pulisá-siddhānta* whose $\pi=3.1416$). Here we have maintained the order adopted by Mr. Kaye.

(iii) Dr. D. E. Smith attributes the value 3.1416 to Ptolemy who takes $3\ 8'\ 30''$ as the approximate value of π. He writes: "Since $3\ 8'\ 30''=3.1416$, his (Ptolemy's) value was very satisfactory."[5] But $3\ 8'\ 30''$ is not equal to 3.1416 but to 3.1417 (correct to 4 places of decimals). It is really equal to 377/120 whereas Ārybahaṭa's value is equal to 3927/1250. So it is difficult to see how the elder Āryabhaṭa's value of π was known to the Alexandrian scholars.[6] Thus the possibility of obtaining this value from their works disappears altogether. Hence Dr. Smith's alternative suggestion[7] that this value may have been found independently in India is the only possible conclusion which stands to reason.

(iv) Mr. Kaye's table of different values of π (*Indian Mathematics*, pp. 32 and 33) is mainly responsible for this argument. It shows that he has failed to correctly understand some of the results of Indian mathematical and astronomical works. The elder Āryabhaṭa uses only one value of π, namely, 62832/20000. But the following additional values have been wrongly attributed to him:[8]

(1) 3; (2) $3393/1080 = 3.14166$;

(3) $600/191 = 3.14136\cdots$ (4) $16/9$;

(5) 1.7.

(1) In the verse immediately preceding the one giving the elder Āryabhaṭa's value of π he states that the chord of the sixth part of the circumference of a circle is equal to the radius. This shows that where accuracy is required the elder Āryabhaṭa cannot take π to be equal to 3. Mr. Kaye has left his readers in the dark as to his reasons for ascribing this value of π to the elder Āryabhaṭa.

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[1] Alberuni's *India*, vol. 1, p. 266; vol. 2, p. 70.

[2] *JASB*, vol. 4 (1908), p. 115.

[3] Alberuni's *India*, vol. 2, p. 69.

[4] *JASB*, vol. 4 (1908), p. 115.

[5] *History of Mathematics*, vol. 2, p. 308.

[6] Also see *JASB*, vol. 22 (1926), p. 38.

[7] *History of Mathematics*, vol. 2, p. 309.

[8] Also see *Indian Education*, vol. 8, p. 351; *JASB*, vol. 4 (1908), pp. 121, 122 foot-note.

(2) The value 3393/1080, which is equal to Ptolemy's value 377/120, has been attributed to Āryabhaṭa because it is stated by Alberuni, on the alleged authority of Brahmagupta,[1] that Āryabhaṭa fixed the circumference as 3393 and the diameter as 1080. Using the value 62832/20000 of π the circumference of a circle of diameter $1080 = (62832/20000) \times 1080 = 3392.928 = 3393$ (to the nearest integer). It will thus be seen that Āryabhaṭa did not use the value 3393/1080 which he does not even mention in his *Gaṇitapāda* (i.e., the chapter on mathematics). The unnatural character of Mr. Kaye's deduction of this value will be clear from the following question: If it be hereafter found that the elder Āryabhaṭa has found, to the nearest integer, the circumferences of 10,000 circles of different diameters, how many different values of π should be attributed to him, seeing that the ratio of the circumference obtained to the corresponding given diameter is different in each case?

(3) Mr. Kaye has followed the same unnatural method in attributing the value 600/191 to the elder Āryabhaṭa. He writes: "Āryabhaṭa sets the radius $= 3438$ minutes whence[2]

$$\pi = (2 \times 90 \times 60) \div 3438 = 600/191 (= 3 \cdot 14136 \cdots)."$$

How has Āryabhaṭa come to take the radius $= 3438$ minutes? He is the first investigator to introduce this innovation.[3] Rodet answers this question thus: $(10800'/1.1416) = 3437'.7 = 3438'$ nearly; and he concludes that the value 3.1416 of π was used. But Mr. Kaye observes:[4] "This is not ingenuous. We might replace this value by Ptolemy's value and then we should have $10800 \times (120/377) = 3437.66 = 3438$, nearly; and we might just as forcibly conclude that Ptolemy's value was used."[5] If Āryabhaṭa had not given a value of π, one would have been justified in suggesting that he might have used Ptolemy's value. Mr. Kaye's observation, therefore, is not only not ingenuous but it also betrays an excessive amount of anti-Indian bias.

(4) Mr. Kaye attributes the value 16/9 to the elder Āryabhaṭa because the latter gives the formula $(\pi r^2)^{3/2}$ for the volume of a sphere of radius r. This is based on the assumption that Āryabhaṭa also knew the volume to be $4\pi r^3/3$.

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[1] Alberuni's *India*, vol. 1, p. 168.

[2] *Indian Education*, vol. 8, p. 351.

[3] Mr. Kaye is not right when he says: "This new measure first occurs in the *Pulisá-siddhānta*, for Alberuni writes (i,275): 'Our calculation is based on this, that the *sinus totus* is3438.' . . . . The source of this calculation . . . . is the *Pulisá-siddhānta*.' " In the last gap Alberuni has used the words "of Balabhadra." He says that he could not find anything of Āryabhata's works (*India*, i, 370). This is why he considers the *Pulisá-siddhānta* as the source of Balabhadra's calculation. It has already been stated that Alberuni's *Pulisā-siddhānta* refers once to Āryabhata and twice to Varāhamihira. In the original *Pulisá-siddhānta* or rather in the one known to Varāhamihira $\pi = \sqrt{10}$ and not $3 \cdot 1416$, and $r = 120$ and not 3438 (*Indian Mathematics*, p. 11).

[4] *JASB*, vol. 4 (1908), p. 125.

[5] Mr. Kaye then says: "Ptolemy's value was most probably used." Thus, according to Mr. Kaye, Āryabhata used both 600/191 (also see *Indian Mathematics*, p. 11) and 377/120 in deducing his measure (namely, 3438') of the radius of a circle.

As there is no justification for this assumption—and Mr. Kaye is not prepared to give Āryabhaṭa credit for a correct expression for the volume of a sphere—, it is wrong to say that in practical applications Āryabhaṭa uses the value 16/9 for π.

(5) 1.7 seems to be a misprint for $1.\dot{7}$ which is the decimal equivalent of 16/9.

Hence it cannot be said that the elder Āryabhaṭa uses various values of π and that he is not uniform in his usage.

For rough calculations where rapidity is preferred to accuracy (e.g., in finding roughly the quantity of grain in a mound) Brahmagupta, Mahāvīra, Śrīdhara, the younger Āryabhaṭa, and Bhāskara take $\pi=3$. For a greater degree of accuracy Brahmagupta, Mahāvīra, Śrīdhara, and the younger Āryabhaṭa use the value $\sqrt{10}$ and Bhāskara 22/7. For further accuracy the younger Āryabhaṭa employs[1] the value 22/7 and Bhāskara 3927/1250. To secure the same degree of accuracy the approximate value of $\sqrt{10}$ to be used must always depend on the magnitude of the number to be multiplied by $\sqrt{10}$. Failing to understand this, Mr. Kaye wrongly attributes the values[2] 22/7 and 721/228 to Brahmagupta and the value[3] 19/6 to Śrīdhara. For a similar reason the value 377/120 has been attributed to Bhāskara.

Mr. Kaye is perhaps inwardly conscious of the weakness of his arguments in favour of a Greek origin of the elder Āryabhaṭa's value of π. So it must be traced to some other non-Indian source. Hence Mr. Kaye writes that possibly the rule giving the value "properly belongs to a later writer, possibly to Āryabhaṭa the younger."[4] If he could establish this point, the credit of discovering this value of π would go to the Arabic mathematician Mohammed Ben Musa who, Mr. Kaye dogmatically asserts,[5] did not copy it from the Hindus.

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[1] Dr. Bibhūtibhūsan Datta infers from the *Mahāsiddhānta*, chapter 16, verse 37, that the younger Āryabhaṭa employed another value of π, namely, 21600/6876(*JASB*, vol. 22,1926, p. 30.) But here the younger Āryabhata gives 6876/21600 or 191/600 as the value of $1/\pi$ which is required in finding the diameter of a circle when the circumference is given. As π is incommensurable, its approximate value is not the same as the reciprocal of the approximate value of $1/\pi$. He finds the circumference by multiplying the diameter by $\sqrt{10}$ or 22/7 and not by 21600/6876.

[2] Mr. Kaye attributes this value to Brahmagupta on the authority of Alberuni who states that Brahmagupta has taken $\pi=\sqrt{10}$ because it is nearly equal to $3\frac{1}{7}$ (*India*, vol. 1, p. 168). If Brahmagupta had been aware of the value $3\frac{1}{7}$, he would not have rejected it in favour of $\sqrt{10}$ which is not only more inconvenient in application but is also not so accurate as $3\frac{1}{7}$.

[3] Śrīdhara gives $\frac{1}{2}d^3[1+(1/18)]$ as the volume of a sphere of diameter d. From this Mr. Kaye infers that Śrīdhara uses the value $3\frac{1}{6}$ of π. The danger of deducing, from results obtained by using certain values of π, other values not actually used by the authors of the results has already been shown. Here also Śridhara uses $\sqrt{10}$ for π as shown below:

$$V=\frac{4}{3}\pi r^3=\frac{d^3}{2}\,\frac{\pi}{3}=\frac{d^3}{2}\,\frac{\sqrt{10}}{3}=\frac{d^3}{2}\left(1+\frac{1}{9}\right)^{1/2}=\frac{d^3}{2}\left(1+\frac{1}{18}\right),$$

nearly.

[4] *Indian Education*, vol. 8, p. 350.

[5] *JASB*, vol. 4, (1908), p. 122.

Mr. Kaye's arguments attributing the elder Āryabhaṭa's value of π to a later Indian writer are as follows:

(a)[1] It is rather extraordinary that the elder Āryabhaṭa himself never utilised this value (*Indian Mathematics*, p. 12; *JRAS*, July, 1910, p. 754).

(b)[1] It was not used by any other Indian mathematician before the twelfth century (*Indian Mathematics*, pp. 12 and 13; *Indian Education*, vol. 8, p. 351; *JRAS*, 1910, p. 754).

(c) No early Indian writer quotes Āryabhaṭa as recording this value (*Indian Mathematics*, p. 13; *Indian Education*, vol. 8, p. 350; *JASB*, vol. 4, 1908, p. 122).

(d) "Āryabhaṭa who is supposed to have discovered, or, at least, introduced the most accurate value of π known in ancient times," gives a wrong rule for the volume of a sphere (*Indian Education*, vol. 8, p. 351).

The first three of the above arguments have been discussed by Dr. Bibhūtibhūṣaṇ Datta who has shown them to be false.[2] The elder Āryabhaṭa has used his value of π in preparing his table of sines and in obtaining 1080 *yojanas* as the measure of the diameter of the earth's wind whose circumference is 3393 *yojanas* and 1050 *yojanas* as the diameter of the earth whose circumference he takes to be 3299 *yojanas*. This value of π was employed also by Varāhamihira, Lalla, the author of the recasted *Pulisá-siddhānta*, and Utpala Bhaṭṭa, all of whom were posterior to the elder Āryabhaṭa and anterior to Alberuni (first half of the 11th century).

The fourth argument need not be taken seriously. A similar remark could be made against many authorities. Rāmānujan has given many incorrect formulae. Yet credit for discovering important results has not been denied him. The fact that Diophantus could not detect the obvious solution ($x=1$) of the equations

$$52x^2 + 12 = \text{a square}, \quad (\textit{Arithmetica}, \text{ III.10}),$$

$$266x^2 - 10 = \text{a square} \quad (\textit{Arithmetica}, \text{ III.11}),$$

does not stand in Mr. Kaye's way of giving him credit[3] for a solution of the equation $ax^2-b=y^2$, although Diophantus himself admits his inability to solve the equation, $15m^2-36=$ a square.[4] Sir Thomas Heath's failure to detect the obvious root 7 of the equation:

$$p^4 - 2p^2 + 197 = \text{a square}^5$$

---

[1] Being misled by Mr. Kaye's writings, Dr. Cajori also has advanced these two arguments (*A History of Mathematics*, 1922, p. 87).

[2] *JASB*, vol. 22 (1926), pp. 26, 27, 34, 35, 36, 39. The second and third arguments have been discussed also by Mr. Nalinbihari Mitra (*Modern Review*, vol. 18, pp. 160 and 161).

[3] *East and West* (Simla), vol. 17, p. 677, footnote.

[4] *Arithmetica*, VI. 14.

[5] *Diophantus* (2nd edition), p. 84, foot-note. This equation may be written as

$$\left(\frac{p^2-1}{2}\right)^2 + 7^2 = \text{a square},$$

whence $(p^2-1)/2=24$, or $p=7$.

does not prevent Mr. Kaye from citing him as an authority. Why, then, should a few incorrect rules given by the elder Āryabhaṭa go to prove his inability to give his value of π?

An examination of the verse giving the value 62832/20000 of π strongly suggests that the author of the verse must be anterior to Varāhamihira who, so far as our present knowledge goes, is undoubtedly the first to introduce the popular custom of expressing numbers in the word-symbol notation based on the principle of place-value. This custom has since been invariably followed by his successors with the exception of the younger Āryabhaṭa who uses his *kaṭapayādi* system of notation in the astronomical chapters of his *Mahāsiddhānta* and the popular word-symbol notation in the arithmetical portion of the work.[1] In the verse under consideration the complicated number 62832, instead of being given in the word-symbol notation as *rada-vasu-kara-rasa* (*rada* = 32, *vasu* = 8, *kara* = 2, *rasa* = 6) or as some other similar equivalent expression, has been stated as follows: Eight times one-hundred-and-four (literally, one hundred increased by four) and sixty-two thousand.

It will appear from the foregoing discussion that, in the absence of fresh evidence to the contrary, the credit of discovering the value 62832/20000 (= 3.1416) of π must, in all fairness, go to the elder Āryabhaṭa.

We do not know how Āryabhaṭa obtained this value of π. He might have got it by actual measurement of the circumference of a circle of radius 10000 or by calculation of the length of the perimeter of a regular polygon of a very large number of sides inscribed in such a circle.

QUESTIONS AND DISCUSSIONS

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSION

I. The Pascal Triangle

By Norman Anning, University of Michigan

It is a tradition that the diagrams for certain proofs of the theorem of Pythagoras need no explanation except "Behold!". In the opinion of the writer most teachers treat Pascal's triangle as something of the same sort. It is presented about like this: "Here is a pretty way of remembering the coefficients of the expansion of $(1+x)^n$ when n is a small positive integer. Observe how the numbers in any line may be readily obtained from those in the line

[1] This fact alone is sufficient to distinguish the *Mahāsiddhanta* of the younger Ārayabhata from the *Āryabhatīyam* of the elder Āryabhata. Dr. D. E. Smith is not right when he says that it is not possible as yet to differentiate clearly between the works of the two Āryabhatas (*History of Mathematics*, vol. 1, p. 156).

above. There is nothing to prevent you from extending the scheme as far as you wish. We call it Pascal's triangle although it was known centuries before his time. *Now* let us turn to the method discovered by Newton for making such a triangle unnecessary." Correct; it is a memory-helper and while the ancients regarded it as something almost magic there is no reason why we should. But there is more that can be said. The purpose of this note is to point out some of the ways in which it can be used to set a spark to student interest.

Inspection of it yields immediately the answers to many problems involving selection. Thus, properly to decide relative ability, two teams will require one game; three teams, three games; six teams, fifteen games; ten teams, forty-five games; etc.

As a second example take the well-known problem: In how many ways can I, without lost motion, go from home to church, j blocks west and k blocks south? The Pascal triangle supplies a table of answers for all values of j and k.

The symmetry of the triangle illustrates a theorem in combinations: Out of n eligibles there are as many ways of choosing k as of choosing $(n-k)$.

In the way the larger numbers are bunched toward the middle of any row the student of statistics has a symbol for his conviction that any measurable physical or mental characteristic possessed by a multitude of people will probably be found to be distributed among them so that the great majority will be close to the average of the group while a few will be noticeably above or below the average.

Suppose the triangle is placed as it appears as part of the large diagram. If we take from any horizontal line a fragment which begins or ends with 1, we can put with the numbers the words *point*, *straight line*, *plane*, *space*, (in this or in the reverse order) along with necessary grammatical connectives to compose the statements of true geometrical theorems. In the examples which follow, "line" means straight line, and "space" means the ordinary, flat, three-dimensional, ante-Einstein space in which all our experiences have been collected. In one plane two lines intersect in one point. In one space two planes intersect in one line. Through one point two lines determine one plane. In one space four planes determine six lines which intersect in four points.[1]

As a last example consider the extract which follows. It is the frame work for the complete description of a figure which is called Desargues' configuration. Translated into English, it reads: In space five planes intersect in ten lines which concur in ten points; there are three points on every line and six points on four lines on every plane. Any plane picture of the lines and points of this

---

[1] Readers who detect something rough and unfamiliar in the wording should be reminded that four given planes are not necessarily in the same space. If we want to have even two planes in the same space we are obliged to say so. In order to determine two general planes we require six points, and six points, if they are independent, determine a *five*-dimensional space. In five-space two given planes will usually have no point in common; in four-space two given planes will usually have just one point in common.

configuration is an answer to the question: How would you plant ten trees in ten rows, three in each row?

$$\begin{array}{c}
1\\
3 \quad 1\\
6 \quad 4 \quad 1\\
10 \quad 10 \quad 5 \quad 1
\end{array}$$

The triangle is extensible upward as well as downward and the upper part is as interesting as the lower.[1] Sir Oliver Lodge says in *Easy Mathematics* (Macmillan and Co., London, 1917, p. 334): "The binomial coefficients can be obtained by interjecting a single one into the middle of a row of noughts and then adding adjacent terms to make a term of the next series, as thus $\cdots$". His figure is the usual one (isosceles triangle of numbers on a horizontal base) but filled up with zeros at the right and left to the level of the interjected one. Now the word "interject" suggests the fiat of an external power. Let us assume, instead, that Sir Oliver's top row, $\cdots$ 0 0 0 1 0 0 0 $\cdots$, arises as a result of the same law of progress which produces all the rows below it. Since 1 in every other place that we find it has been produced by the union of 0 and 1, it is natural that the 1 at the vertex should be so produced. So above it to left

$$\begin{array}{r}
0\\
0 \quad\;\\
0 \quad 0\\
0 \quad 0 \quad\;\\
0 \quad 0 \quad 1\\
0 \quad 0 \quad 1 \quad\;\\
0 \quad 0 \quad 1 \quad -10\\
0 \quad 0 \quad 1 \quad -9 \quad\;\\
0 \quad 0 \quad 1 \quad -8 \quad 36\\
0 \quad 0 \quad 1 \quad -7 \quad 28 \quad\;\\
0 \quad 0 \quad 1 \quad -6 \quad 21 \quad -56\\
0 \quad 0 \quad 1 \quad -5 \quad 15 \quad -35 \quad\;\\
0 \quad 0 \quad 1 \quad -4 \quad 10 \quad -20 \quad 35\\
0 \quad 0 \quad 1 \quad -3 \quad 6 \quad -10 \quad 15 \quad\;\\
0 \quad 0 \quad 1 \quad -2 \quad 3 \quad -4 \quad 5 \quad -6\\
0 \quad 0 \quad 1 \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \quad\;\\
0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\\
0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad\;\\
0 \quad 0 \quad 1 \quad 2 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\\
0 \quad 0 \quad 1 \quad 3 \quad 3 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad\;\\
0 \quad 0 \quad 1 \quad 4 \quad 6 \quad 4 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0\\
0 \quad 0 \quad 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \quad 0 \quad 0 \quad 0 \quad\;\\
0 \quad 0 \quad 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \quad 0 \quad 0 \quad 0\\
0 \quad 0 \quad 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \quad 0 \quad 0 \quad\;
\end{array}$$

and right we place a 0 and a 1. When this step has been taken all the rest that is shown in the figure follows naturally and need not be explained in detail. Notice that every place has been filled so that the chosen law of progress has been nowhere transgressed.

Reading in the horizontal rows we have now the coefficients for all integral powers of a binomial, negative as well as positive. Taking rows running from lower left to upper right we have, in successive rows under the last complete row of zeros, the so-called figurate numbers. These will not be defined or ex-

[1] AUTHOR'S POSTSCRIPT: A figure of the extended triangle is given by Edouard Lucas on p. 22 of his *Théorie des Nombres*.

plained here but it may be mentioned that their definition provides for the wedge of zeros in the place that it occupies. In rows running from upper left to lower right will be found coefficients of the expansion of $(1-x)^n$.

II. The Theory of Least Squares by Vectors

By J. P. Ballantine, University of Washington

In a recent note[1] we showed how Cramer's Rule for the solution of n equations in n unknowns, i.e., the determinantal formula, could be derived graphically. This was done by considering the columns of coefficients as the components of vectors.

When there are more equations than unknowns, a solution is, in general, not possible. The principle of least squares leads to the formation of so called normal equations, one equation for each unknown. The least square values of the unknowns are then obtained by applying Cramer's Rule to the normal equations.

In the present paper, we apply the methods of the previous note to the present situation. We derive the normal equations without recourse to the calculus.

For definiteness, suppose the system of equations to be:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= a_{14}, \\ a_{21}x_1 + a_{22}x_2 &= a_{24}, \\ a_{31}x_1 + a_{32}x_2 &= a_{34}. \end{aligned} \tag{1}$$

Denote the vectors (a_{1i}, a_{2i}, a_{3i}) by A_i for $i=1, 2, 4$. Equations (1) reduce to the single vector equation:

$$A_1x_1 + A_2x_2 = A_4. \tag{2}$$

It is now required to express A_4 as a linear combination of the two vectors A_1, A_2. Since the vector on the left of equation (2) is, for all values of x_1 and x_2 coplaner with A_1 and A_2, a solution of (2) does not in general exist. At this point the principle of least squares states (translated into the present vector notation) that the most satisfactory values of x_1 and x_2 to take are those which make the the vector

$$-A_1x_1 - A_2x_2 + A_4 \tag{3}$$

the shortest. The pair of values for x_1 and x_2 is called the least square solution.

Let A_3 denote any vector parallel to the vector (3), so that (3) is some arbitrary multiple A_3x_3 of A_3. For A_3x_3 to be shortest a necessary (provided $x_3 \neq 0$) and sufficient condition is that A_3 be perpendicular to the plane of the vectors A_1 and A_2. Hence A_3 must be perpendicular to the vectors A_1 and A_2, and we have the equations:

[1] This Monthly, vol. 36 (1929), pp. 439–441.

$$
(4) \qquad \begin{aligned} A_3 \cdot A_1 &= a_{13}a_{11} + a_{23}a_{21} + a_{33}a_{31} = 0, \\ A_3 \cdot A_2 &= a_{13}a_{12} + a_{23}a_{22} + a_{33}a_{32} = 0. \end{aligned}
$$

From the definition of A_3, we have the equation

$$
(5) \qquad A_1x_1 + A_2x_2 + A_3x_3 = A_4.
$$

Equations of components of (5) are

$$
(6) \qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= a_{14}, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= a_{24}, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= a_{34}. \end{aligned}
$$

One possible procedure is to determine the components a_{13}, a_{23}, a_{33} of vector A_3 (except for a factor of proportionality) from (4), and then solve (6) for x_1 and x_2.

A better procedure, however, is to use equations (4) to eliminate x_3 from equations (6). Multiply equations (6) by a_{11}, a_{21}, a_{31}, respectively, and also by a_{12}, a_{22}, a_{32}, and add. The two resulting equations by (4) are free of x_3, and are the well known normal equations. Their solution is the least square solution.

I. Angular Velocity Determined by the Accelerations of Three Points

By E. L. Rees, University of Kentucky

In the December 1916 issue[1] of this "Monthly" Professors Peter Field and Alexander Ziwet discussed briefly the problem of determining the angular velocity of a rigid body given the accelerations of three of its noncollinear points. It was there indicated that the problem has four solutions. It is the purpose of this supplementary note to find these solutions in terms of the given accelerations and to discuss the various special cases.

Denoting by $\boldsymbol{p}$ and $\boldsymbol{q}$ the vectors from one of these points to the other two we have as our assumptions of rigidity,

$$
(1) \qquad \boldsymbol{p}^2 = \text{const.}, \quad \boldsymbol{q}^2 = \text{const.}, \quad \boldsymbol{p}\cdot\boldsymbol{q} = \text{const.}
$$

The points being non-collinear we have also $\boldsymbol{p}\times\boldsymbol{q}\neq 0$.

Differentiating equations (1) twice with respect to the time, we find

$$
(2) \qquad \boldsymbol{p}\cdot\ddot{\boldsymbol{p}} = -\dot{\boldsymbol{p}}^2, \quad \boldsymbol{q}\cdot\ddot{\boldsymbol{q}} = -\dot{\boldsymbol{q}}^2, \quad \boldsymbol{p}\cdot\ddot{\boldsymbol{q}} + \ddot{\boldsymbol{p}}\cdot\boldsymbol{q} = -2\dot{\boldsymbol{p}}\cdot\dot{\boldsymbol{q}}.
$$

Since the accelerations of the three points are given, $\ddot{\boldsymbol{p}}$ and $\ddot{\boldsymbol{q}}$ are known and therefore $\dot{\boldsymbol{p}}^2$, $\dot{\boldsymbol{q}}^2$, and $\dot{\boldsymbol{p}}\cdot\dot{\boldsymbol{q}}$ are also known.

Differentiating the equations $\dot{\boldsymbol{p}} = \boldsymbol{w}\times\boldsymbol{p}$ and $\dot{\boldsymbol{q}} = \boldsymbol{w}\times\boldsymbol{q}$, we have

$$
(3) \qquad \ddot{\boldsymbol{p}} = \dot{\boldsymbol{w}}\times\boldsymbol{p} + \boldsymbol{w}\times(\boldsymbol{w}\times\boldsymbol{p}), \quad \ddot{\boldsymbol{q}} = \dot{\boldsymbol{w}}\times\boldsymbol{q} + \boldsymbol{w}\times(\boldsymbol{w}\times\boldsymbol{q}).
$$

[1] Vol. 23, pp. 371–381.

Eliminating $\dot{\boldsymbol{w}}$ by using $\boldsymbol{p}\cdot$ and $\boldsymbol{q}\cdot$ as multipliers, we obtain

$$
\begin{aligned}
\boldsymbol{p}\cdot\ddot{\boldsymbol{p}} &= \boldsymbol{w} \times (\boldsymbol{w} \times \boldsymbol{p})\cdot\boldsymbol{p} = (\boldsymbol{w}\cdot\boldsymbol{p})^2 - \boldsymbol{w}^2\boldsymbol{p}^2, \\
\boldsymbol{q}\cdot\ddot{\boldsymbol{q}} &= \boldsymbol{w} \times (\boldsymbol{w} \times \boldsymbol{q})\cdot\boldsymbol{q} = (\boldsymbol{w}\cdot\boldsymbol{q})^2 - \boldsymbol{w}^2\boldsymbol{q}^2, \\
\boldsymbol{p}\cdot\ddot{\boldsymbol{q}} + \ddot{\boldsymbol{p}}\cdot\boldsymbol{q} &= 2(\boldsymbol{w}\cdot\boldsymbol{p}\boldsymbol{w}\cdot\boldsymbol{q} - \boldsymbol{p}\cdot\boldsymbol{q}\boldsymbol{w}^2).
\end{aligned} \tag{4}
$$

Finally, eliminating $\boldsymbol{w}\cdot\boldsymbol{p}$ and $\boldsymbol{w}\cdot\boldsymbol{q}$ from equations (4), and replacing $\boldsymbol{p}\cdot\ddot{\boldsymbol{p}}, \boldsymbol{q}\cdot\ddot{\boldsymbol{q}}, \boldsymbol{p}\cdot\ddot{\boldsymbol{q}}+\ddot{\boldsymbol{p}}\cdot\boldsymbol{q}$ by $-\dot{\boldsymbol{p}}^2, -\dot{\boldsymbol{q}}^2, -2\dot{\boldsymbol{p}}\cdot\dot{\boldsymbol{q}}$, respectively, we get

$$
(\boldsymbol{p} \times \boldsymbol{q})^2\boldsymbol{w}^4 - (\boldsymbol{p}^2\dot{\boldsymbol{q}}^2 + \dot{\boldsymbol{p}}^2\boldsymbol{q}^2 - 2\boldsymbol{p}\cdot\boldsymbol{q}\dot{\boldsymbol{p}}\cdot\dot{\boldsymbol{q}})\boldsymbol{w}^2 + (\dot{\boldsymbol{p}} \times \dot{\boldsymbol{q}})^2 = 0, \tag{5}
$$

from which we may find the scalar value of $\boldsymbol{w}$. Let the coefficients in this equation be denoted by A, B and C respectively. Then

$$
\begin{aligned}
A > 0,\ B &= \boldsymbol{p}^2\dot{\boldsymbol{q}}^2 + \dot{\boldsymbol{p}}^2\boldsymbol{q}^2 - 2\boldsymbol{p}\cdot\boldsymbol{q}\dot{\boldsymbol{p}}\cdot\dot{\boldsymbol{q}} \\
&\geqq \boldsymbol{p}^2\dot{\boldsymbol{q}} + \dot{\boldsymbol{p}}\boldsymbol{q} - 2|\boldsymbol{p}|\,|\boldsymbol{q}|\,|\dot{\boldsymbol{p}}|\,|\dot{\boldsymbol{q}}| \geqq 0,\ c \geqq 0.
\end{aligned}
$$

The coefficient B equals zero only if $\dot{\boldsymbol{p}}^2 = \boldsymbol{q}^2 = 0$, *i.e.*, if $\boldsymbol{p}\cdot\ddot{\boldsymbol{p}} = \boldsymbol{q}\cdot\ddot{\boldsymbol{q}} = 0$.

It follows then that the roots of this quadratic in $\boldsymbol{w}^2$ are both positive (or zero) if they are real, *i.e.*, if $B \geqq 2\sqrt{(AC)}$ (radical positive). We shall denote these roots by $\boldsymbol{w}_1{}^2$ and $\boldsymbol{w}_2{}^2$ and assume $\boldsymbol{w}_1{}^2 \geqq \boldsymbol{w}_2{}^2$.

From these inequalities and equations (2) it follows that the accelerations cannot be arbitrarily prescribed, but must satisfy the following conditions:

(a) $\boldsymbol{p}\cdot\ddot{\boldsymbol{p}} \leqq 0,\ \boldsymbol{q}\cdot\ddot{\boldsymbol{q}} \leqq 0$, (b) $c \geqq 0$, i.e., $4\boldsymbol{p}\cdot\ddot{\boldsymbol{p}}\boldsymbol{q}\cdot\ddot{\boldsymbol{q}} \geqq (\boldsymbol{p}\cdot\ddot{\boldsymbol{q}} + \ddot{\boldsymbol{p}}\cdot\boldsymbol{q})^2$,
(c) $B \geqq 2\sqrt{(AC)}$.

It may be shown that $B = 2\sqrt{(AC)}$ if the accelerations satisfy the simple conditions $\dot{\boldsymbol{p}}^2/\boldsymbol{p}^2 = \dot{\boldsymbol{q}}^2/\boldsymbol{q}^2 = \dot{\boldsymbol{p}}\cdot\dot{\boldsymbol{q}}/\boldsymbol{p}\cdot\boldsymbol{q}$, the numerators being the functions of $\ddot{\boldsymbol{p}}$ and $\ddot{\boldsymbol{q}}$ defined by equations (2).

We shall consider first the general case $B > 2\sqrt{(AC)}$ and $C > 0$. From equations (2) and (4) we have

$$
\begin{aligned}
&\boldsymbol{w}\cdot\boldsymbol{p} = \pm\sqrt{(\boldsymbol{w}^2\boldsymbol{p}^2 - \dot{\boldsymbol{p}}^2)},\quad \boldsymbol{w}\cdot\boldsymbol{q} = \pm\sqrt{(\boldsymbol{w}^2\boldsymbol{q}^2 - \dot{\boldsymbol{q}}^2)}, \\
&[\pm\sqrt{(\boldsymbol{w}^2\boldsymbol{p}^2 - \dot{\boldsymbol{p}}^2)}][\pm\sqrt{(\boldsymbol{w}^2\boldsymbol{q}^2 - \dot{\boldsymbol{q}}^2)}] = \boldsymbol{p}\cdot\boldsymbol{q}\boldsymbol{w}^2 - \dot{\boldsymbol{p}}\cdot\dot{\boldsymbol{q}},
\end{aligned} \tag{6}
$$

the signs of the radicals being chosen so as to satisfy the last equation. Obviously there are two ways of choosing the signs for any given w.

Consider the equations

$$
\boldsymbol{r}\cdot\boldsymbol{p} = \pm\sqrt{(\boldsymbol{w}^2\boldsymbol{p}^2 - \dot{\boldsymbol{p}}^2)},\quad \boldsymbol{r}\cdot\boldsymbol{q} = \pm\sqrt{(\boldsymbol{w}^2\boldsymbol{q}^2 - \dot{\boldsymbol{q}}^2)}, \tag{7}
$$

where $\boldsymbol{w}^2$ is a root of equation (5) and $\boldsymbol{r}$ is the running coordinate. Each of these equations represents two planes. Hence these equations, treated simultaneously, with the proper association of signs, represent two lines symmetrically situated with respect to the origin and perpendicular to the plane of the three points.

If these lines intersect the sphere $r^2=w^2$, the vectors terminating in the four points of intersection will be the vectors representing the angular velocities. These vectors will lie along two intersecting lines, two of them being the negatives of the other two.

We shall see that the lines

$$r\cdot p = \pm\sqrt{(w_1^2p^2-\dot{p}^2)},\quad r\cdot q = \pm\sqrt{(w_1^2q^2-\dot{q}^2)}$$

intersect the sphere $r^2=w_1^2$, and we shall find explicit expressions for the w's We shall also see that w_2^2 cannot be used since it leads to an imaginary result.

Letting P denote the perpendicular to one of these lines, we may write $P=lp+mq$, and squaring, $P^2=l^2p^2+2lmp\cdot q+m^2q^2$. Replacing r in equations (7) by this expression for P, we get

$$(8)\quad lp^2+mp\cdot q = \pm\sqrt{(w^2p^2-\dot{p}^2)},\quad lp\cdot q+mq^2 = \pm\sqrt{(w^2q^2-\dot{q}^2)},$$

from which we may find l and m, and therefore P.

To get P^2 without the laborious reductions entailed by direct substitution, we multiply the first and second of these equations by l and m respectively, and add. Then substituting the values of l and m found above, and simplifying by making use of (6), we find

$$P^2=\frac{-p^2\dot{q}^2-\dot{p}^2q^2+2p\cdot q\dot{p}\cdot\dot{q}}{p^2q^2-(p\cdot q)^2}+2w^2=-\frac{B}{A}+2w^2.$$

Indicating by subscripts the corresponding P's and w's ,we have

$$P_1^2=-(w_1^2+w_2^2)+2w_1^2=w_1^2-w_2^2,\quad P_2^2=-(w_1^2+w_2^2)+2w_2^2=w_2^2-w_1^2.$$

Since $w_1^2>w_2^2$, the above equations show that the pair of lines corresponding to w_1 will intersect the sphere $r^2=w_1^2$, and that P_2^2 must be discarded since it is negative.

From a figure which can easily be drawn, it can be seen that $w_1=P_1+np\times q$, where n is an unknown scalar. To determine n, we square and get

$$w_1^2=P_1^2+2[P_1pq]n+(p\times q)^2n^2.$$

But $[P_1pq]=0$ since the vectors are coplanar. Therefore

$$n=\pm\sqrt{\left[\frac{W_1^2-P_1^2}{(p\times q)^2}\right]}=\pm\sqrt{\left[\frac{W_2^2}{(p\times q)^2}\right]}$$

$$=\pm\frac{|W_2|}{|p\times q|},\ \text{and}\ W_1=P_1\pm|W_2|(p\times q)_1,$$

or, if the other signs of the radicals in (7) are used,

$$W_1=-P_1\pm|w_2|(p\times q)_1,$$

$(p\times q)_1$ indicating a unit vector.

Thus we obtain the four values of w.

We thus see that when the accelerations of three non-collinear points of a rigid body are given, subject to the conditions stated above, the magnitude of the angular velocity of the body is determined uniquely, and that the angular velocity may have any one of four directions.

We shall now consider the special cases:

I. The case in which $B>2\sqrt{(AC)}$, $C=0$. In this case one root w_2^2 of equation (5) vanishes. Hence $w_1=\pm \boldsymbol{P}_1$, *i.e.*, the lines are tangent to the sphere and the vectors representing the angular velecities are the radii drawn to the points of tangency. There are thus only two solutions and the $\boldsymbol{w}$'s coincide with the perpendiculars to the two lines, and are therefore in the plane of $\boldsymbol{p}$ and $\boldsymbol{q}$; *i.e.*, the plane of the three points whose accelerations are given. $\boldsymbol{w}$ is a scalar multiple of $\boldsymbol{p}$ if $\ddot{\boldsymbol{p}}=0$, or a scalar multiple of $\boldsymbol{q}$ if $\ddot{\boldsymbol{q}}=0$.

II. The case in which $B=\sqrt{(AC)}$.

(a) $B>0$ ($\boldsymbol{p}\cdot\ddot{\boldsymbol{p}}$ and $\boldsymbol{q}\cdot\ddot{\boldsymbol{q}}$ are not both zero). Here $w_1^2=w_2^2$ and $P_1^2=P_2^2=0$. This shows that the lines pass through the origin and since they are perpendicular to the plane of $\boldsymbol{p}$ and $\boldsymbol{q}$ they are coincident. Hence there will be only two piercing points, and the two values of $\boldsymbol{w}$ are given by

$$\boldsymbol{W} = \pm \ |\boldsymbol{w}|\ (\boldsymbol{p}\times\boldsymbol{q})_1.$$

We conclude that the angular velocity is perpendicular to the plane of the three given points, but that its sense is indeterminate. The velocities $\dot{\boldsymbol{p}}$ and $\dot{\boldsymbol{q}}$ are coplanar with the three points.

(b) $B=0$ ($\boldsymbol{p}\cdot\ddot{\boldsymbol{p}}=\boldsymbol{q}\cdot\ddot{\boldsymbol{q}}=0$). In this case C will also vanish and the roots of equation (5) will be zero. Hence there will be no rotation.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College of the City of New York.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Wells, Webster, and Hart, W. W. *Modern Algebra, Third Semester Course.* New York, D. C. Heath and Company, 1929. viii+266 pages.

Features of this secondary school text are: "diagnostic tests and remedial instruction" in the early review chapters; a chapter on functional relationship; complete treatment of quadratic equations; the elements of trigonometry.

Kellogg, O. D. *Foundations of Potential Theory.* Berlin, Julius Springer, 1929. x+384 pages.

". . . . A two-fold purpose: first, to serve as a introduction for students whose attainments in the calculus include some knowledge of partial derivatives and multiple integrals; and secondly, to provide the reader with the fundamentals of the subject, so that he may proceed immediately to the applications, or to the periodical literature of the day."

Granville, W. A. *Elements of the Differential and Integral Calculus.* Revised by P. F. Smith and W. R. Longley. Boston, Ginn and Company, 1929. xii+516 pages. $3.20.

Some rearrangements of topics have been made; some topics in the older edition have been omitted, and replaced by other topics of importance and interest.

Dickson, L. E. *Introduction to the Theory of Numbers.* University of Chicago Press, 1929. ix+183 pages.

"The aim of this book is not technique, but the central ideas of the subject . . . . . The book is intended for beginners and develops the subject from first principles. [The subjects treated are] divisibility, congruences, quadratic residues, and the reciprocity law; binary quadratic forms; and Diophantine equations."

Kaiser, Ludwig. *Über die Verhältniszahl des Goldenen Schnitts, die Reihe der mit ihr zusammenhängenden ganzen Zahlen und eine aus dieser abgeleitete Reihe.* Leipzig, B. G. Teubner. viii+124 pages.

Lainé, E. *Premières lecons de géomètrie analytique et géomètrie vectorielle.* Paris, Librairie Vuibert, 1929. 48 pages.

Bouligand, Georges. *Complements et exercises sur la mécanique des solides.* Paris, Librairie Vuibert, 1929. viii+132 pages.

Vogel, Kurt. *Die Grundlagen der Ägyptischen Arithmetik, in ihrem zusammenhang mit der 2:n Tabelle des Papyrus Rhind.* München, Michael Beckstein, 1929. vi+212 pages.

Smith, David Eugene. *A Source Book in Mathematics*: Source Books in the History of the Sciences, Gregory D. Walcott, General Editor. New York, McGraw-Hill Book Company, 1929. xxviii+702 pages. $5.00.

Walther, Alwin. *Einführung in die Mathematische Behandlung Naturwissenschaftlicher Fragen.* Erstes Teil, Funktion und Graphische Darstellung, Differential- und Integralrechnung. Berlin, Julius Springer, 1928. viii+220 pages.

A treatise on analytic geometry and the calculus, intended primarily for students of the sciences. Some attention is given to monography, and physical applications are stressed throughout.

Walther, Alwin. *Begriff und Anwendungen des Differentials, mit besonderer Berücksichtigung der Bedürfnisse des Unterrichts und der Naturwissenschaften.* Leipzig, B. G. Teubner, 1929. iv+96 pages.

A similar but briefer work than the foregoing.

REVIEWS

The Theory of Functions of a Real Variable and the Theory of Fourier's Series. By E. W. Hobson. Vol. I, xvi+671 pp., 1921; Vol. II, x+780 pp., 1926. Second Edition. Cambridge University Press.

Hobson's *Theory of Functions* is a unique work, taking its character from the author's lifelong, scholarly, critical and comprehensive study of those parts

of analysis that deal with real functions. As is well known, no other book in this field attempts to present with such detail comparably as many topics. The book is exceptional, too, in having the dual nature of a treatise and a report; much of it is a record of systematizations and extensions attained through long meditation and personal research, while other parts are reëxpositions, not always organically worked in with the rest of the material, of recent journal articles. Thus the work of Hobson, unlike such books as the *Grundzüge* of Hausdorff or the *Reelle Funktionen* of Carathéodory, for example, which are written in a sustained mood, lacks the unity and finish one expects in a treatise or the artistic flow of a *Cours d'Analyse.* This is especially true of the second edition, which undertakes the difficult task of presenting in book form a great diversity of topics repeatedly subject to change of treatment on account of intensive modern research. On this account, too, the work has something of the stimulus which material arouses that has not yet been fashioned in final form.

Throughout there is the somewhat leisurely and truly British style of one who has more than the interest of the narrow specialist; in the manner of a natural philosopher, Hobson is intent on giving the reader the broad scientific bearings of the content. Illuminating historical sidelights, too, come in from time time. While this philosophical point of view is justly impressive, one misses in the occasional "metamathematical" remarks the intuitive insight or profound familiarity which are often the mark of those who, like Zermelo, read little but live largely with ideas coming from within. Thus in reference to the controversies in point sets, Hobson adopts a reasonable middle course; but his own comments on controversial questions are wanting somewhat in incisiveness and conviction.

After the introduction of irrational numbers according to the theories of Dedekind and Cantor, one passes to the descriptive and metric properties of point sets, (the former centering around the notion of limit and the latter around that of content and measure) cardinal number, and order type. The interest here is partly in the subject matter itself and partly in supplying the requisite preliminaries for the theory proper of real functions, which opens with Chap. V, Vol. I. p. 256. In this chapter are considered various types of continuity (for example, uniform, absolute, semi-series, and approximate continuity) functions of limited variation, properties of derivatives, including the comprehensive results of Denjoy and Young, and other things. Chaps. VI, VII and VIII of Vol. I are devoted to Riemann, Lebesgue, and non-absolutely convergent integrals, respectively, the last chapter featuring the Denjoy integral. Vol. II deals principally with series and various ways of representing functions. The theory of trigonometric series, so important in the modern development of the theory of real functions and the beginning of the author's interest in the general field of real functions, constitutes the longest chapter, namely VIII, which covers nearly 250 pages and includes some of the latest developments accessible to the author at the time of writing. Among the numerous topics discussed in Vol. II are, for example, the summation methods of Cesàro,

Hölder, and Riesz, Baire's results on the representation and classification of functions—the considerations here are based on the work of de la Vallée-Poussin—the integration theories of Tonelli and Perron, the construction of functions with assigned singularities, and normal orthogonal functions, in particular the extension to the latter of the Parseval and Riesz-Fischer theorems. Chapter VII, which deals with the representation of functions as limits of integrals, gives special prominence to one of the author's own results, the general convergence theorem, which asserts, under certain conditions, the convergence to zero, as $n \to \infty$, of an integral of type $\int_a^b f(t)\phi(t, x, n,)dt$, and has application to, as it is a summary and extension of, certain known facts concerning integrals occurring, for example, in the theory of Fourier's series.

It is somewhat regrettable that a work devoted to the theory of real functions in its broadest aspects should say so little of properties of unconditioned functions, especially when properties of particular functions, for instance the continuous, may repeatedly be exhibited as specialized cases of properties of arbitrary functions.

Whatever personal impressions one may have of this grand work of Hobson, one must be grateful for the painstaking scholarship of the author, who has made readily accessible to mathematicians the conceptual structure of the theory of real functions and the rich store of its most recent developments.

HENRY BLUMBERG

The New Quantum Mechanics. By George Birtwistle. Cambridge University Press, 1928, xiii+290 p.

The author,[1] a fellow of Pembroke College, Cambridge, England, has already written a book on the Bohr theory of quanta,[2] and now offers an exposition of the quantum theory developed since 1925, known as quantum mechanics. This account is very accurate and contains practically everything that has been done up to the summer of 1927. He gives us, so to speak, original abstracts of the principal papers and allows us a survey of everything that is known. This makes the work not an exposition from one point of view, as is Weyl's new book; it is rather an "impartial" treatment of the methods of the different schools, with credit given to each for its results.

From Birtwistle's book we see quantum mechanics developing in the following fashion. Some years before 1925 Sommerfeld and others had been grappling with the complex problem of the multiplets in spectra and their Zeeman separations. It seemed impossible to account for them on the basis of the Bohr theory of quanta. This theory, unsatisfactory from the beginning through its theoretical inconsistencies, showed itself further insufficient in practical respects.

This led the young German physicist, Heisenberg, to an investigation in

[1] The author died on May 19, 1929 at the age of 52.

[2] *The Quantum Theory of the Atom*, Cambridge, 1926.

1925, in which he developed, from other principles than those of Bohr, a new quantum mechanics, discarding Newtonian mechanics. Heisenberg's leading principle was that the quantum formulae had to contain only experimentally observable magnitudes, such as the frequencies and the intensities of the spectrum. He did not succed in all respects, but his theory was much better than the previous one with its radii of orbits, orbital frequencies, and orbital amplitudes, which probably by their nature can never be observed.

The mathematical tool of the new theory was the so-called matrix calculus, later developed by Heisenberg, himself, Born, and Jordan. In the same period the English physicist Dirac worked out an analogous theory, the so-called q-numbers. The crucial test of both theories was the computation of the lines of the hydrogen spectrum, in which both succeeded.

It was now possible to attack the difficulties in the multiplets. In 1926 Heisenberg and Jordan succeeded in clearing them up on the basis of the new mechanics. They used in their investigation a new notion: of the so-called spinning electron, introduced by Goudsmit and Uhlenbeck (now at Ann Arbor), according to which an electron can spin around like a top.

The new matrix theory was very complicated, its physical meaning very obscure. But in 1926 a quite new development was put forward by Schrödinger, at that time professor at Zürich. Inspired by the ideas of Louis de Broglie's Paris thesis of 1924, in which a wave was associated with every material particle, Schrödinger assumed that the dynamics of an electron can not be those of a point as in classical theory, but must be those of a wave. This wave obeys a linear partial differential equation of the second order. (Newton mechanics leads to a partial differential equation of the first order and second degree.) Such a differential equation admits continuous uniform bounded solutions only for certain discrete values of a parameter (the "energy"), the so-called characteristic values. The critical test of the hydrogen lines was also satisfied in Schrödinger's case.

It was soon clear that the matrix theory of differential equations led to the same results, though the starting point was entirely different. Schrödinger soon showed the reason for this. The main difficulty remained the interpretation of both theories. Schrödinger's way had the advantage of using the highly developed analysis of linear partial differential equations, and therefore had mathematical advantages. Heisenberg used Schrödinger's calculus to solve his matrix equations for the case of the helium atom, where there are two electrons instead of the one electron in the hydrogen atom.

The reason that Schrödinger's calculus was not victorious was the difficulty in the interpretation. He could show that in highly excited states a suitably chosen group of characteristic functions represent a "wave packet" behaving like a point mass of classical mechanics oscillating in a rectilinear path. Heisenberg showed, however, that in general such a wave packet is not stable, that it spreads out over the whole space in the course of time. Schrödinger's theory cannot, therefore, account for mass points.

The matrix school, on the contrary, preferred a statistical interpretaion of the results. What Schrödinger and his school called the "density," is for Heisenberg and his school a "probabil'ity that an electron is in a certain state." The laws of quantum mechanics become, in this theory, statistical laws, but laws of another mathematical structure than the laws of the ordinary theory of probability, underlying for instance the kinetic theory of gases.

Statistics become, therefore, an, essential part of the new mechanics. Here exist three hypotheses, already, on the basis of the old quantum theory, and its application to the theory of radiation; the theories named after Einstein and Bose, Boltzman, and Fermi and Dirac. The second underlies the kinetic theory of gases of Newtonian mechanics. The interpretation of these hypotheses in the many electron problems of the new quantum mechanics is only in a very elementary stage of development.

This is the main content of Birtwistle's book. He ends with an exposition of Heisenberg's "indetermination principle" (1927). One of the characteristics of the new theory is, according to this principle, that it is essentially impossible to get an exact statement about the momentum of an electron if we want an exact statement about its position, and vice versa; the same holds for time and energy, and for all variables and their Fourier transforms. This is a simple corollary of the duality between point-conception and wave-conception.

In a theory in such a rapid state of development a book like Birtwistle's is already incomplete after a year has passed. We will mention here some of the outstanding developments published later than the book: Weyl's investigations on finite groups and quantum mechanics, collected in his new book, *Quantenmechanik und Gruppentheorie*; Dirac's relativistic theory of the spinning electron; Eddington's connection between the so-called exclusion principle of Pauli and the Fermi-Dirac statistics; and the development of the many-electron problem.

Birtwistle's book does not mention the experimental results showing the wave-character of electrons, as do those of Davisson, Germer, Thomson, and others.

D. J. STRUIK

General Mathematics. By C. H. Currier and E. E. Watson. The Macmillan Company, New York, 1929. viii+413 pages.

The first brief review of the book appears, where it properly should, in the authors' preface. We quote from it without their permission: "This book includes the elements of algebra, trigonometry, analytic geometry, and calculus · · · · The review topics, such as the graph, simultaneous equations · · · · are interwoven with and related to the conic sections and other more advanced material. The chief features of the book are the simplicity of method and the selection and order of presentation of the subject matter. The problem material has been selected with reference to utility and interest rather than mathematical completeness. Methods and problems which have proved unsatisfactory [as

tried out in the authors' classrooms] have been eliminated. It is therefore the hope of the authors that this book will prove to be teachable in the hands of others." · · · ·

As textbooks begin to flood the market, reviewers, more especially those who have "authored" one themselves, question the need or the whys for another textbook in the field. "What is there about the newcomer that justifies its crowding the others in the field?" is the usual query. It is many years now since that question has ceased to be original. In the minds of most of us a new one has arisen: "How long will it be before each of us has written his own textbook?" And why not? Almost ever new textbook appears only after it has been "tried out successfully in the author's own classes." Almost every new author claims, usually sincerely and justifiably, that not a textbook on the market meets the needs of either his special style of teaching or the unique requirements of his students. Unwilling to sacrifice himself or his students, what is there left to do but write his own?

Hence, in all seriousness, the reviewer feels that *General Mathematics* by Currier and Watson meets a worth-while need. After a careful reading of the book he finds nothing strikingly objectionable either in content or method. On the other hand it is not likely to set the freshman mathematics world aflame. (Nor do the authors make any such claim.) The reviewer hopes,with the authors, that others may find the book teachable, for despite the ever increasing number of textbooks the teachable ones are scarce indeed.

Evidently the authors' intention is to treat the various branches of elementary mathematics simultaneously. Theirs is distinctly *not* a textbook in "unified", "correlated," or "joined" mathematics. Chapters in algebra are interspersed with those on trigonometry, analytic geometry, calculus, and statistics. The following are the chapter headings given in the order of their appearance:

Functions and Graphs; Trigonometric Functions of an Acute Angle; Exponents—Logarithms; Radian Measure—Trigonometric Functions of Any Angle; Straight Line Formulas; The Quadratic Function; Theory of Equations; First Degree Equations—Use of Determinants; Differentiation of Algebraic Functions; Integration; Relations Among Trigonometric Functions; Polar Coordinates and Allied Topics; Progressions and Series—Interest Formulas —Binomial Theorem; Laws of Growth—Exponential Function; Conic Sections; Space of Three Dimensions; Permutations and Combinations; Theory of Measurements; Complex Numbers.

In all, approximately 130 pages are devoted to algebra, 65 pages to trigonometry, 100 pages to analytic geometry, 65 pages to calculus, and 25 pages to statistics.

The exercises on the whole, are quite excellent. A representative number are "practical" without being too technical. Historical notes abound and in most cases are appropriate and stimulating.

The presentation of new material is all too frequently sketchy and too many proofs are elliptical, to meet the needs of the average student. In partic-

ular, the authors fail to smooth the two rough spots in introductory analytic geometry: (1) Expressing geometric magnitudes or configurations in terms of general or particular coordinates of significant points; (2) "answers" in which x and y remain to the bitter end, thus involving a new notion directly opposed to that of elementary algebra.

Occasionally the lettering of diagrams is not uniform or is too involved. Consider, for example, figure 66, page 95: since a brace is used to indicate the whole of the line y_2-y_1, why is not as much done for x_2-x_1? Also, if the diagram is to aid or elucidate the proof, why not let the student *see* that AP is $y-y_1$, that CP_2 is y_2-y_1, etc.?

What is gained in algebra or analytic geometry by the use of the more or less technical words "rise" and "run" (p. 86)? In $A=\pi r^2$, or $C=2\pi r$, r is *not* an arbitrary constant (page 1). The example on page 12 is confusing. Perhaps the fifth line should read: "If the center of *a* circle is the origin $\cdots$." On p. 41 the term "significant figures" is used without definition or explanation. And lastly, it seems a bit unusual to have a chapter on "Complex Numbers" come last (pages 375–386). Pages 387–392 contain six brief tables. Answers to practically all the exercises are given on pages 397–413.

JOSEPH SEIDLIN

Operational Circuit Analysis. By V. Bush. John Wiley and Sons, New York, 1929. x+392 pages.

In the introduction to this book the author wishes it to be emphasized that he writes as an engineer and that he does not pretend to be a mathematician. However he is writing about a mathematical subject, the solution of linear differential equations (mainly with constant coefficients), and this review is to appear in a mathematical journal, so that we are constrained to regard the mathematical aspects of the book. These we are sorry to say do not command our admiration.

The material treated is Heaviside's operational method together with the more modern modifications, of which the most important is Carson's "infinite integral." The material in pages 1–147 and 189–263 of the book is in our opinion treated more thoroughly and scientifically in an eighteen page article by Paul Lévy.[1] The rest of the book is devoted to treatment by methods of complex variable theory and Fourier integrals. We are inclined to the opinion, purely personal, that the complex variable sortie, led by Wagner, Bromwich, et al., was ill advised. Heaviside's own ideas were more direct and the perfectly rigorous explanation of these (based on Volterra's concept of composition) given by Lévy leaves nothing to be desired. Particularly in the matter of asymptotic solutions there is an irritating feeling through Bush's book of "now you see it, now you don't" (pp. 242–255). The chapter on networks with variable parameters deals with very interesting questions. It is not at all clear to

[1] *Le calcul symbolique d'Heaviside*, Bulletin des sciences mathématiques (2) vol. 50 (1926).

us that the function A of equation (714) will be a function of the difference $t-\lambda$; when the coefficients of the differential equations involved depend on the time t this function A, whose determination solves, in effect, the problem will be a function of both t and λ and not merely of their difference. It is this latter characteristic of equations with constant coefficients that makes them so easy to deal with. In the case in question the variable coefficient arises from a variable resistance and the artifice of imagining the current short circuited across it is resorted to; the result may be correct but it is certainly *not proved.*

Now that we have given our opinion of the book as a mathematician we may be permitted to add that the engineering information given in the book is very good particularly for mathematicians who are unfamiliar with these applications. The book is conscientiously written, well printed, and we feel sure that in engineering circles it will secure a good reputation.

F. D. MURNAGHAN

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEM FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3404. *Proposed by W. J. Greenstreet, Editor of the Mathematical Gazette.*

An endless inextensible weightless string lies in the form of a rhombus $ABCD$; masses each of m lbs. weight are placed at the corners. The particle at A is struck along the diagonal CA away from A by a blow P. What will be the initial velocity of the particle at C?

3405. *Proposed by Paul Wernicke, Washington, D. C.*

Given in a plane three concurrent lines and a point P. Construct an equilateral triangle having its vertices on the three lines and having the point P on one of its sides. Determine the number of solutions.

3406. *Proposed by William P. Parker, Pyengyang, Chosen, Korea.*

Find the condition that $(\alpha x+\beta y+\gamma z)\cdot(\alpha_1 x+\beta_1 y+\gamma_1 z)-(\alpha_2 x+\beta_2 y+\gamma_2 z)^2$ may be resolved into two linear factors.

3407. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

A ray of light emanating from a fixed source L is reflected by a flat mirror

at a point M so that the reflected ray passes through a given point I. Find the locus of M when the mirror revolves about a fixed axis s.

3408. *Proposed by J. V. Uspensky.*

Show that the integral

$$V_n = \int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n} dx_1 dx_2 \cdots dx_n$$

converges to the limit $2/3$ when n increases indefinitely and that the product $n(V_n - 2/3)$ remains bounded.

SOLUTIONS

263 [1917, 177]. *Proposed by J. L. Riley.*

To find positive integral values which verify the equation $x^3+2=y^2$.

Note by the Editors: A solution of this problem is given in the paper, "A Note on the Solution of $x^3+2=y^2$," by P. H. Daus in the Bulletin of the American Mathematical Society, vol. 35 (1929), p. 597. See also the solution in this Monthly by A. Brauer [1928, 494], and the partial solution by L. Hampton [1928, 322].

3333 [1928, 377]. *Proposed by W. H. Roever, Washington University.*

Consider the system of two partial differential equations

$$X_i(f) \equiv a_{1i}\frac{\partial f}{\partial x_1} + a_{2i}\frac{\partial f}{\partial x_2} + a_{3i}\frac{\partial f}{\partial x_3} = 0, \quad i = 1,2,$$

in which the coefficients a_{ji} are functions of x_1, x_2, x_3. From one point of view, a necessary and sufficient condition that these equations have a common integral surface is

$$\Theta \equiv \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ X_1(a_{12}) - X_2(a_{11}) & X_1(a_{22}) - X_2(a_{21}) & X_1(a_{32}) - X_2(a_{31}) \end{vmatrix} \equiv 0,$$

and from another point of view, such a condition is

$$\Omega \equiv U\left(\frac{\partial V}{\partial x_3} - \frac{\partial W}{\partial x_2}\right) + V\left(\frac{\partial W}{\partial x_1} - \frac{\partial U}{\partial x_3}\right) + W\left(\frac{\partial U}{\partial x_2} - \frac{\partial V}{\partial x_1}\right) \equiv 0,$$

in which

$$U = \begin{vmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{vmatrix}, \quad V = \begin{vmatrix} a_{31} & a_{11} \\ a_{32} & a_{12} \end{vmatrix}, \quad W = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}.$$

Show that $\Theta \equiv \Omega$.

Solution by Otto Dunkel, Washington University.

In the determinant Θ multiply the elements of the first row by $-[\partial a_{12}/\partial x_1$

$+\partial a_{22}/\partial x_2+\partial a_{32}/\partial x_3]$, and those of the second row by $[\partial a_{11}/\partial x_1+\partial a_{21}/\partial x_2+\partial a_{31}/\partial x_3]$; add the corresponding products to the elements of the third row. After certain cancellations the resulting third row becomes

$$\partial V/\partial x_3 - \partial W/\partial x_2,\ \partial W/\partial x_1 - \partial U/\partial x_3,\ \partial U/\partial x_2 - \partial V/\partial x_1,$$

and the determinant is now in the form Ω.

3349 [1928, 494]. *Proposed by S. A. Corey, Des Moines, Iowa.*

If $|lmn|$ denote the determinant whose columns are the lth, mth, and nth columns of the array

$$\begin{vmatrix} y & x & -av & au \\ u & bv & x & -by \\ v & -cu & cy & x \end{vmatrix},$$

prove that

$$(x^2+bcy^2+acu^2+abv^2)^3 = |234|^2 + bc|134|^2 + ac|124|^2 + ab|123|^2.$$

Solution by the Proposer.

On expanding the given determinants $(x^2+bcy^2+acu^2+abv^2)$ is found to be their common factor. If we denote this factor by D, the given equation becomes $D^3=x^2D^2+bcy^2D^2+acu^2D^2+abv^2D^2$, an identity.

3357 [1928, 564]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Through a given point to draw a line so that the segment intercepted on it by two given intersecting lines shall be divided in a given ratio by the foot of the perpendicular dropped upon it from the point of intersection of the two given lines.

Solution by Harry Pool, South Dakota State School of Mines.

Suppose the two given intersecting lines are l and m, meeting in a point O. Let the given point, through which the required line is to be drawn, be P.

First construct a triangle whose vertical angle is equal to the angle at O, and whose base is divided in the given ratio by the perpendicular upon it from the vertex. To do this, on any line AB describe a circular segment containing the angle O. Divide the line AB in the given ratio at D. Draw DC at right angles to AB to meet the arc of the segment at C. Then ABC is the triangle required.

Through any point on l draw a line n making an angle equal to angle CAB of the triangle just constructed. The line through P parallel to line n is the line required by the problem.

Also solved by L. W. Johnson, Mark Kurtz, Mark Landan, J. Q. McNatt, W. V. Parker, and S. Pelletier.

3359 [1928, 564]. *Proposed by Clifford N. Mills, Normal, Illinois.*

A, B, C, D are four points in a plane. Let the centroids of the triangles BCD, CDA, DBA, ABC be respectively a, b, c, d. Then Aa, Bb, Cc, Dd meet in

a point, and are divided in the same ratio at this point. Also the quadrilateral *abcd* is similar to *ABCD*.

I. *Solution by T. H. Butchart, Urbana, Ill.*

Consider the triangle ABM, where M is the mid-point of DC. Then b and a lie on AM and BM, respectively, so that $Mb/MA = Ma/MB = ab/AB = 1/3$. Similar reasoning applied to bc, cd, da, shows that $abcd$ has its sides parallel and equal to one third of the corresponding sides of $ABCD$. The two quadrilaterals are therefore similar and similarly placed, and hence the lines joining their corresponding vertices meet in a point O such that

$$Oa/OA = Ob/OB = Oc/OC = Od/OD = 1/3.$$

II. *Solution by E. C. Kiefer, Decatur, Ill.*

Placing the quadrilateral with the side AD on the x-axis and the point A at the origin, let the coördinates of the points be $B(x_1, y_1)$, $C(x_2, y_2)$, $D(x_3,, 0)$. Then the equations of the lines are

$$\begin{array}{ll} Aa, & x(y_1 + y_2) - y(x_1 + x_2 + x_3) \hphantom{{}+ y_1(x_2 + x_3) - x_1y_2} = 0, \\ Bb, & x(y_2 - 3y_1) - y(x_2 + x_3 - 3x_1) + y_1(x_2 + x_3) - x_1y_2 = 0, \\ Cc, & x(y_1 - 3y_2) - y(x_1 + x_3 - 3x_2) + y_2(x_1 + x_3) - x_2y_1 = 0, \\ Dd, & x(y_1 + y_2) - y(x_1 + x_2 - 3x_3) - x_3(y_1 + y_2) \hphantom{{}- x_2y_1} = 0. \end{array}$$

Since the sum of the left sides of these equations is identically zero, the four lines meet in the same point; the first and fourth equations give for its coördinates $\frac{1}{4}(x_1+x_2+x_3)$, $\frac{1}{4}(y_1+y_2)$. This point is easily shown to divide each of the segments Aa, Bb, Cc, Dd in the ratio of $3:1$; it is therefore a center of similitude for the figures $ABCD$ and $abcd$.

Also solved by Rufus Crane, P. S. Dwyer, H. A. DoBell, S. E. Field, A. E. Gault, J. H. Neelley and A. Pelletier.

3361 [1929, 105]. *Proposed by B. F. Finkel, Drury College.*

Find the envelope of a system of circles having for diameter a secant of constant length, $2r$, of a conic.

Solution by Otto J. Ramler, Catholic University, Washington, D. C.

Consider the case of an ellipse with the equation $b^2x^2+a^2y^2-a^2b^2=0$; and let the chord with the length $2r$ have the extremities (x_1, y_1), (x_2, y_2), where

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = 4r^2. \tag{1}$$

Then from the equation of the ellipse we have $b^2(x_2^2-x_1^2)+a^2(y_2^2-y_1^2)=0$. If m is the slope of the chord and (x, y) is its middle point, this equation may be written:

$$b^2x + ma^2y = 0. \tag{2}$$

From (1) and the equation of the chord we have

$$(3) \qquad (y_2 - y_1)/m = x_2 - x_1 = 2r(1 + m^2)^{-1/2}.$$

Combining these results with $2x = x_2 + x_1$, $2y = y_2 + y_1$, we find

$$(4) \qquad y_2 = y + mr(1 + m^2)^{-1/2}, \quad x_2 = x + r(1 + m^2)^{-1/2}.$$

Substituting these values of x_2, y_2 in the equation of the ellipse and eliminating m by means of (2), we obtain

$$(5) \qquad (b^2x^2 + a^2y^2 - a^2b^2)(b^4x^2 + a^4y^2 + a^2b^2r^2) + a^4b^4r^2 = 0,$$

the equation of the locus of the centers of the variable circles.

The envelope of the circles of radius r and centers (x, y) will consist of two curves parallel to (5) whose points are the extremities of the normal to (5) with the constant length r.

For the case of the hyperbola we have merely to replace b^2 in (5) by $-b^2$. The equation resulting from the consideration of a parabola may be obtained in the same manner.

Note by the Editors: This problem is the same as 3310 [1928, 154], a solution of which by W. J. Patterson has been printed [1929, 233].

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus Ohio.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Fourth Carus Monograph on Projective Geometry by Professor J. W. Young of Dartmouth College is now in page proof and will be issued in a few weeks. Members, individual and institutional, should order now through Secretary Cairns at the cost price of one dollar and twenty five cents per copy.

The Nobel Prize in Physics for 1928 has been awarded to Professor O. W. Richardson, of Kings College, London. The prize in physics for 1929 has been awarded to the Duc de Broglie, of Paris.

On the occasion of its 175th anniversary, Columbia University conferred the degree of honorary doctor of philosophy on Dean H. E. Hawkes and Professors C. J. Keyser and D. E. Smith, among others.

Doctor Max Mason has been elected president of the Rockefeller Foundation.

Professor Edwin B. Wilson, of Harvard University, has been elected president of the Social Science Research Council.

Dr. F. R. Bamforth has been appointed assistant professor of mathematics at Cornell University.

Professor M. A. Brumbaugh, of the University of Pennsylvania, has been appointed professor of statistics in the School of Business Administration of the University of Buffalo.

Assistant Professor D. L. Holl, of Iowa State College, has been promoted to an associate professorship.

Mr. W. H. McEwen, who has been teaching mathematics at Regina College Regina, Saskatchewan, is spending the year in graduate study at the University of Minnesota.

Mr. R. H. Marquis has been appointed assistant professor of mathematics at Ohio University, Athens, Ohio.

Dr. E. L. Mickelson has been appointed professor and head of the department of mathematics at New Mexico State Teachers College.

Professor H. W. Tyler, Secretary of the American Association of University Professors, having returned to the Massachusetts Institute of Technology on the expiration of his leave of absence, the management of the Washington office has been assumed by the treasurer, Professor Joseph Mayer, who has been granted leave of absence by Tufts College for this purpose.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.
INDIANA, Earlham College, May 2-3.
IOWA.
KANSAS.
KENTUCKY.
LOUISIANA-MISSISSIPPI, Cleveland, Miss., April.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.
MICHIGAN.
MINNESOTA.
MISSOURI.
NEBRASKA, Peru, Neb., May 9.
OHIO, Columbus, Ohio, April 3.
PHILADELPHIA.
ROCKY MOUNTAIN.
SOUTHEASTERN.
SOUTHERN CALIFORNIA.
TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

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THE MATHEMATICAL ASSOCIATION OF AMERICA

The following twenty-nine persons and one institution have been elected to membership in the Association on applications duly certified:

To Individual Membership

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A. D. BRADLEY, A.M. (Teachers Coll., Columbia). Instr., Hunter Coll., New York, N. Y.

J. F. BUTLER, A.M. (St. Louis Univ.). Instr., Milford Novitiate, Milford, Ohio.

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KAMCHEUNG WOO, 2213 Bancroft Way, Berkeley, Calif.

To Institutional Membership

WILEY COLLEGE, Marshall, Texas.

THE THIRTEENTH ANNUAL MEETING OF THE MISSOURI SECTION

The thirteenth annual meeting of the Missouri Section of the Mathematical Association of America was held at Washington University, St. Louis, on Saturday morning, November 16, 1929. Professor Byron Ingold presided. The at-

tendance was twenty-five, including the following thirteen members of the Association:

Nelson Dunford, Otto Dunkel, Father F. J. Gerst, L. D. Haertter, Byron Ingold, Louis Ingold, Arria Murto, W. O. Pennell, P. R. Rider, W. H. Roever, Eugene Stephens, Jessica M. Young, E. Kathryn Wyant.

The officers who were elected for the coming year are: Chairman, LOUIS INGOLD, University of Missouri; Vice-Chairman, EUGENE STEPHENS, Washington University; Secretary-Treasurer, P. R. RIDER, Washington University.

The following papers were read:

1. "A symbolic method for solving a system of simultaneous ordinary linear differential equations of any order with constant coefficients," by Professor Eugene Stephens, Washington University.

2. "Vector methods in analytic geometry," by Professor Louis Ingold, University of Missouri.

3. "A modification of a proof by Steiner," by Professor Otto Dunkel, Washington University.

Abstracts of these papers follow:

1. The paper by Professor Stephens takes up a new symbolic method for solving a system of simultaneous ordinary linear differential equations of any order with constant coefficients.

The differential operators present in the system are given algebraic or quantitative meaning, upon which the system becomes one of linear algebraic equations in the variables. The matrix of the operators in the system is used intact without the necessity of reduction to an equivalent diagonal system. The theorems applying to the solution of systems of linear algebraic equations are then used upon it. When the solution is obtained it is in algebraic form, with differential and integral operators present. These latter are then interpreted, or used, as their original nature warrants, and when the operations are carried out the final results are the solution of the original system of linear differential equations.

A new theorem is introduced into differential equation theory concerning what is called the "General Complementary Function" from which can be obtained algebraically the "Complementary Functions" of the general solution. For the proof of this theorem some new theorems on "factors of a determinant" had to be built up, which are somewhat analogous to "elementary factors," though more easily obtained than the latter.

2. The paper by Professor Ingold proposes a slight rearrangement of the material in the usual analytic geometry course to make room for a vector treatment of certain topics.

Illustrations of the vector treatment are given and their application in more advanced courses in geometry are indicated.

3. In Crelle's Journal, vol. 24(1842), pp. 96–99 Steiner gave a proof of the theorem that the equilateral triangle has a greater area than any other triangle having the same length of perimeter. In this paper Professor Dunkel gives a

slight modification of this proof and shows how the modified form of reasoning may be applied to prove that the equilateral triangle circumscribing a given circle has a shorter perimeter and a smaller area than any other triangle circumscribing the same circle. These proofs make no use of parallels and hence apply without the Euclidean restriction that the sum of the angles of a triangle shall be 180.°

PAUL R. RIDER, *Secretary.*

THE FOURTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The fourth annual meeting of the Philadelphia Section of the Mathematical Association of America was held in Bennett Hall at the University of Pennsylvania on Saturday, November 30, 1929. There were two sessions, Professor A. H. Wilson of Haverford College presiding at both sessions.

The attendance was fifty-one, including the following thirty-five members of the Association: V. W. Adkisson, E. F. Allen, P. A. Caris, G. G. Chambers, J. W. Clawson, E. S. Crawley, J. E. Davis, Fletcher Durrell, L. P. Eisenhart, Michael Goldberg, O. E. Glenn, J. C. D. Harding, G. A. Harter, Frances H. Jackson, R. W. Jones, C. A. Keeler, J. R. Kline, M. S. Knebelman, P. A. Knedler, K. W. Lamson, W. F. Long, H. M. Lufkin, L. D. McDonough, A. E. Meder Jr., H. H. Mitchell, Richard Morris, C. A. Nelson, A. G. Rau, C. J. Rees, J. F. Ritt, J. H. Roberts, J. A. Roulton, George Rosengarten, Pincus Schub, A. H. Wilson.

At the business meeting the following officers were chosen for next year: Chairman, Professor J. A. MILLER, Swarthmore College; Secretary, P. A. CARIS, University of Pennsylvania; Program Committee, Professors Miller (ex-officio), Fort and Kline. The next meeting of the Section will be held on Saturday, November 29, 1930 at the University of Pennsylvania.

The following papers were presented:

1. "Wave mechanics," by Professor K. W. LAMSON, Lehigh University.

2. "Group characters," by Professor H. H. MITCHELL, University of Pennsylvania.

3. "Dynamical trajectories and geodesics," by Professor L. P. EISENHART, Princeton University.

4. "Integration in finite terms," by Professor J. F. RITT, Columbia University.

Abstracts of the papers follow below, the numbers corresponding to the numbers of the titles:

1. The paper by Professor Lamson mentioned some types of experiments which led to the use of the quantum theory. In that theory only integral values are allowed for some of the variables, which in ordinary mechanics could take on any of a continuous set of values. Schrödinger shows how to replace this assumption by a less startling one. He develops a mechanics whose difference

from ordinary mechanics is analogous to the difference between wave optics and geometrical optics. Corresponding to energy in mechanics is frequency in optics. The whole numbers must enter the theory at some point and Schrödinger gets them by the mild postulate that certain functions must be finite for all values of the variables.

2. The theory of group characters, as developed by Frobenius, Schur, Burnside, and others, plays a fundamental role in the problem of the representation of a given abstract group as a group of linear substitutions. Each such representation that is irreducible determines a set of sums of roots of unity, the "traces" of the substitutions that correspond to the different conjugate sets of operators of the given group. A set of numbers of this sort is termed a "character" of the group, and the number of distinct characters is equal to the number of conjugate sets of operators. A variety of interesting relations hold between the numbers that constitute these characters, by use of which much light can be thrown on the representations of the given group as a group of linear substitutions. An application of the theory to abstract groups is the proof, due to Burnside, that no group whose order is divisible by just two different primes can be simple. Expositions of the theory are now available in several texts.

3. Professor Eisenhart's paper has been published in full in the October 1929 number of the *Annals of Mathematics*.

4. Professor Ritt discussed Liouville's work on the impossibility of performing certain integrations, and solving certain differential equations, in finite terms.

P. A. CARIS, *Secretary*.

THE UNDERGRADUATE MATHEMATICAL CURRICULUM OF THE LIBERAL ARTS COLLEGE*

By F. L. GRIFFIN, Reed College

1.

No curriculum can be intelligently planned, in any subject, without careful consideration of the *human needs*, individual and social, which should be met by instruction in and study of that subject. Colleges of engineering, commerce, and other specialized professions face rather specific demands which largely prescribe their curricula. Even they, however, may well be influenced by the more general considerations which should govern in the college of liberal arts and science. Consequently, portions of the following discussion apply in a measure even to the specialized schools.

The general educated public has as yet scarcely any conception of the rôle which mathematics has played in the development of our present civilization, is playing now in the rapid advance of many sciences, including the life sciences and some social sciences, and is destined to play further in the creation of a more

* Read in substantially this form at the summer meeting of the Mathematical Association of America at Boulder, Colorado, Aug. 26, 1929.

hopeful and rational social order, the enrichment of esthetics and art, and the full liberation of the human spirit.

In all the physical, engineering, and earth sciences, mathematics is an indispensable tool. We are told by one competent to speak that "chemistry has graduated from the class of descriptive sciences into the class of exact sciences and has taken its place by the side of physics and engineering as a branch of mathematics. All chemical phenomena and many biological phenomena are now being interpreted in terms of physical chemistry, and physical chemistry cannot be mastered without a foundation in mathematics which extends through calculus."[1] In biology differential equations are used in studying growth and senescence, digestion, healing and other metabolic activities; elementary calculus, graphical and statistical methods in studying biometry, genetic equilibrium, surface tensions, energetics, and so on. In some biological journals calculus is becoming conspicuous.[2] Similar mathematical methods are rapidly coming into use in psychology and education[3] for the study of foveal vision, the psychology of musical ability, mental growth and learning curves, threshold stimuli for sensations, the Weber-Fechner law, the distribution of mental ability, special types of tests,[4] and various other topics in the new science of psychophysics. A recent writer's allusion to differential equations as a suitable means of expressing the *interaction* of individual and environment in determining human conduct and social evolution, shows the current trend of thought and the possibility of a clarification similar to that introduced into economics by Cournot, Jevons, Walras, and others more recently, through mathematical thinking and the formal use of calculus. The theory of maxima and minima has brought to the business world a great economizing of materials. The theory of probability also is effecting important economies, besides rendering invaluable service in many fields.[5] Some knowledge of the mathematics of finance, and of calculus, is becoming indispensable in handling intricate questions of investment, *e.g.*, long leases with varying rentals and adjustments in the event of building; serial bonds with split rates of interest; etc. Sociological thinking, too, is not only being based increasingly upon statistical methods, but is beginning to be affected fundamentally by the conceptions of the calculus.[6]

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[1] F. Daniels, *Mathematics for students of chemistry*, this Monthly, vol. 35 (1928), pp. 3–9.

[2] O. W. Richards, *The mathematics of biology*, this Monthly vol. 32 (1925), pp. 30–36. See also T. B. Robertson, *The chemical basis of growth and senescence* (Lippincott, 1923); also his *Principles of biochemistry*; and recent volumes of The Journal of General Biology, The Journal of Comparative Neurology, and The Journal of Experimental Medicine.

[3] See recent volumes of The Journal of Experimental Psychology and The Journal of Educational Psychology.

[4] H. M. Walker, *Certain mathematical questions suggested by the true-false test*, this Monthly, vol. 34 (1927), pp. 503–15.

[5] T. C. Fry, *Probability and its engineering applications*. (Van Nostrand, 1928).

[6] Bureau of Census, U. S. Life Tables (1921), pp. 329 ff. See also F. S. Chapin, *Cultural Change* (The Century Co., 1929), pp. 357–384; also R. Pearl, *Biology of Population Growth*, and bibliography (Knopf, 1925).

Among the arts we note the architectural and decorative uses of geometry, the mathematical basis for the choice of graceful curves in drawing and painting,[7] the physico-mathematical basis for the communication of emotion through music, for the criticism of instruments and technique, and for the introduction of new scales.[8] The improved electric recording of music is a notable advance based partly upon the study of differential equations. In games, puzzles and cryptography, mathematics is frequently basic. And now comes a "calculus of variants" to the assistance of the student of literary criticism![9] May we remark further that mathematical manipulation is itself a fine art, requiring skill, judgment, and a sense of appropriateness comparable with like qualities requisite for success in the other arts?

All of this merely hints at what mathematics has contributed thus far toward our external or objective progress. It is also basic for the inner life of speculative philosophy: both as to the problem of knowledge and as to the nature of the universe. Touching the former, it provides a norm of logical structure and rigor, shows the unreliability of intuition in even apparently simple matters, and discloses the relative character of all knowledge. Touching the latter, it exhibits possible varieties of space, generalizes number concepts, clarifies the infinite, the nature of law or functionality, and determines invariants. It even touches questions of religious attitudes and spiritual values.[10]

This progress has been made despite the fact that only a small part of the workers in many fields have had any considerable familiarity with the concepts and methods of mathematical analysis. Doubtless the infiltration of mathematics into some fields will be slower than it has been into fields now saturated with our science. But who can confidently delimit the totality of subjects susceptible of helpful mathematical treatment? May we not reasonably anticipate a great surge of explanation and discovery in many fields when the knowledge and spirit of mathematics shall have seized upon most of the workers?

2.

To most students with other demands upon their time and interest, the freshman course presents the last opportunity for formal mathematical study. Whatever help we are to give the great mass, who will include many educational leaders of the next few decades, must be given in the freshman course. One reason for the past and present general ignorance of the possibilities of mathematics on the part of educated laymen is that there has been *scant opportunity* to acquire the requisite familiarity in the time available. Introductory courses have not adequately exhibited the possibilities of our science, nor sufficiently covered its major concepts and methods, and the latter have been obscured by

[7] J. Ruskin, *The elements of drawing and the elements of perspective*, pp. 155 ff. (Dutton, 1907). See also this Monthly, vol. 25 (1918), p. 192.

[8] J. M. Barbour, *Synthetic musical scales*, this Monthly, vol. 36 (1929), pp. 155–160. See also references given by R. C. Archibald in this Monthly, vol. 31 (1924), pp. 1–25.

[9] W. W. Greg, *The calculus of variants, an essay on textual criticism.* (Clarendon Press, 1927).

[10] D. E. Smith, *Religio mathematici*, this Monthly, vol. 28 (1921), pp. 339–349.

association with topics and technique unnecessary to the proposed objective and better postponed to later courses. Professor Whitehead's warning against reconditeness is much to the point; as is also his statement that "education to be living and effective must be directed to informing pupils with those ideas, and to creating for them those capacities which will enable them to appreciate the current thought of their epoch."[11]

The freshman course can render the great and important service needed only if it includes a thorough treatment of graphs as an instrument of calculation, the elements of differential and integral calculus with a considerable variety of applications, at least enough trigonometry to solve triangles and understand simple periodic oscillations, enough analytic geometry to see its power, solve simple locus problems, and encounter the several conics and some other curves, enough work in logarithmic calculation and in the approximate solution of higher numerical equations to handle these matters effectively, at least the basic ideas of statistics and probability, and some work on investments. Various other topics are desirable, but less essential. The graphical work should stress calculations made by drawing tangents and by measuring areas: in chemistry these have "brought order out of chaos in the study of solutions,"[1] and in many other fields where functional formulas are unknown, they are the sole reliance. The work in calculus should include partial differentiation, which is basic for the understanding of many subjects, from chemical thermodynamics to least squares in educational measurements. Integration should be applied in so many fields that its wonderful possibilities and generality will be sensed, and the procedure for applying it be grasped. "The ability to think in mathematical terms is absolutely essential." Even the work in differentiation falls far short of fruition if integration be slighted.

Such a freshman course can give in a single year a comprehension and working knowledge of elementary mathematical analysis sufficient for studies in many fields. But to deepen the student's insight and extend his horizon, the student's work in the course should be *supplemented by brief lectures* on the signficance of important ideas and processes, and of mathematical analysis as a mode of thought, historical matters, the nature of mathematics as a logical system, its universality and some of its philosophical implications, the logical side of algebra, the parallel postulate and possibly some references to relativity. By all means, a simple explanation should be given of the *idea* of a differential equation of the first order: how it expresses the determination of the rate of change of one varying quantity by some other variable and perhaps by the first quantity itself; what is meant by solving the equation; and so on. (We may fairly question whether a fully satisfactory textbook for such a freshman course has yet been published; but, with continued experimentation and revised editions by various authors, much improvement may doubtless be expected.)

The sophomore course, while serving a much smaller clientele, should again be planned as generally as is feasible. Beyond this the courses will naturally be

[11] A. N. Whitehead, *The aims of education and other essays*, p. 116 ff. (Macmillan, 1929).

more specialized to meet the needs of particular groups. The broad, unified freshman course will prove to be a thoroughly adequate foundation for further study only if it is followed by a course definitely based upon it, which utilizes its gains and fills any gaps left in the freshman year. The greatest need will be drill in technique and practice in making still further applications. To learn to employ calculus in new situations a student must see it at work, and must put it to work, in a variety of fields. Whatever we may think of the doctrine of no-transfer-of-training, we can be sure that the difficulty of applying a familiar process in a field of new ideas will be greatly lessened if it has in the past been applied to novel situations involving similar elements. Thus the calculus should be seen at work not only in geometry and mechanics but also in other branches of physics, in chemistry, in meteorology, in physiology and psychology, in economics and actuarial science. There should be extensive practice in setting up integrals other than those already analyzed in the text, which the student merely reproduces with new limits and new functional formulas to fit the particular case. That is, the student should have actual practice in *analyzing ideas* and reasoning out the type form of the integrand. (He should learn to use the rapid, vigorous, abridged statement with "tiny elements," but should be able to justify his conclusions by a sound limit procedure.) Likewise he should have considerable practice in formulating differential equations as well as in solving them. In applying integration to geometry, he should handle many combinations of elementary surfaces for which no equations are given,—either getting the equations for himself or actually *seeing* the limits of integration from drawings of suitable sections. He should be able to test directly the consistency of two supposed integrals of a common integrand; should learn when to use tables of integrals and when not,—the most efficient procedure always,—and should become familiar with ordinary tables of probability, elliptic integrals, and the gamma function. An introduction to hyperbolic functions is highly important, as are also a brief introduction to Fourier series, and a discussion of mean values for different independent variables, or over a non-linear region. Partial derivatives deserve careful treatment; likewise curvature and motion. The purely geometrical properties of conics may well be postponed for study in the course next to be discussed. (Before leaving the present topic, however, may we ask whether the two-year course so far outlined would not be as satisfactory for engineering students as in a college of liberal arts and science? *The research papers by numerous seniors*, mentioned later, *apparently indicate that unified courses give an adequate foundation for specialized work.*)

3.

In the junior and senior years, at least the following specialized courses are needed to serve various particular groups of students. (Some of these will serve more than one group.) After each title is suggested a suitable number of "units" i.e., year-hours, or the equivalent.

(*a*) *Modern Geometry, Analytic and Synthetic* (*4*). Elementary courses in analytic geometry are generally more "analytic" than they are "geometry"; by

the time the student has learned the basic formulas and equations, and has sensed the method, there is little time left to use the new instrument for the actual investigation of geometric questions. The junior course may well start with a brief concise summary of analytic method, and then apply the methods to *a genuine study of geometry*. The conics offer a veritable mine of theorems relating to diameters, tangents, polars, etc. Several important higher plane curves should be studied, starting from their locus definitions, determining their shapes and special properties, and then returning to them presently in a systematic treatment of the inversion of conics, and pedals of various curves. Much use should be made of construction, accurate or rough, so that the student may see the loci develop, point by point. Likewise, in later parts of the course. This analytic work which vastly enlarges the student's conception of geometry, need not take more than a third of a year. It may well be followed by a brief introduction to projective geometry, chiefly synthetic, occupying even less time, but opening to the student a whole range of new ideas, including Pascal's, Brianchon's and Desargues' theorems. The remaining portion of the year, from a third to a half, can hardly be spent more profitably than upon the type of modern geometry now commonly called "college geometry." This bears far more directly upon the problem of high school teaching than do most of the college courses.

(*b*) *Descriptive Geometry* (*4 or 2*) is another subject of great value to general students as well as engineers, because of its powerful method of studying spatial figures, the training in visualization, and its satisfying logical character. Prospective teachers of elementary or higher geometry should if possible study this subject, at least for a semester.

(*c*) *Higher Algebra* (*2*). Though the student may have had a brief introduction to determinants in his earlier work, and have applied them in his Modern Geometry, he can now advantageously study the general theory, including elimination. This together with work in the theory of equations, including simple group concepts and applications to the problem of the regular n-gon, trisection of the angle, etc., is important equipment for prospective high school and college teachers, and some others. One of my former students, a structural engineer, has found the general solution of the cubic to be of great value in his practice.

(*d*) *Advanced Calculus* (*4*) deserves the attention of mathematical and technical specialists and some others, throughout the senior year. More work in differential equations, ordinary and partial, with applications to vibrating strings and membranes, flow of heat, etc., may well receive a large portion of the time. This involves a definition and very brief study of the Bessel functions, and some further work in Fourier series. Time should also be given to the reduction of elliptic integrals, with many numerical examples and applications; and to devices for evaluating definite integrals, including differentiation with respect to a parameter. (It may be sufficient to state without proof a number of criteria as to the legitimacy of this and other processes, giving references to the litera-

ture where proofs can be found.)[12] There should be a brief introduction to the calculus of variations and to vector analysis, so that students for whom this is the last course in analysis may glimpse these fields, and those who go on may start the graduate courses more readily. Some attention should be given the geometry of surfaces and space curves; also to the complex variable; and to Green's and Stokes' theorems.

(*e*) *Probability and Mathematical Statistics* (*4 or 2*). In the junior and senior year some students will need a course in the calculus of observations. Besides the theory of probability and least squares, finite differences, empirical equations, and further statistical theory, it may cover special methods of numerical calculations.

(*f*) *Mathematics of Finance* (*2*) including insurance, and using the technical procedures not employed in the freshman course.

(*g*) *Spherical Trigonometry and Surveying* (*2*) with introductory work in solid geometry.

(*h*) *Astronomy* (*4*) unless offered in a separate department, should be available, at least a descriptive course with some mathematical calculations.

(*i*) *Analytic Mechanics* (*4*) is an excellent course; but in my opinion should be given in the physics department, illuminated by some experimental work.

(*j*) *History and Foundations* (*2*). At least a brief course of this sort should be a prerequisite for teaching elementary algebra and geometry. Many teachers of algebra are woefully ignorant of its axioms. (For example, recently, in some excellent high schools a number of experienced teachers were unable to supply inquiring pupils with a logical proof that $(-4)(-3)=12$. Instead of reasoning from axioms about *numbers*, they could only mumble about some alleged *analogies*!)

By the way, the "units" suggested for these courses need not necessarily be "hours"; but rather units of total time alloted to the course in class and out. At Reed College an elementary three-unit course may meet four times a week with short assignments, to allow relatively large supervision of the student's progress. An advanced four-unit course may meet but once a week, the student working independently for ten or more hours between. The number of meetings can also be varied during the course, or meetings intermitted, at the option of the instructor. The junior and senior work is under the jurisdiction of a "Division" of the faculty. All this makes for flexibility and effectiveness.

4.

The large college or university can offer a great variety of courses; the small institution has difficulty in providing for as much flexibility. A great help in this direction is to utilize some of the intellectual reserves of the student himself, particularly the advanced student, by offering some of the above courses (*e*)-(*g*) as Special Topics, for individual reading, practice and report. (Group work in

[12] E. J. Moulton, *The content of a second course in calculus*, this Monthly, vol. 25 (1918), pp. 429–34.

special topics for freshmen and sophomores may also profitably be given to supplement the regular courses, covering for instance special statistical methods, use of the slide rule, elements of determinants, some properties of conics, curve tracing with the help of calculus and such devices as Newton's triangle, imaginary regions, etc.) The more advanced work in special topics is substantially of the sort often called honors work, and enjoys the advantage of an engrossing interest, particularly when the student selects the topics himself from time to time.

In my judgment the most valuable experience a senior specialist student can have is to write a thesis, embodying such research as his capacity permits, on a suitably chosen individual problem, and take a final oral examination in his major field. In many institutions this type of work may have to be limited to a group of specially selected honors students. At Reed College, all seniors undertake this work, but to get senior standing a student must pass an intensive Qualifying Examination at the end of the junior year. At that, there is of course a wide variation in the ability of our seniors, and in the independence and value of their theses. An idea of the sort of topics which we have used as thesis subjects in mathematics may be obtained from the appended list.

In some institutions mathematical clubs may be of more value in maintaining student interest and developing initiative than the special methods I have mentioned. But there appears to be no mutual exclusiveness; both activities may be carried on if desired.

In closing let me again emphasize that one of our major tasks is to educate a great multitude in the concepts, modes of procedure, and criteria of mathematical analysis. This task recognized, and suitable action taken in organizing our freshman courses, mathematics will become the key subject in the curricula of the future. We are but at the dawn of the general utilization of mathematics in the amelioration of human life.

A List of Thesis Subjects

Nearly all of these bachelor theses, written at Reed College, embodied some new results of investigation; several broke ground virtually new. Those marked by an asterisk were read, in outline, before the San Francisco Section of the American Mathematical Society.

Integrals and Functions:

On the accuracy of certain approximations to the elliptic integrals.
On the reduction of quotients of quadratic radicals to elliptic integrals.
Imaginary substitutions for elliptic integrals of the first type.
A study of certain functions of a complex variable.
* Generalized functions of sectorial areas.
* Further generalization of the circular and hyperbolic functions of sectorial areas.

Calculus of Variations:

* Concerning geodesics on certain surfaces of revolution.
Some geodesic lines on an ellipsoid of revolution.
Geodesics on a right circular cone.
Some theorems in maxima and minima.

Geometry:

* Curves functionally conjugate with respect to a given curve and origin: Type I. (Others: Types II, III, IV, V, *VI.)

Some properties of normals to an ellipse.

* Certain families of curves having a parabola as their envelope. (Another: the cubical parabola.)

Certain properties of poles and polars.

* Conics through given points.

Concerning certain basic problems in analytic geometry. (Highly original.)

Investigations in the trisection of the angle.

Algebra:

* An insolvable case of the generalized school-girl problem.

* Concerning the solution of cubic and quartic equations.

Equations of higher degree solvable by special artifices and devices.

The construction of a regular 257-gon.

Other Fields:

A mathematical theory of business profits (under special conditions).

* A study of caustic surfaces and of a certain image formed by reflection and refraction.

Accurate measurements with a steel tape or wire.

A new graphical method for determining the positions of a moving body under Newtonian attraction.

The advance of the apsidal lines in central orbits described in the equatorial planes of attracting ellipsoids of revolution.

Relation of secondary mathematics to higher mathematics.

Tentative plan for a course in first year mathematics for secondary schools.

* Some studies in probability and correlation.

THE PROBLEM OF TWO BODIES AND A RELATED SEXTIC[1]

By W. H. GARRETT, Baker University

1. *Equation of orbit.* Given two bodies with masses m_1 and m_2 which attract each other with a force which is proportional to the product of the masses and inversely proportional to the square of the distance between them, the differential equations of motion of one with respect to the other are:

$$d^2x/dt^2 = -k^2Mx/r^3\,;\; d^2y/dt^2 = -k^2My/r^3, \tag{1}$$

where $M = m_1 + m_2$, r = the distance between bodies, and k = acceleration at unit distance.

The integration of this system[2] gives as the polar equation of the orbit,

$$r = c_1^2 k^{-2}M^{-1}/[1 - (1 + c_1^2 c_3 k^{-4}M^{-2})^{1/2} \cos(\theta - c_4)], \tag{2}$$

which is the equation of a conic with the origin at one of the foci.

The ordinary equation of a conic with the pole at the right hand focus is

$$r = p/[1 + e\cos(\theta - \omega)], \tag{3}$$

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[1] Read before the Kansas Section of the Mathematical Association of America, February 4, 1928.

[2] For the separate steps in this integration, see F. R. Moulton's *Celestial Mechanics*, pp. 147–148.

where ω is the angle between the polar axis and the major axis of the conic.

Comparing (2) with (3) it is seen that

$$(4)\qquad p = c_1^2 k^{-2} M^{-1}\ ;\ e^2 = 1 + c_1^2 c_3 k^{-4} M^{-2}\ ;\ \omega = c_4 - \pi.$$

2. *An important property of the orbit.* If $e<1$, the orbit is an ellipse and $p=a(1-e^2)$. Hence, combining this relation with the first two of equations (4), there is obtained:

$$c_3 = -\ k^2 M(1 - e^2)p^{-1} = -\ k^2 M a^{-1}.$$

Now one of the integrals of (1) is

$$(dx/dt)^2 + (dy/dt)^2 = 2k^2 M r^{-1} + c_3.$$

Hence the velocity V of one body relative to the other is given by the relation,

$$V^2 = k^2 M(2r^{-1} - a^{-1}).$$

Solving for a,

$$a = k^2 r M/(2k^2 M - V^2 r).$$

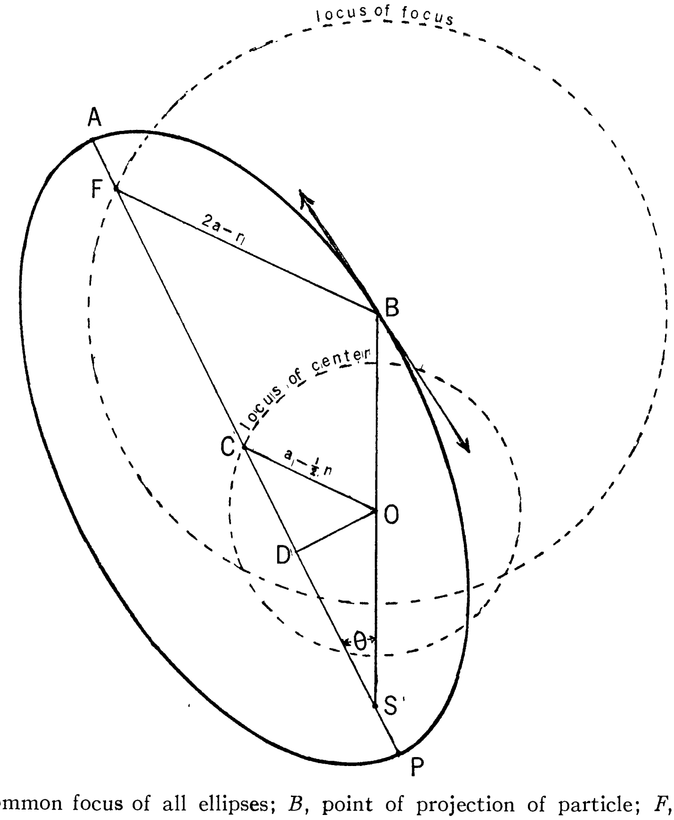

FIG. 1. S, common focus of all ellipses; B, point of projection of particle; F, second focus; A, aphelion point; P, perihelion point.

This establishes the following important property: If a body m_1 at a distance r from a body m_2 is projected in different directions with the same velocity V, $[V<(2k^2Mr^{-1})^{1/2}$, the parabolic velocity$]$, then the orbits described by m_1 relative to m_2 will be ellipses with major axes equal in length.

3. *Loci of focus and center.* Let B (Fig. 1) be the point of projection of m_1, and S, the position of m_2, the fixed focus. The locus of the other focus F will be a circle with center at B and radius $2a-r$, since $SB+BF=2a$ and $SB=r$.

From this it follows that the locus of centers of all conics will be on a circle with center at O, the midpoint of SB, and radius $=a-\frac{1}{2}r$.

4. *Loci of Aphelion and Perihelion Points.*[3] In Fig. 1, let A be the aphelion point and P the perihelion point. Take the pole at S and let SB be the polar axis. Draw the perpendicular OD from O to the line SF. Then the loci of A and P in terms of the polar coordinates ρ and θ may be obtained as follows:

$$\rho = SD + DC \pm CA. \quad \text{Then, since } DC = \left(\overline{CO}^2 - \overline{DO}^2\right)^{1/2},$$
$$\rho = \tfrac{1}{2}r\cos\theta + \left[(a - \tfrac{1}{2}r)^2 - (\tfrac{1}{2}r\sin\theta)^2\right]^{1/2} \pm a,$$

which becomes

$$\rho^2 + 2a\rho + ar - r(\rho + a)\cos\theta = 0, \tag{5}$$

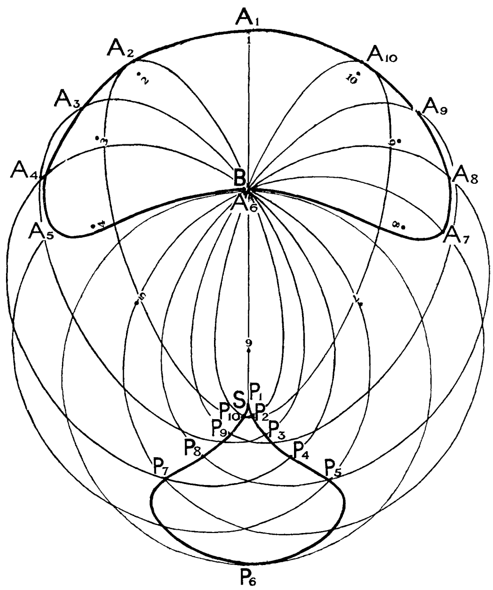

FIG. 2. $r=8$, $a=7$; S, common focus of all ellipses; B, point of projection; 1, 2, $\cdots$, 10, second foci; A_1, A_2, $\cdots$, A_{10}, aphelion points; P_1, P_2, $\cdots$, P_{10}, perihelion points.

[3] This problem appears in F. R. Moulton's *Celestial Mechanics*, page 155. It was however originally solved by the writer prior to the appearance of the first edition, while he was a member of Dr. Moulton's class in Celestial Mechanics.

which is the polar equation of the loci of the aphelion and perihelion points.

Changing to rectangular coordinates, the following sextic equation of the loci is obtained:

$$
\begin{aligned}
(6) \qquad & x^6 - 2rx^5 + (r^2 + 3y^2 + 2ar - 4a^2)x^4 + (4a^2r - 2ar^2 - 4ry^2)x^3 \\
& + (3y^4 + 4ary^2 - 8a^2y^2 + r^2y^2)x^2 + (4a^2ry^2 - 2ar^2y^2 - 2ry^4)x \\
& + y^6 + (2ar - 4a^2)y^4 + a^2r^2y^2 = 0.
\end{aligned}
$$

In Figs. 2 and 3, the loci of the aphelion and perihelion points are constructed for the two cases $a<r$, and $a>r$. ($r=8$ and $a=7$ and 9).

Special case, $a=r$: If $a=r$ then equations (5) and (6) become

$$(\rho + r)(\rho + r - r\cos\theta) = 0,$$

$$(x^2 + y^2 - r^2)(x^4 - 2rx^3 + 2x^2y^2 - 2rxy^2 + y^4 - r^2y^2) = 0,$$

which represent the circle and the cardioid. These loci are shown in Fig. 4 and one may easily trace the transition from the loci of aphelion and perihelion

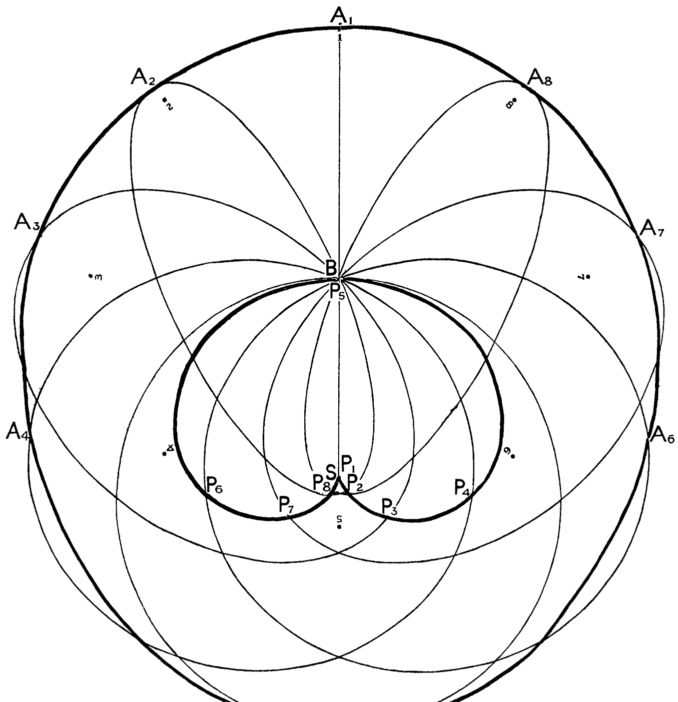

FIG. 3. $r=8$, $a=9$; other points as in FIG. 2.

points in Fig. 2 to the corresponding loci in Fig. 3 by means of this connecting link.

A further consideration of Fig. 4 also shows that here one member of the family of ellipses is a circle with radius r and any point on this circle may be considered as either a perihelion or aphelion point.

5. *Mechanical Construction of Sextic.* Several members of the family of sextics are shown in Fig. 5. These curves were constructed mechanically as follows:

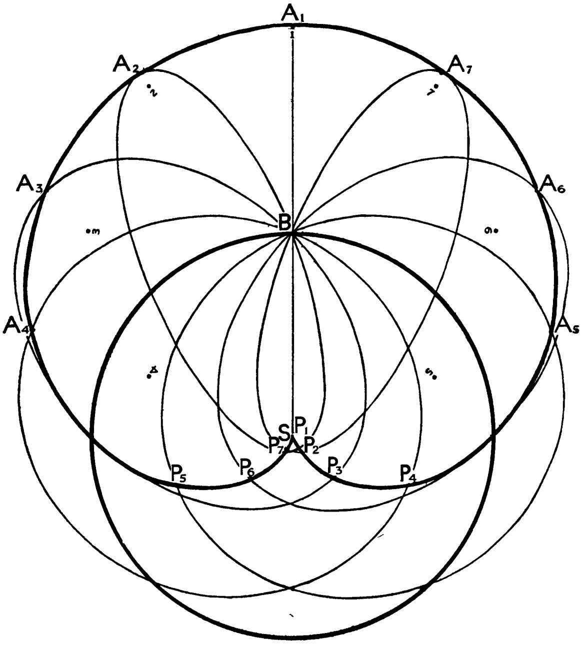

FIG. 4. $r=8$, $a=8$; other points as in FIG. 2.

Two pencils at A and P were held at a distance of $2a$ apart by means of two narrow strips of wood. Two stove bolts clamped the strips together, holding the pencils firmly in proper position.

A short cross arm pivoted at one end to a fixed nail at O[4] and at the other end to C, the midpoint of AP, allowed C to rotate around O at a distance of $a-\frac{1}{2}r$.

The groove between the parallel strips was placed over a fixed nail at S at a distance of $\frac{1}{2}r$ from O, and kept A, S, and P always on the same straight line, the major axis of the ellipse.

As the point C was revolved around O, the pencils at A and P described the locus of the sextic, that is the loci of the aphelion and perihelion points.

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[4] See Fig. 1.

The different members of the family were obtained by letting the parameter a take the values 6, $6\frac{1}{2}$, 7, $7\frac{1}{2}$, 8, $8\frac{1}{2}$, 9, with $r=8$.

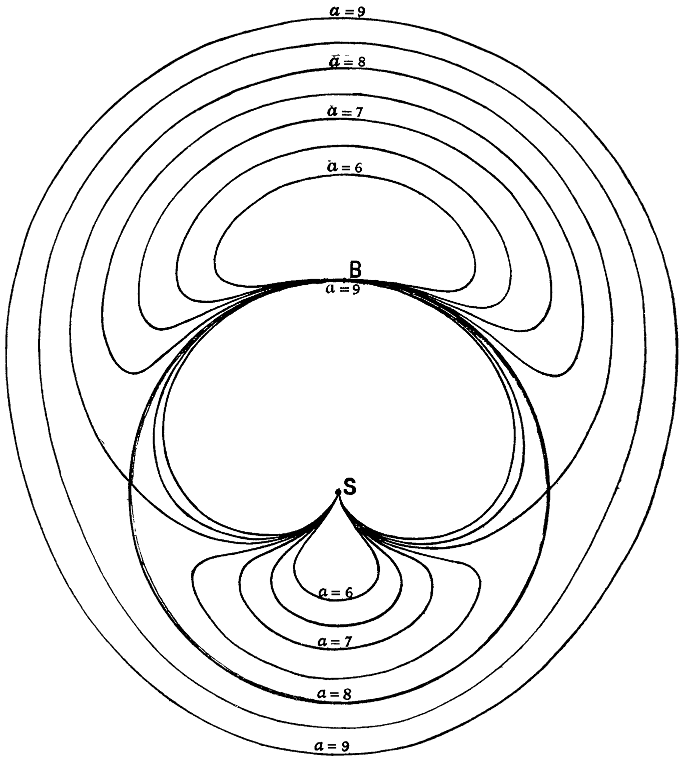

FIG. 5. Family of sextics for values $r=8$, $a=6$, $6\frac{1}{2}$, 7, $7\frac{1}{2}$, 8, $8\frac{1}{2}$, 9. In the special case $a=8$, the sextic becomes a circle and a cardioid.

THE RECTANGULAR HEXAGON

By PAUL WERNICKE, Washington, D. C.

The theorems here to be considered concern the figure of a hexagon in space (S_3), the consecutive sides of which meet at right angles. It is determined by three lines, P, Q, R, not in general all parallel to a plane. Following J. Petersen[1] we call P, Q, R the *sides* and the common perpendiculars (shortest distances) of the pairs QR, RP, PQ the *edges* of this "trilateral figure." The sides are determined by the edges as well as vice-versa, so that the hexagon may be considered as a trilateral figure in two ways.

The methods of proof are based on Study's coordinates of the straight line

[1] L'Intermédiaire des Mathématiciens, vol. 5 (1898) p. 36.

(Strahlen-Coordinaten[2]), the system of which, in turn, is built on Plücker's homogeneous (and redundant) coordinates of the right line as determined by two points, in Cartesian rectangular coordinates: $x=(x_1/x_0, x_2/x_0, x_3/x_0)$ and $x'=(x_1'/x_0', x_2'/x_0', x_3'/x_0')$:

$$rX_{0i} = x_0x_i' - x_ix_0', \quad rX_{jk} = x_jx_k' - x_kx_j',$$

where i, j, k are different, $i, j, k=1, 2, 3$, and r is a factor of proportionality.

Departing slightly from Study in order to use vector algebra,we combine the first three into the *vector* of X:

$$X_v = iX_{01} + jX_{02} + kX_{03};$$

and the last three into its *moment* (about the origin of coordinates):

$$X_m = iX_{23} + jX_{31} + kX_{12}.$$

We further write $X=X_v+\epsilon X_m$, where ϵ is a symbol to be defined.
Since the determinant

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_0' & x_1' & x_2' & x_3' \\ x_0 & x_1 & x_2 & x_3 \\ x_0' & x_1' & x_2' & x_3' \end{vmatrix}$$

vanishes we find that the scalar product $X_v \cdot X_m=0$. This is Plücker's equation. If

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_0' & x_1' & x_2' & x_3' \\ y_0 & y_1 & y_2 & y_3 \\ y_0' & y_1' & y_2' & y_3' \end{vmatrix} \equiv X_v \cdot Y_m + X_mY_v = 0,$$

the lines X, Y intersect.

Dividing the rows of this determinant by x_0, x_0', y_0, y_0', respectively, and subtracting the third row from the fourth, then the first from the second and third, we obtain the negative volume of the parallelopiped having x, x', y, y' for summits—negative, because X_v, Y_v, $X_v \times Y_v$, Y_v form a positive sequence of vectors while the three rows referring to them in the determinant occur in the order X_v, $X_v \times Y_v$, Y_v. The base of this parallelopiped being $X_v \times Y_v$ its altitude is the distance of Y from X.

$$X_v \cdot Y_m + X_m \cdot Y_v = - \sqrt{X_v^2}\sqrt{Y_v^2} \sin(X_v, Y_v) \operatorname{dist}(X, Y).$$

Now if the components of the two vectors in $P=P_v+\epsilon P_m$ are *any* six quantities, not in general satisfying Plücker's equation, then, with an indeterminate

[2] Study, Géométrie der Dynamen, Leipzig (1903), p. 201.

line X (for which $X_v \cdot X_m = 0$), the equation $P_v \cdot X_m + P_m \cdot X_v = 0$ is well known to represent a *nullsystem* (a "screw" in R. S. Ball's terminology).

Putting $U_v = P_v^3$ and $U_m = P_v^2 P_m + (P_v \cdot P_m) P_v$ we have $U_v \cdot U_m = 0$. There exists, therefore, a line $U = U_v + \epsilon U_m$. It is the principal axis of the null system, and remains unchanged if P be replaced by $P' + aP_v + \epsilon(aP_m + bP_v)$, for $U_v' = a^3 P_v^3$ and $U_m' = a^3[P_v^2 P_m - (P_v P_m) P_v]$, and division by a^3 is permissible. We must have $a \neq 0$ but otherwise a, b may be any (real) numbers.

Defining ϵ as a unit in the nature of an infinitesimal, so that $\epsilon^2 = 0$ while the first power of ϵ must be carried in the work, P' arises from P through multiplication by $a + \epsilon b$, a complex number which we call a *scalar dualite* while $P = P_v + \epsilon P_m$ is a *vectorial dualite*.[3]

Multiplying P by any scalar dualite $a + \epsilon b$ (where $a \neq 0$), we obtain a null system coaxial to P. Study now takes the step of using P, subject to such multiplication, as the representative of the right line which is the axis of $P_v \cdot X_m + P_m \cdot X_v = 0$.

The dualite components of P,

$$P_1 = P_{01} + \epsilon P_{23}, \quad P_2 = P_{02} + \epsilon P_{31}, \quad P_3 = P_{03} + \epsilon P_{12}$$

are *Study's coordinates* of the right line (Strahlencoordinaten). Multiplication by

$$P_v^2 + \epsilon(P_v \cdot P_m)$$

at any time effects a reversion to Plücker's coordinates. The *scalar product of vectorial dualites* $P \cdot Q$ is

$$P \cdot Q = (P_v + \epsilon P_m)(Q_v + \epsilon Q_m) = P_v \cdot Q_v + \epsilon(P_v \cdot Q_m + P_m \cdot Q_v).$$

$P \cdot Q = 0$ carries with it the separate vanishing of the terms $P_v \cdot Q_v$ (meaning perpendicularity) and $P_v \cdot Q_m + P_m \cdot Q_v$ (meaning intersection). P and Q, therefore, intersect at right angles. *The vectorial product*,

$$P \times Q = P_v \times Q_v + \epsilon(P_v \times Q_m + P_m \times Q_v)$$

represents the common normal of P and Q since $P_v \cdot (P \times Q) = Q_v \cdot (P \cdot Q) = 0$, as is readily verified. We have $P \times Q = -Q \times P$, whence $Q \times Q = 0$.

For $P \cdot (Q \times R) = Q \cdot (R \times P) = R \cdot (P \times Q)$ we write more briefly (with Study), (PQR). The vanishing of this scalar product evidently means that P, Q, R have a common normal.

$$\begin{aligned} P \times (Q \times R) = {} & P_v \times (Q_v \times R_v) + \epsilon[P_v \times (Q_v \times R_m) + P_v \times (Q_m \times R_v) \\ & + P_m \times (Q_v \times R_v)] = Q_v(R_v \cdot P_v) - R_v(P_v \cdot Q_v) + \epsilon[Q_v(R_m \cdot P_v) \\ & - R_m(P_v \cdot Q_v) + Q_m(R_v \cdot P_v) - R_v(P_v \cdot Q_m) + Q_v(R_v \cdot P_m) \\ & - R_v(P_m \cdot Q_v)] = Q_v(R \cdot P) - R_v(P \cdot Q) + \epsilon[Q_m(R_v \cdot P_m) \\ & - R_m(P_v \cdot Q_v)], \end{aligned}$$

[3] Study speaks of "dual numbers" and uses the terms "scalar" and "vectorial" in a different meaning.

whence by the addition of $\epsilon^2[Q_m(R_v\cdot P_m+R_m\cdot P_v)-R_m(P_v\cdot Q_m+P_mQ_v)=0$ we get $P\times(Q\times R)=Q(R\cdot P)-R(P\cdot Q)$, the *common normal* of P and $Q\times R$.

The sides of our rectangular hexagon or "trilateral" being P, Q, R, its edges become $Q\times R$, $R\times P$, $P\times Q$. The theorem that the three normals between side and opposite edge have a common normal was proved by F. Morley[4] in 1898 and announced by J. Petersen[5] in the same year, to be proved later.[6]

The vector product of two of these normals is

$$[Q\times(R\times P)]\times[R\times(P\times Q] = (Q\times R)(R\cdot P)(P\cdot Q)$$
$$+(R\times P)(P\cdot Q)(Q\cdot R)+(P\times Q)(Q\cdot R)(R\cdot P),$$

an expression obtained by Study for the common normal of the three which he calls Morley's line. Its invariance under transpositions of P, Q, R shows that it is the normal of any two of the lines $P\times(Q\times R)$, $Q\times(R\times P)$, $R\times(P\times Q)$.

A theorem on a *type of perspective* of two rectangular hexagons is given by J. Petersen in the same volume of L'Intermédiaire:

When in two trilaterals the three normals of corresponding sides have a common normal, those of the corresponding edges have one likewise.

For a proof let the sides of the first hexagon be P, Q, R, its edges, $Q\times R$, $R\times P$, $P\times R$, while the sides of the other are denoted by $Q'\times R'$, $R'\times P'$, $P'\times Q'$ and its edges by P', Q', R'. Normals of corresponding sides are:

$$P\times(Q'\times R'),\ Q\times(R'\times P'),\ R\times(P'\times Q')$$

and of corresponding edges:

$$P'\times(Q\times R),\ Q'\times(R\times P),\ R'\times(P\times Q).$$

The scalar products:

$$(P\times[Q'\times R'],\ Q\times[R'\times P'],\ R\times[P'\times Q'])$$
$$=(P'Q'R')[(Q'\cdot R)(R'\cdot P)(P'\cdot Q)-(Q\cdot R')(R\cdot P')(P\cdot Q')]$$
$$(P'\times[Q\times R],\ Q'\times[R\times P],\ R'\times[P\times Q])$$
$$=(PQR)[(Q\cdot R')(R\cdot P')(P\cdot Q')-(Q'\cdot R)(R'\cdot P)(P'\cdot Q)]$$

evidently vanish together. This proves the proposition.

The *cosine of the angle* between two *vectors* Q_v, R_v is given by the expression $Q_v\cdot R_v/(\sqrt{Q_v^2}\sqrt{R_v^2})$. We correspondingly form $Q\cdot R/(\sqrt{Q\cdot Q}\sqrt{R\cdot R})$ where the denominator may be written $\sqrt{Q_v^2}\sqrt{R_v^2}$.

$$Q\cdot R=\cos(Q_v,R_v)-\sin(Q_v,R_v)\operatorname{dist}(Q,R).$$

Accordingly, Study defines a *dualite angle*:

$$\operatorname{ang}(Q,R)=\operatorname{ang}(Q_v,R_v)+\epsilon\operatorname{dist}(Q,R)$$

[4] Proceedings of the London Mathematical Society vol. 29 (1898), p. 670.
[5] L'Intermédiaire des Mathématiciens vol. 5 (1898), p. 36.
[6] Nyt Tijdskrift for Mathematiker vol. 2 (1899).

of which $Q \cdot R - \sqrt{Q \cdot Q}\sqrt{R \cdot R}$ is the cosine. Its sine can either be found directly or by the usual formula $\sin x = \sqrt{(1-\cos^2 x)}$. It is

$$\sqrt{[(Q \times R) \cdot (Q \times R)]} \div [\sqrt{(Q \cdot Q)}\sqrt{(R \cdot R)}].$$

A "transversal" line U divides the angle ang (Q, R) in the sine-ratio:

$$\begin{aligned} &\sin \text{ang } (Q, U)/\sin \text{ang } (U, R) \\ &= \sqrt{[(Q \times U) \cdot (Q \times U)]}\sqrt{(R \cdot R)}/\sqrt{[(U \times R)(U \times R)(Q \cdot Q)]} \\ &= -\sin \text{ang } (Q, U)/\sin \text{ang } (R, U). \end{aligned}$$

Evidently *the three sine-ratios in which the angles ang* (Q, R), *ang* (R, P), *ang* (P, Q) *are divided by* U *have negative unity for their product.* This corresponds to the theorem of Ceva for plane triangles and, when stated for the dualite angles between the edges instead of the sides, to the theorem of Menelaus.

We have, thus, a system of perspective based on that of plane geometry, in which *intersections* of lines are replaced by their *common normals* and the rectangular hexagon (trilateral figure) takes the place of the triangle. As Petersen suggests, a projective geometry may be based thereon.

A PROBLEM IN THE CALCULUS OF VARIATIONS

By C. E. RHODES, University of Cincinnati

In presenting the calculus of variations for the first time to a student, it is customary to give the well known classical examples of the minimum distance, the brachistochrone, and the minimum surface of revolution. Such problems can not be multiplied too much to give to the beginner a thorough understanding of certain ideas which are later generalized. It should therefore be of interest to present an additional problem which not only has a concrete setting, but which also brings out certain properties of the fields used in the sufficiency proofs.

A statement of the problem: Given a stream with straight parallel banks, such that the direction of flow is everywhere parallel to the banks. The velocity of the current is assumed to be an even function of the distance from midstream, having an absolute maximum, w_0, at midstream, and an absolute minimum, w_1, along either bank. A boat is to go from a fixed point on one bank to a fixed point on the opposite bank. If the speed of the boat relative to the water is a positive constant, u, determine the course of the boat so that the crossing can be made in the minimum time.

To set up the problem analytically, let us suppose the river to be 2 units wide. Let the distance between the projections of the end points of the course on the center line of the river be $2a$. (If the general direction of the course is downstream, a is positive; if upstream, a is negative. See Fig. 2.) If we take co-ordinate axes as shown in Fig. 2, the course will be symmetric with respect

to the origin, since the current, w, is an even function of x. We shall therefore consider only half the path, viz., from O to A. Let v denote the resultant of u and w, y the ordinate of any point on the course, s the distance along the course measured from O, θ the angle between v and the X-axis, and T the time required to go from O to A along the course. Projecting w and u onto v as

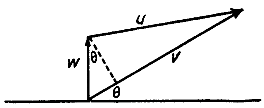

FIG. 1

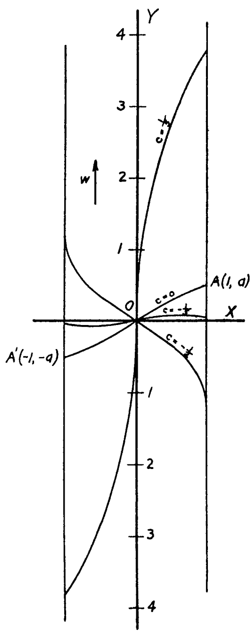

FIG. 2

shown in Fig. 1, we see that $v = w \cdot \sin\theta + \sqrt{(u^2 - w^2\cos^2\theta)}$. Since the resultant velocity, v, is the direction of the tangent to the course, $\tan\theta = dy/dx = y'$. Using this, we see that v may be put in the form

$$v = \frac{[wy' + \sqrt{(u^2 - w^2 + u^2y'^2)}]}{\sqrt{(1 + y'^2)}}.$$

From this it follows that

$$(1) \qquad T = \int_0^s \frac{ds}{v} = \int_0^1 \frac{(1 + y'^2)dx}{wy' + \sqrt{(u^2 - w^2 + u^2y'^2)}}$$
$$= \int_0^1 \frac{wy' - \sqrt{(u^2 - w^2 + u^2y'^2)}}{w^2 - u^2} dx.$$

Since by hypothesis u is a constant and w is a function of x alone, the integrand function in (1) is a function of x alone, and the Euler equation which determines our extremal curves becomes

$$w - [u^2y'/\sqrt{(u^2 - w^2 + u^2y'^2)}] = c(w^2 - u^2),$$

where c is a constant of integration. Solving this equation for y', and integrating again, we can express our two parameter family of extremals in the form

$$y = \int \frac{(1+cw)w + cu^2}{u\sqrt{[(1-cw)^2 - c^2u^2]}}\,dx + b\,,$$

b and c being the parameters. From our choice of axes, all members of this family must pass through the origin. This condition reduces the family of extremals to a one parameter family which can be written

$$y = \int_0^x \frac{(1-cw)w + cu^2}{u\sqrt{[(1-cw)^2 - c^2u^2]}}\,dx\,. \tag{2}$$

To insure a real integrand function, we must have

$$(1-cw)^2 - c^2u^2 = (1-cw+cu)(1-cw-cu) \geqq 0\,,$$

whence, if $w > u$, either

$$c \leqq 1/(w+u) < 1/(w-u) \qquad \text{or} \qquad c \geqq 1/(w-u) > 1/(w+u)\,.$$

If $w < u$, as would usually be the case, either

$$1/(w-u) \leqq c \leqq 1/(w+u) \qquad \text{or} \qquad c \leqq 1/(w-u)\,, \quad c \geqq 1/(w+u)\,.$$

This last is impossible, and hence, considering the range of the variable, w, we have for $w < u$,

$$1/(w_1 - u) \leqq c \leqq 1/(w_0 + u)\,. \tag{3}$$

For values of c within these limits, the integrand function is real, and equation (2) represents a real curve.

The second partial derivative with respect to y' of the integrand function in (1) turns out to be $u^2/(u^2 - w^2 + u^2y'^2)^{3/2}$ which is greater than zero for all values of x, y, y'. Hence both the Legendre and Weierstrass conditions are satisfied. For the Jacobi condition, there must be no point on the extremal between O and A that is conjugate to O, or, in other words, the particular extremal must not touch the envelope of the family (2) between O and A. From (2) we obtain

$$\frac{\partial y}{\partial c} = \int_0^x \frac{u\,dx}{[(1-cw)^2 - c^2u^2]^{3/2}}$$

Since this integrand function is always positive, $\partial y/\partial c$ never vanishes, save at $x = 0$, there is no envelope, and hence no conjugate point. Furthermore, since $\partial y/\partial c$ is always positive, there can not be more than one member of the family (2) passing through any point, save at the origin. Hence there is a region of the plane simply covered by the family (2), and we have a direct proof of the existence of a field of extremals. Solving the equation,

$$a = \int_0^1 \frac{(1 - cw)w + cu^2}{u\sqrt{[(1 - cw)^2 - c^2u^2]}}dx,$$

for c, our solution must then be unique. Substituting this value of c in equation (2), we obtain the extremal curve which actually furnishes a strong relative minimum.

It is interesting to note that this field of extremals (2) may or may not cover the entire plane. The critical points occur when c is equal to one of its limits (3) for then the denominator in (2) vanishes. The field will have an upper bound if, and only if, the integral

$$\int_0^1 \frac{(1 - cw)w + cu^2}{u\sqrt{[(1 - cw)^2 - c^2u^2]}}dx$$

exists with $c = 1/(w_0 + u)$, and a lower bound provided this same integral exists with $c = 1/(w_1 - u)$. In the latter case the above integral takes the form

$$\int_0^1 \frac{(w_1 - w - u)w + u^2}{u\sqrt{(w - w_1 + 2u)}}\cdot\frac{dx}{\sqrt{(w - w_1)}}.$$

Hence a necessary and sufficient condition for the field to have a lower bound is that the integral $\int_0^1 dx/\sqrt{(w - w_1)}$ exist. In like manner, it follows that a necessary and sufficient condition for the field to have an upper bound is that the integral $\int_0^1 dx/\sqrt{(w_0 - w)}$ exist. Experimental measurements[1] give for the current function of a straight stream the empirical form $w = w_1 + (w_0 - w_1)\sqrt{(1 - x^2)}$. With this function, our field of extremals would have a lower bound, but no upper bound.

By making the substitution $\xi = (1 - cw)/cu$ in (2), we obtain the form

$$y = sgn(c)\left[\frac{1}{cu}\int_0^x \frac{\xi dx}{\sqrt{(\xi^2 - 1)}} - \int_0^x \sqrt{(\xi^2 - 1)}dx\right]. \tag{4}$$

For the current function, $w = 2 - x$, $dx = ud\xi$, and (4) is readily integrated into

$$y = sgn(c)\cdot\frac{u}{2}\left[\cosh^{-1}\xi + \left(\frac{2}{cu} - \xi\right)\sqrt{(\xi^2 - 1)}\right]_{\xi_0}^{\xi}.$$

From this, the curves in Fig. 2 were drawn for the values $u = 3$, and c between the limits $-\frac{1}{2} \leqq c \leqq \frac{1}{5}$. This gives the extremal curves only for positive values of x, the other half being drawn from symmetry. Other interesting sets of curves can be obtained by taking u so that $w_0 > u > w_1$.

This problem suggests at once the extension to the case of a curved stream in which both the magnitude and direction of the current would be a function of the two variables x and y. Here we could also consider the problem with a variable end point, viz., starting from a fixed point on one bank, to determine

[1] M. Merriman, *Treatise on Hydraulics*, 10th edition, p. 321.

the course of the boat so that the opposite bank could be reached in minimum time, with no specified landing point. Going into three dimensions, we could consider the course of minimum time for an airplane between two points. The Euler equations for the extremal curves would be more difficult to handle in general, but might be simplified for certain special current functions. The vector calculus should prove useful in this connection.

SOME CURIOUS GEOMETRIC PROPERTIES CONNECTED WITH POLYGENIC FUNCTIONS

By FREDERIC H. MILLER, Columbia University

In this note we consider some geometric relations among certain direction angles in the z-, γ- and σ-planes belonging to polygenic functions[1] and their first and second derivatives. By a polygenic function (in contrast with a monogenic function) of the complex variable $z=x+iy$ is meant any function of the form

$$w = \phi(x,y) + i\psi(x,y) \tag{1}$$

for which the Cauchy-Riemann equations are not obeyed. Here we assume that for some region the components ϕ and ψ of w and their first and second derivatives are continuous real functions of the real variables x and y. The derivative dw/dz then depends not only on the value of z but also on the slope $y'=dy/dx$; thus, we have:[2]

$$\frac{dw}{dz} = \gamma = \alpha + i\beta = \frac{w_x + w_y y'}{1 + iy'}\,. \tag{2}$$

The quantity γ therefore has infinitely many complex values at a point. Differentiation of (2) leads to the second total derivative of w with respect to z:[3]

$$\frac{d^2w}{dz^2} = \sigma = \xi + i\eta = \frac{w_{xx} + 2w_{xy}y' + w_{yy}y'^2}{(1 + iy')^2} + \frac{w_y - iw_x}{(1 + iy')^3}y''\,. \tag{3}$$

Evidently σ is a function of z and of the differential element (y', y'') of the second order along which the point is approached; that is, it depends on both the direction of approach and the curvature, as well as on the point in the

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[1] So named and extensively studied by Edward Kasner in a series of papers, among which the following are hereinafter referred to by number: I. *A new theory of polygenic (or non-monogenic) functions*, Science, vol. 66 (1927), pp. 581–582; II. *General theory of polygenic or non-monogenic functions.* The derivative congruence of circles, Proceedings of the National Academy of Science, vol. 14 (1928), pp. 75–82; III. *The second derivative of a polygenic function*, Transactions of the American Mathematical Society, vol. 30 (1928), pp. 803–818.

[2] II, p. 77.

[3] III, p. 804.

z-plane. Hence d^2w/dz^2 takes on ∞^2 values for each value of z, the derivatives y' and y'' varying simultaneously. But since the expression for σ is an integral linear function of y'', it is seen that for a fixed value of y' the locus in the σ-plane will be a straight line.

If we replace the partial derivatives of w in equations (2) and (3) by their values from (1) and separate into real and imaginary components, we obtain equations for α and β involving y' and equations for ξ and η containing both y' and y''. Elimination of y' from the former pair of equations leads to the equation of the derivative circle,[4]

$$(\alpha - H)^2 + (\beta - K)^2 = h^2 + k^2 = R^2, \tag{4}$$

the locus of points γ corresponding to a given point z. Here

$$\begin{aligned} 2H &= \phi_x + \psi_y, & 2K &= -\phi_y + \psi_x, \\ 2h &= \phi_x - \psi_y, & 2k &= \phi_y + \psi_x. \end{aligned} \tag{5}$$

Letting θ_z denote the direction angle of the element in the plane of $z=x+iy$, so that $\tan\theta_z = y' = m$, it is easily found that the inclination θ_γ of the vector from the center (H, K) of the derivative circle to a point on it is given by

$$\tan\theta_\gamma = \frac{\beta - K}{\alpha - H} = \frac{km^2 + 2hm - k}{hm^2 - 2km - h}. \tag{6}$$

Likewise, elimination of y'' from the equations for ξ and η secures us the equation of the σ-line,[5] whose slope is

$$\tan\theta_\sigma = \frac{km^3 + 3hm^2 - 3km - h}{hm^3 - 3km^2 - 3hm + k}. \tag{7}$$

Now it may be readily verified that a simple relation connects these three direction angles; we have, in fact,

$$\tan(\theta_z - \theta_\gamma) = \cot\theta_\sigma. \tag{8}$$

Assigning merely direction but not a sense to the three elements in their respective planes, we may restrict the direction angles to values $0° \leqq \theta < 180°$. With this understanding, it is clear that

$$\theta_z + \theta_\sigma = \theta_\gamma + (2n+1)90°, \tag{9}$$

where n must have one of the values -1, 0, 1. It is of particular interest to note that this interdependence among the three angles is not affected by the function w or the values assumed by z.

We proceed to the implications of equation (9). For better comparison of

[4] I, p. 581.

[5] III, p. 812.

angular relations, let us suppose the three planes superimposed; then the following geometric properties are evinced:

(a) If the z-element is horizontal, the γ-radius will be perpendicular to the σ-line. The γ-vector then assumes the position of the phase vector[6] $h+ik$; thus, if we denote by tan θ_c the ratio k/h (the slope of the phase vector), we now have for $\theta_z=0°$, $\theta_\gamma=\theta_c$ and $\theta_\sigma=\theta_c+(2n+1)90°$.

(b) If the z-element is in a vertical position, equation (9) tells us that the γ-radius and σ-line are parallel. Since the vector in the γ-plane moves through twice the angle traversed by the z-element and in the opposite direction,[7] we have $\theta_\gamma=\theta_\sigma=\theta_c$ when $\theta_z=90°$.

(c) If the element in the z-plane is parallel to the radius in the γ-plane, the σ-line is vertical.

(d) If the z-element and the γ-radius are at right angles, the σ-line is horizontal.

(e) Suppose now that θ_z is increased from any particular value by an amount δ. Then θ_γ will decrease by 2δ, and equation (9) shows[8] immediately that θ_σ must be lessened by 3δ. Thus the three inclinations change in the ratio $1:-2:-3$.

(f) Due to this ratio of rotation, the z-element and σ-line will be parallel when $\theta_z=\theta_\sigma=[\theta_c+(2m+1)90°]/4$, where m is some integer. Combining this with (9), we find

$$(2m - 4n - 1)90° - \theta_\gamma = \theta_\gamma - \theta_c\,; \tag{10}$$

hence, when the z-element and σ-line are parallel, the γ-radius will bisect one of the angles between the phase vector $h+ik$ and the vertical.

(g) By rotating the z-element an odd-numbered multiple of 22.5° from the position of (f), it will become perpendicular to the σ-line. Then, for p some integer, we get from (9) and (10), $\theta_z-\theta_\sigma=(2p+1)90°$ and

$$(m - 2n - p - 1)180° - \theta_\gamma = \theta_\gamma - \theta_c, \tag{11}$$

whence we see that the γ-radius now bisects one of the angles between the phase vector and the horizontal.

(h) When the γ-vector is in a horizontal position, the z-element and σ-line are symmetrically situated with respect to the 45° line.

(i) Finally, if the γ-radius is vertical, the element in the z-plane and the σ-line are symmetrically situated with respect to the vertical.

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[6] I, p. 582.

[7] II, p. 80.

[8] For this result, otherwise obtained, see also III, p. 812.

DOUBLY HOMOGENEOUS FUNCTIONAL EQUATIONS[1]

By JOSEPH D. GRANT, James Millikin University

An algebraic addition theorem[2] is defined as a polynomial relation between $F(x)$, $F(y)$, and $F(x+y)$. This may be generalized to a polynomial relation between the functions of various linear homogeneous arguments which we shall call a homogeneous functional equation. The purpose of this paper is to show that, if such equations have constant coefficients and may be normalized, the solution may be made to depend upon the solution of an ordinary differential equation.

Theorems III and IV of the first paper hold and Theorem II becomes: *A solution $F(x)$ implies a solution $F(mx)$.* Theorem I takes the following form for this generalized situation:

Theorem V. *A necessary condition that a homogeneous functional equation have a solution which is analytic at zero*

(a) *except for a pole of order $v>0$ is that the terms of highest degree form an equation having the solution $A_{-v}x^{-v}$.*

(b) *and has there a zero of order $v>0$ is that the terms of lowest degree form an equation having the solution $A_v x^v$.*

(c) *and has there the finite value $k\neq 0$ is that k be a root in the equation in $F(0)$ formed by making all the variables zero, and that the linear terms resulting from the substitution $F(x)=k+G(x)$ form an equation having a solution $A_v x^v$, $v>0$.*

In part (c), k is a solution in any case, but will be trivial if it stands alone without a second term in the series. The test for a second term given in part (c) applies equally well to parts (a) and (b) except that it leads to the consideration of equations whose coefficients are not constants.

As examples we may consider, in order, the usual equations for cotangent, tangent, and cosine

$$V(\text{a}): \qquad F(x)F(y) - F(x)F(x+y) - F(y)F(x+y) = 1$$

for which

$$F(x)F(y) - F(x)F(x+y) - F(y)F(x+y) = 0$$

has the solution $A_{-1}x^{-1}$.

$$V(\text{b}): \qquad F(x+y) - F(x) - F(y) = F(x)F(y)F(x+y)$$

for which

$$F(x+y) - F(x) - F(y) = 0$$

has the solution $A_1 x$.

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[1] This is the author's second paper with this title. The first one was published in this Monthly, vol. 36 (1929), pp. 267–273.

[2] Forsyth's *Theory of Functions* (1918), Chapter 13; also Hancock's *Theory of Elliptic Functions* (1910), chapter 2.

$$V(\text{c}): \qquad F(x+y)+F(x-y) = 2F(x)F(y)$$

for which

$$x = y = 0, \quad F(0) = k \neq 0, \quad 2(k^2-k) = 0, \quad k = 1$$

and

$$G(x+y)+G(x-y) - 2G(x) - 2G(y) = 0$$

has the solution A_2x^2.

As for Theorem I, the proof follows from a consideration of the identities which arise upon the substitution of a power series with undetermined coefficients.

That the homogeneous equations define the same range of functions as those which are doubly homogeneous is given by

Theorem VI. *For every homogeneous functional equation there exists a doubly homogeneous equation which includes among its solutions all those of the homogeneous equation.*

To prove this it will suffice to show an example of the method of formation of the doubly homogeneous equation and to note that the method is general. The following equation is satisfied by the Weierstrass "P" function:[3]

$$[F(x+y)+F(x-y)][F(x)-F(y)]^2 - 2F(x)F(y)[F(x)+F(y)] + \tfrac{1}{2}g_2[F(x)+F(y)] + g_3 = 0.$$

From this equation form two more having the same solutions by replacing x by $x+y$ and by $x-y$. The condition that these three equations be consistent in $\frac{1}{2}g_2$ and g_3 is the vanishing of the following determinant:

$$\begin{vmatrix} [F(x)+F(x-2y)][F(x-y)-F(y)]^2 - 2F(x-y)F(y)[F(x-y)+F(y)], & F(x-y)+F(y), & 1 \\ [F(x+y)+F(x-y)][F(x)-F(y)]^2 - 2F(x)F(y)[F(x)+F(y)], & F(x)+F(y), & 1 \\ [F(x+2y)+F(x)][F(x+y)-F(y)]^2 - 2F(x+y)F(y)[F(x+y)+F(y)], & F(x+y)+F(y), & 1 \end{vmatrix}$$

The result is clearly doubly homogeneous and is satisfied by any solution of the original. In general, one groups together terms of like degree, writes by repeated substitutions as many equations as there are different degrees for the terms, and eliminates the coefficients of the groups of terms to obtain the required doubly homogeneous equation.

If some variable, say x, enters each of the arguments with the same coefficient which may be taken as unity the equation will be said to be in normal form. It is possible to normalize some equations as is shown by the following example: $F(x+5y)F(2x)-F(x-3y)F(2x+4y)=0$ has solution e^{nx^2}.

Form a second equation by replacing x by $x+2y$ and a third by replacing y by $2y$. The condition that these three equations be consistent in $F(2x)$, $F(2x+4y)$, and $-F(2x+8y)$ is

[3] Whittaker and Watson's *Modern Analysis*, 3rd edition, p. 442.

$$\begin{vmatrix} F(x+5y), & -F(x-3y), & 0 \\ 0, & F(x+7y), & F(x-y) \\ F(x+10y), & 0, & F(x-6y) \end{vmatrix} = 0.$$

The resulting equation is clearly in normal form and has the solution $ke^{mx}e^{nx^2}$. In general if the function of arguments requiring elimination enter linearly the above process will suffice. The remarkable simplicity of normal form equations is accounted for, in part at least, by the following theorem.

Theorem VII. *For normal form equations, a solution $F(x)$ implies a solution $F(x+a)$.*

To prove this one may replace x by $x+a$ and then $F(z+a)$ by $G(z)$. $G(x)$ is seen to satisfy the same equation as $F(x)$, which proves the theorem.

Several interesting corollaries appear when this theorem is considered in connection with theorem one which stated that the first term of an ascending power series representing a solution must be a monomial solution. If $F(x+a)$ be expanded in Taylor's series in powers of x, the first term will be $F(a)$. Since $F(a)$ will not in general be zero or infinite, an equation having a solution will have an arbitrary constant solution. If $F(x)$ has a zero or pole of order v, in the finite plane there will be some value of "a" which will place it at zero so that the first term of the power series, and hence a monomial solution, will be x^v or x^{-v}. Since a doubly periodic function must have poles in the finite plane,[4] a necessary condition for the existence of doubly periodic solutions is that there be a solution $x^{-v}(v>0)$. One also notes that complementary trigonometric functions will appear as solutions of the same equations.

It will be convenient to give the method of solution in detail for the general second degree equation in normal form in two variables which is

$$\sum_{i=1}^{n} c_i F(x+b_i y)F(x+d_i y) = 0. \tag{1}$$

The solution is effected by means of related differential equations. Expand each function in equation (1) by Taylor's series in powers of y and arrange the result in ascending powers of y to obtain

$$\sum_{k=0}^{\infty} \phi_k y^k = 0 \tag{2}$$

wherein

$$\phi_0 = (0,0)Z^2,\ \phi_1 = (0,1)ZZ',\ \phi_k = \sum_{j=0}^{t}(j,i-j)\frac{Z^{(j)}Z^{(i-j)}}{j!(i-j)!}, \tag{3}$$

$$(p,q) = \sum_{i=1}^{n} c_i(b_i^p b_i^q + b_i^q d_i^p),\ (p,p) = \sum_{i=1}^{n} c_i b_i^p d_i^p. \tag{4}$$

$Z^{(0)} = F(x)$, t is $\frac{1}{2}k+1$ if k is even and $\frac{1}{2}(k+1)$ if k is odd.

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[4] Whittaker and Watson's *Modern Analysis*, 3rd edition, p. 432.

THEOREM VIII. *A necessary and sufficient condition that $F(x)$ satisfy equation* (1) *is that it satisfy each of the differential equations $\phi_k=0$.*

The proof of this follows from the equivalence of the series (2) and the equation (1) and the fact that both must be satisfied identically in y by a solution.

To obtain the solution of the functional equation it is not (of course) necessary to solve the infinite set of equations $\phi_k=0$. In any particular case one finds $\phi_k\equiv 0$, $k=0, 1, \cdots, m-1$ and $\phi_m\not\equiv 0$, that is, some coefficient in ϕ_m is not zero. Differential equations of this type have been shown always to possess solutions analytic at zero which involve as many arbitrary parameters as the order of the highest derivative which appears in the equation.[5] The solutions of $\phi_m=0$ must then be substituted in the functional equation as they will not in general satisfy all the equations $\phi_k=0$.

The essential properties of normal form equations, such as solutions and form, are invariant under the transformation

$$x = x' + \alpha y' \; ; \; y = \beta y'.$$

Before carrying out the computations for any particular example, it is desirable to have the equation in as simple a form as possible.

THEOREM IX. *For every normal form equation, there exists one of the same degree having the same ϕ_m which is invariant under the transformation $y=-y'$.*

To prove this change y to $-y$ in equation (1) and add the result to equation (1) if m is even or subtract if m is odd. The resulting equation will then satisfy the conditions of the theorem.

The normal form equations in two variables thus separate into two types which may be designated as even and odd from their analogy to the equations $F(y)-F(-y)=0$ and $F(y)+F(-y)=0$, respectively. The general even equation of second degree is

$$\sum_{i=1}^{n} c_i\{F(x+b_iy)F(x+d_iy) - F(x-b_iy)F(x-d_iy)\} = 0.$$

$$(p,q) = \sum_{i=1}^{n} c_i\{b_i^p d_i^q + b_i^q d_i^p - (-b_i)^p(-d_i)^q - (-b_i)^q(-d_i)^p\}, \quad (p,p) \equiv 0.$$

One notices that $\phi_{2r}\equiv 0$ for all r's.

Similarly for the general odd equation of second degree,

$$\sum_{i=1}^{n} c_i\{F(x+b_iy)F(x+d_iy) + F(x-b_iy)F(x-d_iy)\} = 0.$$

One notices that $\phi_{2r+1}\equiv 0$ for all r's. As a special subclass of the odd equations one has those in which every term is invariant.

To illustrate the method we shall solve the following even equation:

[5] Forsyth's *Differential Equations* (1900), vol. 2, p. 26.

$$
\begin{aligned}
&F(x-y)F^2(x)F(x+y)F(x+2y)-F(x-2y)F(x-y)F^2(x)F(x+y)\\
&\quad+F(x-2y)F^3(x-y)F(x+y)-F(x-y)F^3(x+y)F(x+2y)\\
&\quad+F(x-2y)F(x)F^2(x+y)F(x+2y)-F(x-2y)F^2(x-y)F(x)F(x+2y)\\
&\quad+F^3(x-y)F(x)F(x+y)-F(x-y)F(x)F^3(x+y)\\
&\quad+F^2(x)F^2(x+y)F(x+2y)-F(x-2y)F^2(x-y)F^2(x)\\
&\quad+F(x+2y)F^2(x-y)F^2(x+y)-F^2(x-y)F^2(x+y)F(x+2y)=0.
\end{aligned}\tag{5}
$$

One sees that a solution $F(x)$ implies a solution $[F(x)]^{-1}$. For, if one replaces $F(x)$ by $1/G(x)$ and multiplies throughout by $G(x-2y)G^3(x-y)G^2(x)G^3(x+y)G(x+2y)$, $G(x)$ is seen to satisfy the same equation as $F(x)$.

$$\phi_m = Z(Z^2Z'Z^{(4)} - Z^2Z''Z''' - 3ZZ'^2Z''' + 3Z'^3Z'') = 0.$$

By three quadratures this is reduced to $Z'^2=B_0+B_2Z^2+B_4Z^4$ whose general solution[6] is $c\cdot sn(ax+b, k)$. There are also a number of degenerate solutions, some of which are

$c\cdot\sin(ax+b)$, $B_4=0$; $c\cdot\tan(ax+b)$, $B_2^2=4B_0B_4$; $c\cdot\csc(ax+b)$, $B_0=0$;

$c\cdot e^{ax}$, $B_0=B_4=0$; $(ax+b)$, $B_2=B_4=0$; $(ax+b)^{-1}$, $B_0=B_2=0$.

To complete the solution we must show that $sn(x, k)$ is a solution of equation (2). Pierce's integral tables give, as number 725,

$$sn\,(x+y)\,sn\,(x-y)\{1-k^2\,sn^2\,x\,sn^2\,y\} = sn^2\,x - sn^2\,y.$$

From this form two equations by replacing x by $x+y$ and by $x-y$. The quantities k^2sn^2y and sn^2y enter linearly into the three equations and their elimination results in equation (5).

As an example of an odd equation we shall solve

$$
\begin{aligned}
&F^2(x+y)F(x-2y)+F^2(x-y)F(x+2y)\\
&\qquad+F^3(x)-F(x+2y)F(x)F(x-2y)-2F(x+y)F(x)F(x-y)=0.
\end{aligned}
$$

A solution $F(x)$ implies a solution $e^{nx}F(x)$, by Theorem III:

$$\phi_m = ZZ''Z^{(4)} - Z'^2Z^{(4)} - ZZ'''^2 + 2Z'Z''Z''' - Z''^3 = 0.$$

Two quadratures will put this in the form

$$Z'' - 2nZ' + (n^2+a^2)Z = 0,$$

whose general solution is $c\cdot e^{nx}\sin(ax+b)$. One easily checks the solution $\sin x$.

However, it is not always true that every solution of $\phi_m=0$ is a solution of the functional equation. To see this we may consider the following odd equation which is symmetric term by term:

---

[6] $sn(x+b, k)$ is defined as the solution of $z'^2=(1-z^2)(1-k^2z^2)$.

$$F(x+2y)F(x-2y)-4F(x+y)F(x-y)+3F^2(x)=0. \tag{6}$$

$$\phi_m = ZZ^{(4)} - 4Z'Z''' + 3Z''^2 = 0,$$

which may be written

$$Z^2\left\{\frac{d^4}{dx^4}(\log Z) + 6\left[\frac{d^2}{dx^2}(\log Z)\right]^2\right\} = 0,$$

whose general solution is seen to be $c\cdot e^{nx}\sigma(x+b, 0, g_3)$. To check the solution $\sigma(x, 0, g_3)$, one would need to know properties in addition to its differential equation. If it is not a solution, it may be easier to find one of the $\phi_k=0$ which it does not satisfy. In this case $\sigma(x, 0, g_3)$ is not a solution of

$$\phi_6 = ZZ^{(6)} - 6Z'Z^{(5)} + 15Z''Z^{(4)} - 10Z'''^2 = 0.$$

The general solution of (6) will then be seen to be the degenerate solution of $\phi_m=0$, namely $e^{nx}(ax+b)$.

It will be of interest as a further illustration to determine all the odd second degree normal form equations in three arguments which can exist. The general equation is

$$\begin{aligned} &c_0F^2(x)+c_1F(x+y)F(x-y)+c_2[F^2(x+y)+F^2(x-y)]\\ &\qquad +c_3F(x)[F(x+y)+F(x-y)]=0. \end{aligned} \tag{7}$$

A necessary condition that there be any solution analytic at zero is $(0, 0) = c_0+c_1+2c_2+2c_3=0$. The other coefficients one needs to consider are

$$\begin{array}{lll} (0,2) = 2c_1+4c_2+2c_3\,; & (1,3) = 2(1,1)\,; & (1,5) = 2(1,1)\,; \\ (1,1) = -c_1+2c_2\,; & (2,2) = c_1+2c_2\,; & (2,4) = 2(2,2)\,; \\ (0,4) = (0,2)\,; & (0,6) = (0,2)\,; & (3,3) = (1,1). \end{array}$$

If the equation (7) have factors rational in F, the problem is not one of second degree equations and may be omitted from consideration. The condition for such factors is

$$\Delta = \begin{vmatrix} 2c_2 & c_1 & c_3 \\ c_1 & 2c_2 & c_3 \\ c_3 & c_3 & 2c_0 \end{vmatrix} = 0.$$

Substituting the value of c_0 from $(0, 0)=0$, one finds

$$\Delta = (c_1-2c_2)(c_1+2c_2+c_3)^2 = 0.$$
$$\Delta = (1,1)(0,2)^2 = 0.$$

From this one sees that the necessary and sufficient condition that the equation (7) shall not have factors rational in F is that neither (0, 2) nor (1, 1) be zero.

Let us write, then, $(0, 2)=a\neq 0$; $2(1, 1)=b\neq 0$; $6(2, 2)=n$.
The differential equations having the coefficients are

$$
\begin{aligned}
&\phi_2 = aZZ'' + bZ'^2 = 0,\\
(8)\qquad &\phi_4 = aZZ^{(4)} + 4bZ'Z''' + nZ''^2 = 0,\\
&\phi_6 = aZZ^{(6)} + 4bZ'Z^{(5)} + 5nZ''Z^{(4)} + 10bZ'''^2 = 0.
\end{aligned}
$$

Since any solution of equation (7) is a solution of each of the equations (8), it is necessary that the equations (8) be consistent.

The consistency condition for $\phi_2=0$, $\phi_2'=0$, $\phi_2''=0$, $\phi_4=0$ is

$$
(9)\qquad nb = (2a - b)(a + 2b).
$$

The consistency condition for $\phi_2=0$, $\phi_2'=0$, $\phi_2''=0$, $\phi_2'''=0$, $\phi_2^{(4)}=0$, $\phi_6=0$, after writing for n its value from (9), is

$$
(a - b)(a + b)(a + 2b)(2a + b) = 0.
$$

Hence there can not be more than four odd second degree normal form equations in three arguments which do not have rational factors in F. The four equations exist and are unique. The following list then comprises all possibilities.

I. $a - b = 0$, $\qquad ZZ'' + Z'^2 = 0$, $\qquad Z = (px + q)^{1/2}$.

$$
F^2(x + y) + F^2(x - y) - 2F^2(x) = 0.
$$

II. $a + b = 0$, $\qquad ZZ'' - Z'^2 = 0$, $\qquad Z = qe^{px}$.

$$
F(x + y)F(x - y) - F^2(x) = 0.
$$

III. $2a + b = 0$, $\qquad ZZ'' - 2Z'^2 = 0$, $\qquad Z = (px + q)^{-1}$.

$$
2F(x + y)F(x - y) - F(x)F(x + y) - F(x)F(x - y) = 0.
$$

IV. $a + 2b = 0$, $\qquad 2ZZ'' - Z'^2 = 0$, $\qquad Z = (px + q)^2$.

$$
\begin{aligned}
F^2(x + y) + F^2(x - y) - 2F(x + y)F(x - y)&\\
- 16F^2(x) + 8F(x)F(x - y) + 8F(x)F(x + y) &= 0.
\end{aligned}
$$

The problem of exponential multipliers for solutions of normal form equations is easily handled. Before stating the theorem it will be desirable to show the existence of an equation for which a solution $F(x)$ implies a solution $e^{nx^v}F(x)$ for each positive integer v. For equations in one variable this is seen from $F(2x) = F^{2^v}(x)$. The two term normal form equation

$$
\prod_{i=1}^{r} F(x + a_i y) = \prod_{i=1}^{r} F(x + b_i y)
$$

will have this property if[7] $a_i \overset{v}{=} b_i$. A few sets of such numbers will be of interest

$$-1, +1 \overset{1}{=} 0,0,$$
$$-1, -1, +2 \overset{2}{=} +1, +1, -2$$
$$-3, -4, +3, +4 \overset{3}{=} -5, +5,0,0,$$
$$-1, -5, -9, +5, +8 \overset{4}{=} -7, -8,1,5,9,$$
$$-5, -6, -11, +5, +6, +11 \overset{5}{=} -1, -9, -10, +1, +9, +10$$

That sets will exist for all positive integers v is shown by a theorem due to E. B. Escott.[8]

$$a_i \overset{v}{=} b_i,\ k \neq 0 \text{ implies } a_i,\ b_i + k \overset{v+1}{=\!=} a_i + k,\ b_i.$$

For the two term equation

$$\phi_m = Z^r \frac{d^{v+1}}{dx^{v+1}} (\log Z) = 0.$$

Theorem X: *The necessary and sufficient condition that for the equation*

$$\sum_{j=1}^{n} c_j \prod_{i=1}^{r} F(x + b_{ij} y) = 0$$

a solution $F(x)$ *imply a solution* $e^{x^v} F(x)$ *is that* $\sum_{i=1}^{r} b_{ij}^{w} = M_w$ *(independent of* j*) for each* $w \leqq v$.

The proof follows by direct substitution.

Two corollaries follow at once. Since the conditions are symmetric functions of the b_{ij}, $v+1 \leqq r$. Also a multiplier e^{x^v} implies a multiplier $e^{x^{v-1}}$.

The following equation for the sigma function will further illustrate the theorem

$$\begin{aligned} &F(x + 3y)F(x + y)F^4(x - y) + F(x + 2y)F(x + y)F^3(x)F(x - 3y) \\ &\quad + F(x + 2y)F^2(x + y)F(x)F^2(x - 2y) - F(x - 3x)F(x - y)F^4(x + y) \\ &\quad - F(x - 2y)F(x - y)F^3(x)F(x + 3y) - F(x - 2y)F^2(x - y)F(x)F^2(x + 2y) \\ &\quad = 0 . \end{aligned}$$

The sum of b_{ij} is zero for each term and the sum of b_{ij}^2 for each term is **14**. The solution is $ke^{nx}e^{mx^2}\sigma(x+b,\ g_2,\ g_3)$.

---

[7] L. E. Dickson's *History of the Theory of Numbers*, vol. 2, chapter 24. The symbol $\overset{v}{=}$ means that the sums of all positive powers of the two sets of integers are equal up to and including the power v.

[8] *Quarterly Journal of Pure and Applied Mathematics*, vol. 41 (1910), p. 144.

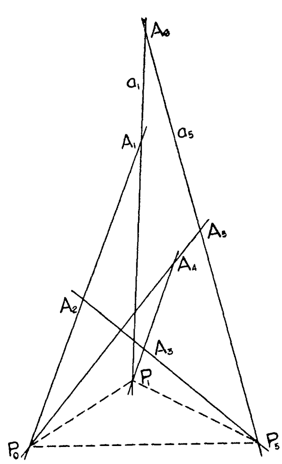
GENERALIZATIONS OF PASCAL'S AND BRIANCHON'S THEOREMS

By GLENN JAMES, University of California at Los Angeles

For convenience we first consider the converse of Pascal's theorem and state it in the following form: *If the intersections of opposite sides of a hexagon are always collinear any vertex describes the conic determined by the other five.* We shall consider five of the vertices, hence four of the sides, as fixed. One pair of opposite sides will then be fixed and their intersection will be the base of a pencil described by the line on the intersections of the other two pairs of opposite sides. The base of this pencil will, of course, be the center of perspectivity of the two ranges cut off by the pencil on the other two fixed sides. We shall speak of this pencil and any ranges on these fixed sides, whether or not they are perspective, as the Pascal pencil and the Pascal ranges. The varying vertex of the hexagon will be said to be opposite the line joining corresponding points on the Pascal ranges. The object of this paper is to determine the loci of the varying vertex when the intersections of opposite sides of the hexagon are not necessarily collinear but certain restrictions have been placed upon the triangle which they form. Obviously, the Pascal ranges may then be projective and not perspective and all six vertices of the hexagon may not lie on the same conic. The conic determined by the five fixed vertices and the triangle formed by the intersections of opposite sides of the hexagon will be termed respectively the base conic and the defining triangle.

Theorem I: *If the line on corresponding points of the Pascal ranges envelopes a conic tangent to the bases of the ranges, the opposite vertex of the hexagon describes a conic which has the two neighboring vertices in common with the base conic, and conversely.*

Proof: Denote the fixed vertices of the hexagon by A_1, A_2, A_3, A_4, A_5,



the varying vertex by A_6 and the Pascal ranges on the sides opposite A_1A_6 and A_5A_6, respectively by P_1 and P_5. Then the pencils A_1A_6 and A_5A_6 are projective whenever the ranges P_1 and P_2 are since the ranges are sections of the pencils.

This simple theorem includes the Pascal theorem and its converse; for when the ranges P_1 and P_2 are perspective A_3 is a self-corresponding point and $A_1A_2-A_4A_5$ is the center of perspectivity. The dual of this theorem bears a similar relation to Brianchon's theorem.

Theorem II: *If the determinant* $|X_1,\ Y_2,\ 1|$, *where* X_iY_i *are the cartesian coordinates of the intersections of opposite sides of a hexagon is equal to a constant then any vertex describes a conic which cuts its base conic in four real points, two on each of the moving sides when these are parallel to their opposite sides, and conversely.*

Proof: Since the coordinates of the moving point enter linearly[1] in the coefficients of the equations of the moving sides, the equation of the locus of the moving point is of the form

$$\Delta = KD_1D_2D_3, \tag{1}$$

where K is some constant, $\Delta=0$ is the equation of the base conic, D_1 is the determinant of the equations of the pair of fixed opposite sides and D_2 and D_3 are respectively the same functions of the other two pairs of opposite sides. The determinants D_2 and D_3 are linear in the coordinates of the moving point. Since (1) is quadratic, its locus obviously passes through all points on the base conic whose coordinates make either D_2 or D_3 zero. When the moving vertex coincides with a neighboring vertex of the hexagon, one of the moving sides is indeterminate. Hence D_2 or D_3, as the case may be, is zero. Each of these determinants is also zero when the varying line, whose equation enters into it, is parallel to its opposite side. To prove the converse, it suffices to note that any fifth point on the given conic, other than the four given points, will determine a real value of K.

It is of interest to note that when the moving point is on an intersection of the two conics the defining triangle, with an area not necessarily zero, reduces to an infinite segment of a straight line. In other words the area has taken the form $0\cdot\infty$, which is defined to have the value chosen for the area of the triangle. In stating the above theorem, it has been assumed that the vertices of the defining triangle would be taken in a given order. If, however, we merely take the numerical value of the area of this triangle, (1) becomes a quartic whose locus is two conics, both of which are given by the following construction.

To construct points on the conic defined by (1), which we shall call a principal conic, we proceed as follows: Using the notation of Theorem I,

[1] Because of this relation, it is evident that one can impose many different conditions upon the defining triangle which will restrict A_6 to lie upon a conic. However, the one used in this theorem is typical and seems to be the most interesting.

draw through one of the fixed vertices, say A_5, an arbitrary line. Join the intersection of this line and A_2A_3 to the point $A_2A_3-A_4A_5$. Divide this line segment into twice the constant given for the area of the defining triangle. Then construct parallels to the line on the line segment at distances numerically equal to this quotient. The points where these parallels intersect A_3A_4, taken in turn with A_1 determine lines which intersect the arbitrary line through A_5 in two points, one on each of the principal conics defined by taking the vertices of the defining triangle in different orders.

It follows from equation (1) that the configuration of twelve conics (or twenty-four if we disregard algebraic sign) defined by Theorem 2 reduces to a single conic when the area of the defining triangle is zero. The dual of this theorem, that is the corresponding generalization of Brianchon's Theorem, is as follows:

Theorem 2. *If the determinant* $|A_1, B_2, 1|$, *where* $A_iX+B_iY+1=0$ *are the equations of the lines joining opposite vertices of a hexagon is equal to a constant, any side of the hexagon envelopes a conic which has four real tangents in common with its base conic, two on each of the moving points when these are ideal points, and conversely.*

QUESTIONS AND DISCUSSIONS

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSION

A Note on Horner's Method of Solving an Algebraic Equation

By C. W. Bruce, Wesleyan College, Macon, Ga.

In the determination of incommensurable roots of an algebraic equation by Horner's method after the roots have been isolated by some one of the various methods, the procedure as given in text books is as follows: Suppose one root lies between 2 and 3. First, the roots of the equation are diminished by the whole number 2. The resulting equation will have a root between 0 and 1. An approximate value of this root is found by neglecting the terms of second degree and higher, and setting the first degree term equal to the negative of the known term. The question arises: will this approximate value of the variable be an approximation to the desired root, or to some other root of the equation? There is no obvious reason why this should give one root rather than another. In answer to the above question let us consider the cubic $x^3+ax^2+bx+c=0$, which has incommensurable roots.

Case I. All the roots positive. Let the roots be r, r', and r'' where $r<r'<r''$ and r is not integral. Neglecting the first two terms of the equation we get

$$x = -c/b = rr'r''/(rr' + rr'' + r'r'').$$

If x is to be an approximation to r rather than to r' or r'', then must

$$\frac{rr'r''}{rr' + rr'' + r'r''} - r < r' - \frac{rr'r''}{rr' + rr'' + r'r''}.$$

Clearing of fractions and omitting terms which cancel gives $-r^2r' - rr'' < rr'^2 + r'^2r''$ which is evidently true, showing that when all the roots are positive the method gives an approximation to the smallest root.

Case II. One root negative, two positive. Let the roots be r, r', and r'', where $r < r' < r''$ and r' is not integral.

As in case I $x = rr'r''/(rr' + rr'' + r'r'')$. The numerator is here negative. The two terms rr' and rr'' in the denominator are negative. The sign of x will be positive if $|rr' + rr''| > r'r''$. This may or may not be the case. Hence the usual process may or may not lead to the desired result in this case.

Case III. Two roots negative, one positive. Let the roots be r, r', and r'' where $r < r' < r''$ and r'' is not integral.

Again $x = rr'r''/(rr' + rr'' + r'r'')$. The numerator here is positive. In the denominator rr' is positive, $r'r'' + rr''$ is negative. x will be positive if $rr' > |r'r'' + rr''|$.

If this inequality is not true the method again fails.

The method will always work for the smallest root provided the first transformation of the equation makes all the roots positive. The method may or may not work for the remaining roots. Since the numerical value of the root looked for is made small in the first transformation of the equation, the method works more often than it fails.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College of the City of the New York.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

The Pastures of Wonder. By Cassius Jackson Keyser. Columbia University Press, New York, 1929. xii+208 pages. \$2.75.

In this book the author has carried to the extreme the tendencies shown in his previous writings concerning mathematics, and has attached here also his views of science. He has departed widely from his view, once expressed, that science is "a sublimated form of play, the austere and lofty analogue of the kitten playing with the entangled skein or of the eaglet sporting with the mountain winds."

He discusses two "realms": that of mathematics, and that of science. That of mathematics, in brief, he asserts is nothing more than deductive logic. That of science he proposes to limit to the establishment of categorical pro-

positions. Mathematics is an edifice of hypotheticals, science an edifice of categoricals. With the first idea many will take issue. The proposal many will reject. That some mathematicians and some others have wished to limit mathematics to its purely logical aspect is of course true, but mathematicians generally refuse to agree that what is thus excluded is not mathematics. That the deductive reasonings from hypotheses in science do not constitute a legitimate part of science few will agree with. There are of course scientists who would limit science to accurate and ordered accounts of phenomena. But this limitation appeals to few. The author undertakes to meet certain challenges to his definitions, and of course disposes of them to his own satisfaction. The book is a natural outcome of the notions of the "logistic" development by Russell and others. The antidotes for this disease to be found in the writings of Poincaré and others need to be more widely spread. One of the most serious challenges to logistic is the simple question: What constructive mathematics has it ever produced? To confine mathematics to logistic is worse than to confine it to "the study of integers and what can be got out of them, and nothing else."

To quote an example which the author himself gives, the proposition that "the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides" is a categorical proposition, and is not therefore mathematical. This is, according to the author, the conclusion from a set of unstated "Euclidean" postulates, and a conclusion by itself is not a mathematical proposition. The mathematical proposition is the statement that this conclusion follows from the original postulates. On this basis all conclusions in mathematics would seem to be in science, not mathematics. The only thing left to mathematics is the barren statement that the conclusion has been "proved." The premises may be false, they may be mere empty symbols, and the conclusion may be nowhere applicable, but if the logic is correct (*and this is a big assumption*) then we have a mathematical statement. It is the old and absurd statement of Russell, that in mathematics we do not know what we are talking about nor if our conclusions are true. But one needs only to notice that we have struck here an endless sequence from which we cannot escape. For we should say above; *if the process of logic used in demonstrating Pythagoras' theorem is valid, then the conclusion is a consequence of the postulates.* Now this is itself a hypothetical proposition, and *its* conclusion is a *categorical proposition.* Thus the original statement that the theorem of Pythagoras follows from the postulates of Euclid is categorical and so not mathematical. This can be extended ad libitum. In fact, every hypothetical proposition is also a categorical proposition, and the distinction the author desires to make breaks down.

The statement of Benjamin Peirce, made years before Russell's work, that mathematics is the science that draws necessary conclusions, is the basis for these logistic arguments. But the other statement of Peirce is almost invariably omitted. It is that the processes of logic cannot be applied without being transmuted into various forms, and this transmutation is the mathematical process

in the inquiry. This is very important, for a large part of mathematical investigation does not consist in drawing conclusions, but in building structures of thinking. Most of mathematics consists in the development of ideal structures. The brilliant son of Benjamin Peirce, Charles Saunders Peirce, gave as the definition of mathematics, in the Century Dictionary, "the study of ideal constructions (often applicable to real problems), and the discovery thereby of relations between the parts of these constructions, before unknown." He adds, "The observations being upon objects of imagination merely, the discoveries of mathematics are susceptible of being rendered quite certain." Such observations are categorical assertions.

The author of the book under review makes much of the *forms* of propositions and elaborates this with various examples of syllogism. But he evidently fails to see that forms themselves are objects, and assertions about them are categorical assertions. It was Kempe who wished to define mathematics as the science of pure form, but he was quite clear in his vision of forms as ideal objects, in practically the same sense as C. S. Peirce. As the author says on page 99, mathematical verities are eternally true, but not because they are mere logic. Mere logic itself is an ideal creation of the human mind, and may be utterly changed. Witness the work of Brouwer on the law of excluded middle. They are true because the human mind has the ability to see the universal characters in its own creations.

As for the proposed definition of science, we may leave the scientist to accept or reject it. The discussion of the proposal with imaginary scientists, of course frames answers for the objections already set. The proposal seems to have as one aim to exclude mathematics from science. This has already been done if we accept the common modern view that science is based on the observation of the natural world. Also he tries to exclude the mathematical ideals of methodology and thought from science. This is being rapidly accomplished by modern physics in some directions, but is being equally rapidly shown to be impossible in others. The reviewer is referring to Bridgman's *Logic of Modern Physics*, and to all the present exposition of atoms as matrices. The author of course admits the necessity for the scientist to use mathematics, and probably would not deny the fact that many advances in mathematics have originated in scientific problems.

The title, "Pastures of Wonder," is ascribed in the preface to the existence of two great realms, the categorical and the possible (*equivalent to hypothetical?*) and the wonder these two create. But the two worlds are also called propositional, and one may well ask if the universe consists merely of propositions, for human experience would certainly deny this statement. And propositions later are defined to be only those statements which can be tagged as true or false. Are there no others of use? And the old, old question, which Russell gave up, still remains: What does true or false mean, and why choose one rather than the other?

The universe is indeed filled with pastures of wonder. One of these is the

pasture of beauty. One can defend the play of the mathematician as the loitering of an enchanted spirit in this pasture. And this pasture is not the rarefied ether, empty of all content. Mathematics would have died of inanition long ago were it that. For all creative mathematicians are chiefly concerned in making categorical assertions about the ideal world in which they live. This is what the "educated layman" to whom the author addresses his essay should understand as the meaning of the term mathematics. If the layman will pick up almost any book on mathematics he will find that the writer is chiefly concerned with the properties, the characters, the structure, the relations of the mathematical objects he is considering. And he should not be misled into thinking that the only feature of the book which is *mathematical* is the empty assertion that the conclusions follow from the premisses. He will find indeed that the greater part of the book is derived from direct intuition and not from logic at all. If he succeeds in seeing the structures discussed as the writer sees them he will be filled with wonder at their beauty.

JAMES BYRNIE SHAW

Elementary Differential Equations. By Thornton C. Fry. D. Van Nostrand Co., New York, 1929. x+255 pages.

A textbook on elementary differential equations consisting of 250 generously sized pages, written by one who is prominently connected with the mathematical research side of electrical engineering is sufficient to pique one's curiosity and justify the expectations that a book particularly suited to introducing engineers and engineering students to the desirability and fascination of differential equations has appeared. The book, as stated in the introduction, is actually based on out-of-hour courses of the Bell Telephone Laboratories. This fact is, of course, responsible for some of the idiosyncrasies of the book, more detailed explanation of some of the elementary processes, slighting of some topics, and overemphasis on others, especially linear differential equations with constant coefficients, and many fascinating side remarks, which enliven the text and the footnotes.

The amount of actual material in differential equations covered in the book is a little disappointing in its meagerness, when one considers the size of the book. Analysis shows that the book covers little more than one frequently finds in the closing chapters of a good first book on calculus: i.e., the standard methods for solving ordinary differential equations of the first order, a passing mention of the solvable cases of the second order, where dependent or independent variable are absent (treated as a parting shot in the chapter on differential equations of the first order), and linear differential equations with constant coefficients. The material not usually found in a book on calculus is a treatment of singular solutions of first order differential equations (why important for the engineer?) a rather extensive group of applications, the exposition of the symbolic method of solving linear differential equations with con-

avoid confusion, if entirely different letters were used for the two concepts. After all, what is the matter with D as a symbol for derivative, as contrasted with p?

The book should prove stimulating and helpful to the engineering student of differential equations, provided he is warned in advance that not all differential equations are linear with constant coefficients.

T. H. HILDEBRANDT

Seven Place Natural Trigonometric Functions, together with many miscellaneous tables and appendices on the adjustment of the engineer's transit and level, area computation, vertical curves, simple curves, and determination of latitude, longitude and azimuth. By Howard C. Ives. John Wiley and Sons, New York, 1929. vi+222 pages. $2.50.

The major part of this book consists of seven-place tables of the natural trigonometric functions. It is pointed out in the preface that the widespread adoption of calculating machines is having the effect of rendering logarithms to some extent obsolete, and necessitates the use of tables of natural functions.

Tables I–V give the six trigonometric functions, versed sines and exsecants, chords and co-chords, for angles differing by one minute. Table VI is the inevitable table of lengths of circular arcs with unit radius, that is to say a mere multiplication table for π; and table VII, transforming degrees to radians, merely repeats a part of VI. There is a table of powers, root, and reciprocals; and the remaining tables are chiefly of interest to the engineer.

An excellent syllabus of trigonometry, mensuration, and other topics needed by the engineer, together with a section on adjustments of instruments, should render this a most useful little volume.

To the mathematician the most obvious fault of the book is the lack of any suggestion for interpolation except by the usual method of first differences. In fact, in the explanation of the tables there are explicit directions for the use of this method. When we are dealing with seven places of decimals there is evidently danger that simple interpolation will not be accurate; the possible error is of course of the order of one-eighth of the second differences. It turns out that simple interpolation is accurate for all sines and cosines; and perhaps for all trigonometric functions less than unity. For tangents and secants of angles greater than 45°, however, correct results can be obtained only by using second differences, and directions for this process should accompany the tables.

R. A. J.

Plane Trigonometry (*With Tables*). By Ernest Jackson Oglesby and Hollis Raymond Cooley. Prentice-Hall, Inc., New York, 1929. xii+226 pages.

In the construction of this text the authors seem to have not only carefully considered the subject matter that should be included but also the best method of presenting and developing this subject matter that it might be attractive and interesting to the student as well as teachable for the teacher. One of the very

striking features of the book is the introduction of each new chapter with preliminary leading questions in order to put the student on the alert for the discussion which is to follow. The good effect of this scheme is enhanced by means of the occasional appearance of significant questions throughout the body of each chapter.

Additional features which the authors "think merit the special consideration of teachers of mathematics" are the introduction of "the trigonometric functions of the general angle before proceeding with the special application to the acute angle," the treatment of the general triangle, and the two last chapters of the book which are devoted to polar coordinates and complex numbers respectively. In the discussion of the general triangle the law of sines has been introduced and then the far-reaching application of this law and the need for other formulae are shown in the discussion of the four cases that arise in the solution of the general triangle. The addition formulae are introduced as a means by which the law of tangents may be derived, the purpose being to illustrate at least one practical application of these formulae.

The introduction to the use of logarithmic tables is very well developed and seems to present the necessary instruction in a carefully planned manner. The book also contains a rather complete and usable set of logarithmic and trigonometric tables.

Last, but by no means least, the printers have done an excellent job in the printing of this text. The type is such that the reading will not be a strain on one's eyes, the figures and tables are clear and well constructed, and a good grade of paper has been used.

In general, the text impresses one as being a book that deserves the careful examination of those teachers who desire texts that will not only contribute to the information of their students but also tend to develop their interest in mathematics as well as to cultivate an appreciation of mathematics.

F. L. WREN

Statistical Mechanics. By R. H. Fowler. Cambridge University Press, 1929. The Macmillan Company, New York. viii+570 pp. 35 shillings.

So long as a dynamical system has a reasonably small number of degrees of freedom, it is practical to study the evolution of the system by an examination or actual solution of the equations of motion. For dynamical systems of very many degrees of freedom, such as a gas, this detailed study is out of the question, and one utilizes the methods of statistical mechanics. If the number of degrees of freedom is, for example, n, a complexion of the system is completely specified by giving the location of a point in a so-called "phase space" of $2n$ dimensions, n of the variables corresponding to the positional coordinates $q_1 \cdots q_n$ of the dynamical system, while the remaining n variables correspond to the associated momental coordinates $p_1 \cdots p_n$. A single point in the phase space thus represents a system, and the trajectory of this single point gives

the successive configurations through which the system passes. It is, moreover, an essential part of the technique of statistical mechanics to consider not this representative point of a single system, but a whole dust of points in the phase space corresponding to a very large number or "ensemble" of systems, alike in their constitutions but differing in their phases.

Having assumed a certain distribution of such representative points in the phase space, one then seeks to determine what physical properties are enjoyed by an overwhelming majority of the dynamical systems which the various points represent. If a certain property is characteristic of all but an infinitesimal[1] fraction of the systems, one then concludes that this property will be possessed by one actual system during all but a negligibly small fraction of the time. There are two subtle difficulties here, neither of which has ever been satisfactorily analyzed; viz., there is no really satisfactory basis for the choice of density of the representative points in the phase space, and there is no satisfactory justification for the identification of the number average over the ensemble with the time average for one system.

From a mathematical point of view, the two matters just mentioned are among the most interesting aspects of statistical mechanics. The treatise under review is not concerned with these difficulties; although the author, in a clear and illuminating preliminary statement, calls attention to them. The volume is, in fact, not a critical study of the foundations of statistical mechanics, but rather a rich and varied development of the applications of statistical mechanics to problems of modern physics. Some idea of the scope and importance of these applications is gained from the following partial list of subjects treated, taken from the table of contents: assemblies of permanent systems; specific heats of simple gases; partition functions for temperature radiations and for crystals; dissociation and evaporation; the relationship of the equilibrium theory to classical thermodynamics; Nernst's heat theorem and the chemical constants; the theory of imperfect gases; interatomic forces; applications of the equilibrium theory to thermoionics; the dielectric, dia- and paramagnetic constants of gases; the properties of dilute solutions; atmospheric problems; applications to stellar interiors; chemical kinetics in gaseous systems; fluctuations. Any discussion here of the five hundred sixty pages of detailed analysis applied to these problems is both impossible and inappropriate. It must suffice to say that the problems treated are of first rate importance, and that the exposition is clear and accurate, although it naturally demands a considerable amount of physical and mathematical sophistication on the part of the reader. The treatise is a brilliant and monumental piece of work which is certain to rank, for many years, as one of the most useful, complete, and scholarly books ever written in this field.

Although the reviewer does not feel it wise to discuss at this place the more purely physical aspects of this book, there is, nevertheless, one such matter

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[1] Infinitesimal, in the accurate sense, as the number of dynamical systems becomes infinite.

of principal importance which should be at least mentioned. It is one of the distinguishing features of this treatise that the quantum aspects of the statistical problems are introduced *ab initio* so that classical statistics appears as a natural limiting case of the more general quantum statistics, rather than that quantum statistics appears as a bizarre departure from classical statistics.

It was mentioned above that Professor Fowler had been interested in this book in applying statistical mechanics, rather than studying statistical mechanics as such. In the process of application, however, he has made a great improvement in technique and has added materially to the elegance, logical accuracy, and ease with which the calculations may be carried out. This matter is of particular mathematical interest and deserves discussion here. To simplify the discussion, suppose that the type of dynamical system under consideration is a gas of given volume, etc., and that one is investigating the law for the distribution of velocities of the various molecules forming one of the systems. Given a distribution of representative points in the phase space, one calculated, according to the older procedure, the most probable distribution of velocities; meaning the distribution of velocities which would most probably be found to obtain in a gas sample picked at random from the ensemble of samples. One then argued that this most probable law for the distribution of velocities was not only the most probable but that it (or one differing negligibly from it) was characteristic of all but an infinitesimal fraction of the systems forming the ensemble. When these two points had both been argued, one called (following Jeans) the property in question a "normal" property.

Of the two steps just noted, the latter offers considerably more difficulty, and it has become not infrequent for authors to omit this portion of the argument entirely, partly because it is hard to make it accurate and partly because of the general optimistic belief, based on experience, that a most probable property is likely to be also a normal property. Professor Fowler, using the new procedure inaugurated by Darwin and himself, considers the normal properties of an ensemble to be the average properties rather than the properties of maximum frequency of occurrence; and in the calculation of these average properties, Darwin and Fowler have introduced a most clever and convenient method of calculation. The calculation of these averages, in fact, requires the calculation of complicated sums of factorials. In the older procedure, such factorials were approximated by the Stirling formula, and the disagreeable task of determining the degree of approximation in the final results was usually neglected. A typical, although simple, example of the sort of factorial sums which enter is

$$C = \sum_{a,b} \frac{M!}{a_0!a_1! \cdots} \frac{N!}{b_0!b_1! \cdots},$$

summed for positive or zero integral values of a and b, subject to

$$\sum_r a_r = M, \quad \sum_s b_s = N, \quad \sum_r r a_r \epsilon + \sum_s s b_s \eta = E,$$

where M, N, ϵ, η, and E are constants whose physical meaning need not concern us here. Darwin and Fowler noticed that C is equal to the coefficient of z^E in

$$(1 - z^\epsilon)^{-M}(1 - z^\eta)^{-N},$$

which coefficient, in turn, may be written as

$$\frac{1}{2\pi i}\int \frac{dz}{z^{E+1}} \frac{1}{(1 - z^\epsilon)^M(1 - z^\eta)^N},$$

the path of integration circulating once counter-clockwise around $z=0$. So far the calculation is exact; but in the actual physical cases one is interested in the asymptotic value of this integral as M, N, and E tend toward infinity in fixed ratios. This asymptotic value can be readily computed by the so-called method of steepest descent.[2] This method which first came into prominence when used by Debye in 1909 to obtain asymptotic expressions for certain cylinder functions, depends upon the fact that the integrand above has, along the real axis, a minimum at a point $z=\theta$ between $z=0$ and $z=1$; this same point being, however, the location of a sharp maximum as one traverses the circle, with center at the origin, which passes through $z=\theta$. If one uses this circle as the contour along which the above integral is to be extended, then the value of the integral arises, sensibly entirely, from the immediate neighborhood of $z=\theta$. The asymptotic value of the integral, and hence the original factorial sum, can thus be computed with great simplicity and with complete rigor. This elegant method is used with great skill and effectiveness throughout the volume.

WARREN WEAVER

Statistics. By William Vernon Lovitt and Henry F. Holtzclaw. Prentice-Hall, Inc., New York, 1929. 304 Pages. $4.00

Mathematics Preparatory to Statistics and Finance. By George N. Bauer. The Macmillan Co. New York, 1929. 337 Pages.

The Lovitt and Holtzclaw text is rather unique in that it could be used either in a first course in mathematical statistics or in a course in statistics in business schools. In the latter case, the author's claim that "some of the more difficult mathematical passages may be omitted without destroying the continuity of the course" would probably be accepted fairly often, while in the former case, much of the first six chapters would have to be served in concentrated doses.

The text includes the usual treatment of averages and dispersion and not only an unusually full treatment of the theory of correlation itself but also references to the various modifications of the Pearsonian coefficient.

[2] Called, in German, "die Sattelpunktmethode"; and, in French, "méthode du col."

Index numbers and the standard treatment of time series are given an unusual amount of attention. The text includes also appropriate parts of college algebra and simple curve fitting including the normal probability curve and its bearing upon the meaning and use of various types of probable errors. The book has been carefully prepared; only a few errors, of a trivial character, were noted.

The Bauer text is essentially a very acceptable elementary text covering both analytic geometry and college algebra but draws fairly consistently upon the general field of finance and statistics for its applications and exercises. While a little of the theory of finance and of statistics is included it is admittedly of a very elementary character.

Some topics of a more advanced character, such as least squares, are taken up in a very mechanical or rule-of-thumb way but even though the treatment is not complete in such cases this fact is frankly admitted and the work is evidently included mainly for the sake of a much desired drill and not as an essential part of the fundamental theory.

Logarithms are treated in great detail and, in general, the whole text is well adapted and seems to have been prepared much for those who are a little "slow" in their mathematics.

While the reviewer is not entirely in sympathy with the use of terms like "the straight line law," "the law of the parabola," "accumulated amount" (for the amount of an annuity), or with the definition of the coefficient of correlation which "is not the usual definition," etc., the purpose back of them is recognized and, for the most part, good and by the time a student finishes this text he will not only be well "prepared" for a regular course in finance or in statistics but also probably very much interested.

C. H. Forsyth

An Introduction to Mechanics. By J. W. Campbell. Houghton Mifflin Company, Boston, 1929. XIV+384 pages. $3.50.

This is an admirable text on the subject. Its appeal will probably be more direct to the young student of mathematics, although the young engineering student would profit immensely if the opportunity were offered to him to elect it after his first course in the calculus.

The author begins his preface thus: "The statement has been made and does not seem to have been challenged, that the elementary principles of mechanics are more poorly understood by students than the elementary principles of any other subject." Later he observes: "In my opinion a fundamental principle when presented should be in such form that, although a student will probably on a first study of it get only a few of its important features, he will when occasion later arises find that the treatment stands analysis. His appreciation of it will grow and he will have nothing to unlearn." The author does not lose sight of these ideas in the development of his text.

The calculus is freely and naturally used throughout. Topics recur instead of being "exhausted" upon their first presentation. Relative motion, for example, is introduced in the first chapter and further developed as late as chapter XIII. So with other topics. The catenary types receive especial attention and are presented in an unusually attractive manner. The author happily elected to solve the problem of the freely hanging pulley which is itself in motion, being attached to a fixed pulley—a problem illustrating so well the fundamental relation between force and motion. There are many excellent features—large, clear type, well drawn figures, more than six hundred well selected problems and, best of all, the delightful manner of the author's presentation of his subject.

At the end there are eight appendices: principles of computation, definite integrals, "force", Simpson's rule and other topics, closing with interesting remarks on the growth and conception of mechanics.

One cannot predict how a new book will work out with a class until he has used it at least once, but this book bids fair to succeed if one may judge from its content and manner of presentation.

G. H. HUNT

Numerical Tables of Hyperbolic and Other Functions. By J W. Campbell. Houghton Mifflin Company, Boston, 1929. 76 pages. $1.25.

The tables by Professor Campbell are fourteen in number and include hyperbolic functions (four decimal places and five significant figures), $(\sinh x)/x$, $(\cosh x-1)/x$, $(\cosh x)/x$, trigonometric functions for both radian and degree values of the angle, $\log_{10}x$, $\log_e x$, x^2, x^3, $1/x$, together with notes on the tables and a useful tabulation of catenary and transmission line formulae, also formulae for solving cubic equations, with remarks on each. These tables would naturally accompany the author's text on Mechanics, but are valuable in themselves. The type is good.

G. H. HUNT

Elementary Mechanics. By Joseph B. Reynolds. Prentice-Hall, Inc., New York, 1928. VIII+250 pages. $2.50.

In this text the leading topics of the elements of mechanics are developed without the aid of the calculus. The text, therefore, is especially adapted to classes of students who have not studied the calculus. It is, however, not a shop book or in any sense a handbook of rules. The mathematical principles are remarkably well presented. Algebra, trigonometry and geometry are freely used and, to replace the calculus, the principles of limits. The selection of problems and in general the discussion of the subject emphasizes the engineering side of mechanics.

The chapters: resolution and composition of forces, equilibrium of rigid

bodies, center of gravity, friction, work, kinematics, kinetics, energy, impact and simple harmonic motion, indicate the order of treatment.

The author says in his preface: "This text in elementary mechanics is intended particularly for students who desire a working knowledge of the easier applications of mechanics" . . . . . "An attempt has been made to avoid problems involving complicated arithmetic because of the data given." How many students have lost all sense of principle involved when in beginning a new subject problems were too frequently introduced which served merely to bewilder them in a maze of complicated numerical calculations.

This book is superior to most texts which have substituted the language of the theory of limits for the symbols of the calculus. In that the author has evidently conscientiously striven to present his theory rigorously the question arises as to whether it will prove successful with the classes for which it is intended. There is every reason to suppose that this will be the case.

G. H. Hunt

Plane Trigonometry. By J. B. Rosenbach and E. A. Whitman. John Wiley & Sons, New York, 1929. 216 pages. $2.00

The advertising circulars of this book emphasize that it is a thoroughly teachable book. The teachable and teaching qualities are those which have impressed the reader as being the most outstanding of its good features. It is easily seen that the text is the result of long classroom experience. Detailed proofs and explanations are given for theorems and illustrative examples and there are many diagrams with more than the usual number of significant details. The exercises, with answers at the back for the odd numbered ones, are numerous, varied, and well arranged. Thus, during the course of a chapter there are examples for drill and written work, and at the end of the chapter a set of general exercises.

Besides the topics usually found in a text on trigonometry, there are sections on the periodicity of trigonometric functions; various types of trigonometric and inverse trigonometric equations; drawing graphs by the composition of ordinates, and graphic solutions of equations involving trigonometric functions of one angle. There is a chapter on the theory and use of logarithms, but complete logarithmic tables are not included.

In the first chapter the trigonometric ratios are defined for the general angle. Later these definitions are restated, for the acute angle, in terms of the sides of a right triangle. Radian measure follows directly after the definition of the (general) angle. At the end of the book tables are given converting degrees, minutes, and seconds to radians, and inversely,—also a table of sines, cosines, and tangents for angles expressed in radians.

The only objection which has occurred to the reader is that important facts and theorems do not stand out with sufficient emphasis among the great number of illustrative examples and the variety of types of cases given. The

text would not be recommended for use in a general course which contains a unit of trigonometry, but it should prove most satisfactory for colleges and schools at which a full course in trigonometry is offered.

LAURA GUGGENBÜHL

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3409. *Proposed by Norman Anning, University of Michigan.*

When $3b^2 > 8ac$, the curve, $y = ax^4 + bx^3 + cx^2 + dx + e$, has two real inflectional tangents. Prove that they cut off equal areas.

3410. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Construct a triangle given the base, the difference of the base angles, and the bisector of the vertical angle.

3411. *Proposed by J. Rosenbaum, Milford, Conn.*

A particle within a closed plane curve is attracted by every point of the curve with a force proportional to the distance from the particle to the point. Find the position of equilibrium of the particle.

UNSOLVED PROBLEMS

2954 [1922, 81]. *Proposed by C. N. Mills.*

A machine-gun is placed on an armored train which is moving with a velocity v feet per second along a straight horizontal track. The muzzle velocity of the bullets is v feet per second. Find the greatest range, (1) in front and (2) behind the train.

3034 [1923, 276]. *Proposed by J. L. Riley, Stephenville, Texas.*

If every root of the equation $f'(x) = 0$ be subtracted from every root of the equation $f(x) = 0$, find the sum of the reciprocals of the differences.

3124 [1925, 138]. *Proposed by M. B. Porter, University of Texas.*

$f(x)$ and $\phi(x)$ are polynomials and all the zeros of $\phi(x)$ are real. Let $P(x) = \phi(x)[f^{(i)}(x)]^r$ where $f^{(i)}$ stands for the ith derivative of $f(x)$.

Prove that $[P'(0)]^2 < P''(0)P(0)$ (1) is a sufficient condition that $f(x)$ has imaginary zeros. Newton's test[1] $C_i^2 \leqq C_{i-1}C_{i+1}$ where C_i is the coefficient of x^i in $f(x)$ is a special case of (1).

SOLUTIONS

3045 [1923, 449]. *Proposed by S. A. Corey, Des Moines, Iowa.*

If in the equation

$$s!\left[\frac{1}{(s+1)!}+\frac{c_3}{4!(s-1)!}+\frac{c_5}{6!(s-3)!}+\frac{c_7}{8!(s-5)!}+\cdots+\frac{c_s}{(s+1)!2!}\right]=0 \quad (1)$$

$c_3, c_5, c_7, \cdots c_s$ be given and retain such constant values that (1) is satisfied for all positive odd integral values of s, $(s>1)$, prove that if s be decreased by unity (so that $s=2n$), then the left member will become equal to $\pm B_n$, according as n is odd or even, B_n being Bernoulli's nth number. Also show how any one of the constants c may be found without first finding all the preceding constants.

Solution by Otto Dunkel, Washington University.

The Bernoulli numbers, B_n, occur in the coefficients of a number of expansions of functions, and we shall use the following expansion for the derivation of the required relations

$$(1) \qquad y = \frac{x}{e^x-1} = 1 - \frac{1}{2}x + B_1\frac{x^2}{2!} + \cdots + (-1)^{i-1}B_i\frac{x^{2i}}{(2i)!} + \cdots .$$

Writing $y(e^x-1)=x$ and differentiating both sides m times, we obtain

$$(2) \qquad e^x[y + {}_mC_1y' + \cdots + {}_mC_{m-1}y^{(m-1)}] + (e^x-1)y^{(m)} = 0, \quad m \geqq 2,$$
$$= 1, \quad m = 1,$$

where $y^{(i)} = d^iy/dx^i$ and ${}_mC_i$ is the binomial coefficient. Setting $x=0$, $y_0=1$, $y_0' = -\frac{1}{2}$, $y_0^{(2i+1)}=0$, and $y_0^{(2i)}=(-1)^{i-1}B_i$, we shall make use of three equations obtained by setting $m=2n$, $2n+1$, $2n+2$. The first of these equations is

$$(3)\ 1 - \tfrac{1}{2}\,{}_{2n}C_1 + {}_{2n}C_2B_1 + \cdots + (-1)^{i-1}{}_{2n}C_{2i}B_i + \cdots + (-1)^{n-2}{}_{2n}C_{2n-2}B_{n-1} = 0,$$

while the second and third are similar but contain a final term in B_n. Multiply the first equation by $2n+1$, the second by $-2n$, and add the results; divide this result by $(2n+1)!$, and split the last term into two terms by writing the $2n$ in its coefficient as $(2n-1)+1$. There results:

[1] See Netto's *Vorlesungen über Algebra* (1896), vol. 1, p. 234, for an account of Sylvester's paper.

$$(4)\quad \frac{1}{(2n+1)!}-\frac{B_1}{(2n-1)!2!}+\frac{3B_2}{(2n-3)!4!}-\cdots$$
$$+(-1)^n\frac{(2n-1)B_n}{1!(2n)!}+(-1)^n\frac{B_n}{(2n)!}=0.$$

Now multiply the second equation by $2n+2$, the third by $-(2n+1)$, add the results, and divide the resulting equation by $(2n+2)!$. We shall thus obtain a second equation which may be written down from the above by omitting the last term and by increasing by unity the numbers in the first factorial factors of the denominators. By comparing these results with the equations of the problem we see that $c_{2i+1}=(-1)^i(2i-1)\ (2i+1)\ (2i+2)B_i$, and it follows that any method of calculating the B's gives by a very slight modification a method of calculating the c's, and conversely. One method is obviously given by solving a set of the linear equations in the B's or the c's by the use of determinants. Or we may proceed thus. Write (1) in the form $y=\log_e[1+(e^x-1)]/(e^x-1)$, and then develop in powers of (e^x-1). We have then

$$(5)\qquad y=\sum_{m=0}^{\infty}(-1)^m\frac{(e^x-1)^m}{m+1}.$$

We may now write the expansion of (e^x-1) in powers of x, obtain the mth power of the series, and collect the terms. The coefficient of x^{2n} will then be $(-1)^{n-1}B_n/(2n)!$. Or we may obtain a result which is easier to write by first expanding by the binomial theorem $(e^x-1)^m$ and then developing into series each term of the result, noting that the last term to be so treated is given by $m=2n$. The result may be written

$$B_n=(-1)^n\left[\frac{\Delta_{2n}}{2}-\frac{\overset{2}{\Delta}_{2n}}{3}+\cdots-\frac{\overset{2n}{\Delta}_{2n}}{2n+1}\right],$$

where

$$\overset{m}{\Delta}_{2n}=\sum_{\gamma=1}^{m}(-1)^{m-\gamma}{}_mC_\gamma\gamma^{2n}.$$

It is clear from the method that $\Delta_{2n}=1$, $\overset{2n}{\Delta}_{2n}=(2n)!$. If we form the first difference $(x+1)^{2n}-x^{2n}$ and then set $x=0$, we shall obtain Δ_{2n}; then forming the second difference $(x+2)^{2n}-2(x+1)^{2n}+x^{2n}$ and setting $x=0$, we shall obtain $\overset{2}{\Delta}_{2n}$; and finally $\overset{k}{\Delta}_{2n}$ is the value of the difference of the kth order of x^{2n} after setting $x=0$. We may obtain a homogeneous equation in the B's by multiplying the second equation corresponding to (4) by $-(2n+2)$ and adding the result to the equation written. We then obtain

$$B_1-{}_{2n}C_2B_2+{}_{2n}C_4B_3-\cdots+(-1)^n{}_{2n}C_{2n-4}B_{n-1}+(-1)^{n+1}(2n+1)(n-1)B_n=0.$$

3362 [1929, 105]. *Proposed by R. E. Gaines, University of Richmond.*

A triangle ABC circumscribes an ellipse of axes $2a$ and $2b$ so that A lies on one directrix and B on the other, while AB, BC, CA touch the ellipse at P, Q, R. Show that the envelope of QR and the locus of C are ellipses with $2a$ as the major axis of one and the minor axis of the other, while $2b$ is the mean proportional between the other axes.

Solution by William Hoover, Columbus, Ohio.

Let the equation of the given ellipse be $b^2x^2+a^2y^2-a^2b^2=0$, and the equation of the line AB be $lx+my-n=0$; then for tangency of AB we must have $n^2=a^2l^2+b^2m^2$. A line through the intersection of AB and the directrix $x-ae^{-1}=0$ has the equation $lx+my-n-k(x-ae^{-1})=0$. Applying to this line the condition for tangency above, we find that $ka(1-e^2)=2e(n-ael)$; and inserting this value of k in the equation above, we obtain for AC the equation,

$$[al(1+e^2)-2ne]x+ma(1-e^2)y+an(1+e^2)-2a^2el=0.$$

The equation for BC is obtained by replacing in this equation e by $-e$. Solving the two equations thus obtained, we find

$$l/n=-x/a^2,\quad m/n=-(1-e^2)y/(1+e^2)b^2\ ;$$

and, inserting the values of these two ratios in the equation of condition above, we obtain for the locus of C,

$$x^2/a^2+[(1-e^2)/(1+e^2)]^2(y^2/b^2)=1. \tag{1}$$

This is the equation of an ellipse having for its minor axis the major axis $2a$ of the given ellipse.

Since the envelope of QR is the polar reciprocal of this ellipse (1) with respect to the given ellipse, it must also be a conic, which is evidently an ellipse within the given ellipse and tangent to it at the extremities of its major axis. Denote an extremity of the major axis of (1) by $(0, y_1)$; then the polar of this point with respect to the given ellipse is $yy_1-b^2=0$. Then if $(0, \pm y_2)$ are the extremities of the minor axis of the envelope, we have $y_2y_1=b^2$; and this completes the proof.

This problem is a special case of more general theorems which have interested British geometers such as Salmon, Cayley, and Casey. See Salmon's *Conic Sections*, 6th ed. (1879), pp. 250, 257, 319, 349, 350. In the last place referred to the theory of invariants is used, and there is reference to the Quarterly Journal of Mathematics, vol. 1 (1857).

Also solved by Lawrence Hampton and the proposer.

3367 [1929, 169]. *Proposed by Harry Langman, Arverne, L. I., N. Y.*

Given any triangle. On each side construct an equilateral triangle externally. The centers of these triangles determine another equilateral triangle A. Similarly an equilateral triangle B is determined by constructing the equilateral

triangle internally. Show that the difference between the areas of the triangles A and B is equal to the area of the given triangle.

Solution by J. W. Peters, University of Illinois.

Let x_1, x_2, x_3, representing any three points in the complex plane, be the coordinates of the vertices of the given triangle. Let y_1 be the point forming with x_2 and x_3 the equilateral triangle drawn externally on the side x_2x_3. Since the triangle y_1, x_2, x_3 is equilateral,

$$\begin{vmatrix} y_1 & x_2 & x_3 \\ 1 & \omega^2 & \omega \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

where ω and ω^2 are the complex cube roots of unity. From this relation, we find $y_1 = -\omega(x_3 + \omega x_2)$. Similarly, if y_2 and y_3 are the vertices of the equilateral triangles drawn on the sides x_3x_1 and x_1x_2, respectively, we have:

$$y_2 = -\omega(x_1 + \omega x_3) \text{ and } y_3 = -\omega(x_2 + \omega x_1).$$

If a_1, a_2, a_3 are the centroids of the three equilateral triangles, then

$$a_1 = \tfrac{1}{3}(1 - \omega)(x_3 - \omega^2 x_2),$$
$$a_2 = \tfrac{1}{3}(1 - \omega)(x_1 - \omega^2 x_3),$$
$$a_3 = \tfrac{1}{3}(1 - \omega)(x_2 - \omega^2 x_1).$$

These are the vertices of triangle A. By finding the quotients like

$$(a_1 - a_2)/(a_3 - a_2)$$

we see that the angles of A are all equal. Hence, A is equilateral. The area is given by

$$-4iA = \left| a\, \bar{a}\, 1 \right| = \begin{vmatrix} a_1 & \bar{a}_1 & 1 \\ a_2 & \bar{a}_2 & 1 \\ a_3 & \bar{a}_3 & 1 \end{vmatrix},$$

where the bars denote the conjugate quantities.

If z_1, z_2, z_3 are the vertices of the equilateral triangles drawn internally, we have $z_1 = -\omega(x_2 + \omega x_3)$, $z_2 = -\omega(x_3 + \omega x_1)$, and $z_3 = -\omega(x_1 + \omega x_2)$. If b_1, b_2, b_3 are the centroids of these three equilateral triangles, then

$$b_1 = \tfrac{1}{3}(1 - \omega)(x_2 - \omega^2 x_3),$$
$$b_2 = \tfrac{1}{3}(1 - \omega)(x_3 - \omega^2 x_1),$$
$$b_3 = \tfrac{1}{3}(1 - \omega)(x_1 - \omega^2 x_2),$$

These are the vertices of triangle B. It is easily verified that B is equilateral. The area of B is given by $4iB = \left| b\bar{b}1 \right|$.

If in the equations giving the areas of the triangles A and B, we replace

the a's and b's by their values in terms of the x's, and expand the determinants, we find on subtracting that

$$-4i(A-B)=\left| x\,\bar{x}\,1 \right|.$$

The determinant on the right is $-4iC$, where C is the area of the given triangle.

Dr. H. W. Bailey, my colleague, suspected some further interesting facts about this configuration, which we have verified. The triangles A and B have the same centroid and it is the centroid of the given triangle. Furthermore the angle of intersection of the triangles A and B is invariant under homologies, i.e., transformations of the type $y=ax+b$. The proof follows: The angle of intersection is given by the amplitude of $(b_2-b_3)/(a_2-a_3)$ or $-(x_1+\omega x_2+\omega^2x_3)/(x_1+\omega^2x_2+\omega x_3)$. The quantities $x_1+\omega x_2+\omega^2x_3$ and $x_1+\omega^2x_2+\omega x_3$ are the Lagrange resolvents of the cubic equation which has x_1, x_2, x_3 for its roots. The quotient of these two resolvents is invariant under homologies. Hence the angle is invariant.

Also solved by W. E. Buker, J. W. Clawson, P. J. Federico, W. W. Johnson, J. H. Neelley, and A. Pelletier.

3368 [1929, 169]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

About a given quadrilateral to circumscribe a rhombus similar to a given rhombus.

Solution by L. W. Johnson, The University of Oklahoma.

Let $ABCD$ be the given quadrilateral and $KLMN$ the given rhombus. On AB and CD construct segments of circles external to $ABCD$ in which the equal angles K and M of the given rhombus may be inscribed. Let E and F, respectively, be the mid-points of these arcs and E' and F', respectively, the mid-points of the arcs which complete the circles of which the arcs already constructed are a part. Let the join of E' and F' intersect the arcs upon which E and F lie in the points G' and H', respectively. Join G' to A and B, and H' to C and D, calling R the intersection of $G'A$ with $H'D$ and S the intersection of $G'B$ with $H'C$. $G'H'RS$ is the required rhombus.

Proof: $G'H'RS$ is a parallelogram since its opposite sides make equal angles with $G'H'$. It is a rhombus since the triangle $G'H'S$ is isosceles.

Discussion: The join of E and F determines two points G and H analogous to the points G' and H' used above which may therefore be joined, respectively, to A and B and to C and D to determine another rhombus satisfying the conditions of the problem. Since what has been said of one pair of opposite sides of the given quadrilateral applies to the other pair, there are in general four solutions to this problem. If G' and H' coincide the number of solutions will be infinite.

Also solved by A. Pelletier.

3370 [1929, 169]. *Proposed by Paul Wernicke, Washington, D. C.*

Write down an orthogonal transformation from rectangular Cartesian coordinates X, Y, Z to x, y, z having the same origin such that the z-axis be-

comes the line $X=Y=Z$ and that the y-axis lies in the plane through the Y and z axes.

Solution by J. H. Neelley, Carnegie Institute of Technology.

Let $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3$ be, respectively, the direction angles of the three mutually perpendicular lines ox, oy, oz. Then the six conditions on these nine angles together with the special conditions given in the problem make $\alpha_3=\beta_3=\gamma_3$ and $\alpha_2=\gamma_2$ with $\beta_2=\frac{1}{2}\pi\pm\beta_3$ and $\alpha_1=\pi-\gamma_1$, $\beta_1=\frac{1}{2}\pi$. So the possible transformations are

$$\begin{aligned} X &= ax + by + cz, \\ Y &= \quad - 2by + cz, \\ Z &= - ax + by + cz, \end{aligned}$$

where $a=\pm 2^{-1/2}$, $b=\pm 6^{-1/2}$ and $c=\pm 3^{-1/2}$. All groupings of the signs are possible.

Also solved by A. Pelletier.

3371 [1929, 169]. *Proposed by Harry Langman, New York City.*

Let ABCD be any simple quadrilateral (convex or cross) inscribed in the circle whose center is O. Let AB and DC meet in F, BC and AD in E. Let M be the midpoint of the third diagonal, EF, and MU and MV tangents at U and V. Let EU and FV meet in P; EV and FU meet in Q. Take the point G on EF so that $\angle DGF=\angle DAF$, and let AC cut OG in the point R. Let the secants GA, GB, GC, GD, cut the circle again in the points A', B', C', D', respectively. Take $OA=r$. Then prove the following:

(a) G is the Clifford point of the quadrilateral $ABCD$—i.e., the common intersection of the circles about the four possible triangles formed by the sides of the quadrilateral.

(b) The square of EF is equal to the sum of the squares of the tangents to O from E and F (Casey: *Sequel to Euclid*).

(c) The circle on EF as diameter cuts the circle O orthogonally (Casey).

(d) $AC'//BD'//A'C//B'D//EF\perp OG$, proving the theorem that the Clifford point of an inscribed quadrilateral is the foot of the perpendicular from the center to the third diagonal.

(e) P and Q are the intersections of OG and the circle O.

(f) BD and UV pass through R, proving the theorem that the perpendicular from the center on the third diagonal of an inscribed quadrilateral passes through the intersection of the other two diagonals.

(g) If a quadrilateral be circumscribed about a circle at the vertices of an inscribed quadrilateral, the two pairs of diagonals intersect in a common point.

(h) Any obtuse-angled triangle may be the self-conjugate triangle of an inscribed quadrilateral. If the triangle be given, the center and radius of the circle are determined; but, when one side of the triangle has been chosen as the third diagonal of an inscribed quadrilateral, the quadrilateral is not thereby determined, there being one degree of freedom.

Solution by J. W. Clawson, Ursinus College.

(a) Assume $\angle DCB \geqq \angle DAB$. $\angle EGD = \angle BAD = \angle ECD$. Hence D, E, G, C are concyclic. Similarly it may be shown that the other groups of four points are concyclic. Hence G is the "Clifford" or "Miquel" or "focal" point of the quadrilateral $ABCD$.

(b) This theorem applies only to the case of a convex quadrilateral. $ED \cdot EA = EG \cdot EF$, since A, D, G, F are concyclic. $FB \cdot FA = FG \cdot FE$, since A, B, G, E are concyclic. Hence, the square of the tangent from E to the given circle plus the square of the tangent from F to the given circle $= EG \cdot EF + FG \cdot FE = EF^2$.

(c) Draw OE and let K be the foot of the perpendicular from F to OE. Then FK is the polar of E, since E and F are conjugate points with respect to (O). Hence $OE \cdot OK = r^2$. Let the circle (M) on EF as a diameter cut (O) in U and V. Since it passes through K, $OK \cdot OE$ is also the square of the tangent from O to (M). Hence $OU = OV = r$ are tangents to (M). This shows that any circle having two conjugate points of the circle (O) as extremities of a diameter is orthogonal to (O).

(d) From the inscribed quadrilateral $DCBD'$ we have $\angle BD'D = \angle BCD (= \angle DCE)$; and since E, G, C, D are concyclic $\angle DCE = \angle DGE$. Hence $BD' \| EF$. Similar proofs apply to AC', DB', CA'. Since BD' and DB' are parallel chords of (O) and DD' and BB' meet in G, the triangles GBD' and $GB'D$ are isosceles. Hence the perpendicular from G to these chords bisects them and passes through O. Therefore GO is perpendicular to EF.

(e) A circle can be passed through U, V, P, Q; and from the complete quadrilateral we see that E and F are conjugate with respect to it. Therefore the circle (M) is cut orthogonally by it at U and V; this circle must therefore be (O), and PQ is a diameter. In the triangle EFP, the altitudes FU and EV meet in its orthocenter Q; hence POQ is the third altitude perpendicular to EF. From (d) we see that it cuts EF in G.

(f) The pole of EF is determined by the original quadrilateral as the intersection R of BD and AC. From the quadrilateral $UVPQ$, it follows that the pole R must also be the intersection of PQ and UV.

(g) If tangents at A and B intersect in I, at B and C in J, at C and D in K, at D and A in L, the polars of K and I pass through F, hence the polar of F is KI. But, since the polar of R passes through F, the polar of F passes through R. Hence KI passes through R. Similarly LJ passes through R.

(h) EFR is a self-conjugate triangle. O is the orthocenter of the triangle. $OR \cdot OG = r^2$.

Hence, starting with any obtuse-angled triangle EFR, locate the orthocenter O. Let OR cut EF at G. Find the mean proportional between OR and OG. This gives r, the radius of the circle. Now draw any line through E cutting the circle at D and A. Join DF, AF cutting circle at C and B. Let ER cut DC, AB in H and N. Then $(ANBF)$ and $(DHCF)$ are harmonic ranges. Therefore AD, NH, BC are concurrent. Hence BC passes through E.

Also solved by A. Pelletier.

3372 [1929, 232]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Given a point of a conic, the tangent at this point, an axis, and the center (at a finite, or infinite distance), the following two lines are constructed: (1) the symmetric of the given tangent with respect to the parallel to the given axis through the given point, and (2) the diameter through the symmetric of the given point with respect to the axis. Prove that if at the point of intersection of these two lines the perpendicular is erected to the first line, it will meet the normal to the conic at the given point in the center of curvature of the conic at this point.

Solution by Margaret M. Young, Hunter College of the City of New York.

Consider the ellipse $b^2x^2+a^2y^2=a^2b^2$ and one of its points (x_0, y_0); then the center of curvature for this point has the coördinates $X=x_0-m(1+m^2)/m'$, $Y=y_0+(1+m^2)/m'$, where $m=y_0'$, $m'=y_0''$. Here $b^2x_0+a^2y_0m=0$, and by the elimination of b we find $m(x_0^2-a^2)=x_0y_0$. Solving this last equation for a^2 and differentiating the result we find $x_0y_0m'=m(y_0-mx_0)$. Inserting this value of m' we obtain

$$(mx_0 - y_0)X = mx_0(x_0 + my_0), \qquad m(mx_0 - y_0)Y = - y_0(x_0 + my_0).$$

The equation of line (1) is $y-y_0=-m(x-x_0)$; while that of line (2) is $x_0y=-xy_0$. Their point of intersection (x_1, y_1) is given by $x_1(mx_0-y_0)=x_0(y_0+mx_0)$, $y_1(mx_0-y_0)=-y_0(y_0+mx_0)$. The equation of the line perpendicular to (1) at (x_1, y_1) is

$$m(mx_0 - y_0)y + my_0(y_0 + mx_0) = (mx_0 - y_0)x - x_0(y_0 + mx_0).$$

The values of X and Y given above satisfy this equation and the proof of the construction is complete. The proof is similar for the other two forms of the conic.

Also solved by A. Pelletier.

Note by the Editors. This construction easily follows from the theorem: *The angle between the common tangent and the common chord of a conic and its osculating circle is bisected by the perpendicular from the point of contact to an axis of the conic.* A proof is given in a Note [1924, 51] following the solution of 2990 [1922, 356]. If P is the given point on the conic and PT is the tangent at this point cutting in T the axis AO of the conic, and if P' is the point of the conic symmetric to AO with respect to this axis, then $P'T$ is tangent at P', and the diameter $P'O$ bisects all chords parallel to $P'T$. Hence if we draw the chord PC of the conic parallel to $P'T$, this chord will also be the chord of the osculating circle by the above theorem, and it will be cut in its middle point M by $P'O$. Therefore the perpendicular to PC at M cuts the normal at P in the center of curvature I of the conic.

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus Ohio.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohic.

At the thirty-sixth annual meeting of the American Mathematical Society, which opened at Lehigh University on December 26, Professor W. C. Graustein, of Harvard University, and Professor E. P. Lane, of the University of Chicago, were elected Vice-presidents.

Princeton University announces the appointment of Dr. J. Von Neumann, of the University of Berlin, and Dr. E. Wigner, of the Kaiser Wilhelm Institut für Physikalische Chemie, as visiting lecturers in Mathematical Physics for the second semester of the present academic year.

The one thousand dollar prize offered annually by the American Association for the Advancement of Science for a noteworthy contribution to science presented at a meeting was awarded at Des Moines to Professor Arthur J. Dempster, of the University of Chicago, for work on wave characteristics of the proton.

Professor Arthur Haas, of the University of Vienna, and F. G. Donnan, of the University of London have been invited by the Gibbs Committee (representing the Departments of Physics and Chemistry of Yale University) to be editors of a commentary on the works of J. Willard Gibbs.

On account of the large sale of Cajori's "History of Mathematical Notations," both in Europe and America, The Open Court Publishing Co. expects to publish a new edition next summer.

At Syracuse University Professor Alan D. Campbell has been made head of the department of mathematics in the College of Liberal Arts.

Professor E. F. Cox, of West Virginia Collegiate Institute, has been appointed to an associate professorship at Howard University.

Assistant Professor E. B. Lytle has been promoted to an Associate Professorship of the teaching of mathematics at the University of Illinois.

Dr. G. M. Merriman has been appointed assistant professor of mathematics at the University of Cincinnati.

Mr. C. A. Spicer has been appointed professor of mathematics at Western Maryland College.

Professor J. C. Tinner, of Wilberforce University, has been appointed to a Professorship at Bishop College.

The following appointments to instructorships are announced:
Cornell University, Mr. J. M. Clarkson
Crane Junior College, Miss Edna M. Feltges
Houston Junior College, Mr. W. A. Rees
Union College, Mr. A. H. Fox and Mr. F. W. Lerch
State College of Washington, Mr. John Biggerstaff and Mr. L. G. Butler
Yale University, Dr. A. K. Mitchell
Hunter College of the City of New York, Mr. A. D. Bradley and Miss Harriet Griffin
South Dakota State College, Mr. H. B. MacDougal, Mr. Gordon Fuller, and Miss Ruth Rasmusen

Dr. Edward Drake Roe, Jr., head of the mathematics department of the College of Liberal Arts of Syracuse University, died suddenly from a heart attack, on December 11, at the age of 70. Dr. Roe was graduated from Syracuse University, received an A.M. degree at Harvard and a Ph.D. degree at Erlangen (doing much of his work there under Gordan). He taught at Harvard, Boston University, and Oberlin College before coming to Syracuse. He was a professor at Syracuse from 1900 until his death; since 1919 he was also director of the astronomical observatory and head of the mathematics department. He founded the mathematical fraternity of Pi Mu Epsilon in 1914. He published many papers on mathematical research (principally in algebra), several papers on astronomy and on philosophy, also a text-book in trigonometry and one in algebra. He always stood for high scholarship. He was a very thorough teacher, leaving a lasting impression on the students who took his courses and inspiring in them his own deep love for mathematics and for astronomy.

The Chauvenet Prize

In the year 1925, the ASSOCIATION established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period 1929-1931.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.
INDIANA, Earlham College, May 2-3.
IOWA.
KANSAS.
KENTUCKY.
LOUISIANA-MISSISSIPPI, Cleveland, Miss., April.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.
MICHIGAN.
MINNESOTA.
MISSOURI.
NEBRASKA, Peru, Neb., May 9.
OHIO, Columbus, Ohio, April 3.
PHILADELPHIA.
ROCKY MOUNTAIN.
SOUTHEASTERN.
SOUTHERN CALIFORIA, Los Angeles, Calif., March 8.
TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE FOURTEENTH ANNUAL MEETING OF THE ASSOCIATION

The fourteenth annual meeting of the Mathematical Association of America was held at Des Moines, Iowa, on Tuesday and Wednesday, December 31, 1929, and January 1, 1930, in affiliation with the American Association for the Advancement of Science and the American Mathematical Society. One hundred forty were in attendance at the meetings, including the following one hundred four members of the Association:

Harriet Anderson, Grand Island College
R. C. Archibald, Brown University
C. H. Ashton, University of Kansas
C. S. Atchison, Washington and Jefferson College

R. W. Babcock, DePauw University
Frances E. Baker, Creston (Iowa) Junior College
R. P. Baker, University of Iowa
A. K. Bettinger, Creighton University
A. H. Blue, Western Union College
F. A. Brandner, Louisiana State College
W. C. Brenke, University of Nebraska
M. Sue Burney, St. Joseph (Mo.) Junior College
W. H. Bussey, University of Minnesota

W. D. Cairns, Oberlin College
E. H. Carus, La Salle, Illinois
E. W. Chittenden, University of Iowa
L. M. Coffin, Coe College
Julia T. Colpitts, Iowa State College
C. E. Comstock, Bradley Polytechnic Institute
I. S. Condit, Iowa State Teachers College
Byron Cosby, Missouri State Teachers College
D. R. Curtiss, Northwestern University

Marian E. Daniells, Iowa State Teachers College
R. D. Daugherty, University of Iowa
L. L. Dines, University of Saskatchewan

C. W. Emmons, Michigan State College
H. P. Evans, University of Wisconsin
H. S. Everett, University of Chicago

B. F. Finkel, Drury College
Irving Fisher, Yale University
Annie W. Fleming, Iowa State College
M. M. Flood, University of Nebraska

M. G. Gaba, University of Nebraska
J. S. Gold, Bucknell University
G. W. Gorrell, University of Denver
C. Gouwens, Iowa State College

LAURENCE HAMPTON, Alabama Polytechnic Institute
W. L. HART, University of Minnesota
J. O. HASSLER, University of Oklahoma
E. R. HEDRICK, University of California at Los Angeles
MARY E. HELWIG, Kansas City (Kans.) High School
GERTRUDE A. HERR, Iowa State College
H. M. HOSFORD, University of Arkansas
J. M. HOWIE, Nebraska Wesleyan University
JEWELL C. HUGHES, University of Arkansas

BYRON INGOLD, Culver-Stockton College
LOUIS INGOLD, University of Missouri
MARK H. INGRAHAM, University of Wisconsin

DUNHAM JACKSON, University of Minnesota
G. H. JAMISON, Missouri State Teachers College

DORA E. KEARNEY, Iowa State Teachers College
O. D. KELLOGG, Harvard University
A. E. KENNELLY, Harvard University
A. F. KOVARIK, Yale University

E. P. LANE, University of Chicago
R. E. LANGER, University of Wisconsin
E. B. LYTLE, University of Illinois

S. L. MACDONALD, Colorado State Agricultural College
W. D. MACMILLAN, University of Chicago
R. B. MCCLENON, Grinnell College
F. M. MCGAW, Cornell College
J. V. MCKELVEY, Iowa State College
MRS. J. V. MCKELVEY, Iowa State College
A. D. MICHAL, California Institute of Technology
U. G. MITCHELL, University of Kansas
F. R. MOULTON, Utilities Power & Light Corporation, Chicago
C. N. MOORE, University of Cincinnati
PAUL MUEHLMAN, St. Louis University
SIGURD MUNDHJELD, Waldorf College

I. F. NEFF, Drake University

E. A. PATTENGILL, Iowa State College
E. W. PEHRSON, University of Utah
H. P. PETTIT, Marquette University

G. E. RAYNOR, University of Oklahoma
J. F. REILLY, University of Iowa
B. L. REMICK, Kansas State Agricultural College
H. L. RIETZ, University of Iowa
B. D. ROBERTS, Parsons College
G. EDNA ROBINSON, Osark Wesleyan College
W. H. ROEVER, Washington University
C. F. ROOS, Cornell University

G. T. SELLEW, Knox College
W. A. SHEWHART, Bell Telephone Laboratories, New York
J. A. G. SHIRK, Kansas State Teachers College
H. E. SLAUGHT, University of Chicago
E. R. SMITH, Iowa State College
G. W. SMITH, University of Kansas
G. W. SNEDECOR, Iowa State College
EUGENE STEPHENS, Washington University
E. B. STOUFFER, University of Kansas

A. HELEN TAPPAN, Western College for Women
J. S. TURNER, Iowa State College

L. E. WARD, University of Iowa
WARREN WEAVER, University of Wisconsin
MARIE J. WEISS, National Research Fellow, University of Chicago
J. J. WHEELER, University of Kansas
A. E. WHITE, Kansas State Agricultural College
K. P. WILLIAMS, Indiana University
F. E. WOOD, Northwestern University
ROSCOE WOODS, University of Iowa
KATHRYN WYANT, University of Missouri

JESSICA M. YOUNG, Washington University
J. W. YOUNG, Dartmouth College

JOHN ZIMMERMAN, University of Dubuque

The sessions of the A.A.A.S. began on Friday evening with an address in Shrine Temple Auditorium by the retiring president, Professor Henry Fairfield Osborn, on "The discovery of tertiary man," and the general reception which is held regularly on the opening evening. Among other addresses of a general character were a number which had mathematical bearings, for example, "Earthquakes and what they tell us" by J. B. Macelwane of St. Louis Univer-

sity; "The relation between the size of the energy atom and its physiological effect" by W. T. Bovie; "Some aspects of celestial evolution" by Professor E. B. Frost of Yerkes Observatory; "The laws of racing fatigue in men and horses" by Professor A. E. Kennelly of Harvard University; "The alleged sins of science" by Dr. R. A. Millikan, president of the American Association for 1929; and a stereopticon lecture on the new "Max Adler Planetarium of Chicago" by Professor Philip Fox, recently of Northwestern University and now Director of the Planetarium. On Tuesday afternoon the seventh Josiah Willard Gibbs lecture was given by Professor Irving Fisher of Yale University on "The application of mathematics to the social sciences." This address was given under the joint auspices of the American Association and the American Mathematical Society; an audience of over four hundred indicated the general interest in the subject and the speaker.

The scientific exhibition was held in the large registration hall of the Shrine Temple and included the exhibits of numerous publishers of college and university texts and of makers of scientific apparatus. Of especial interest to mathematicians was an exhibit of old books and incunabula on mathematics, astronomy and physics from the library of Professor J. S. Turner of Iowa State College shown at Des Moines Public Library; this was made possible through the generosity of Professor Turner and the cooperation of the department of mathematics of Iowa State College.

The groups of mathematicians stayed for the most part in Hotel Fort Des Moines, and the mathematics meetings were all held in suitable rooms in this hotel. As a consequence there was unusual opportunity for the informal social aspects of the meetings. About one hundred thirty attended the joint dinner on Tuesday evening in the Oak Room, Professor W. H. Bussey, editor of the MONTHLY, acting as toastmaster and calling on various representatives of the two organizations. Professor Rietz welcomed the guests to Iowa and spoke of the importance of the meetings for that region. Professor J. W. Young, coming from New England, spoke as president of the Association and mentioned the difficulty experienced by mathematicians in keeping in touch with each other in our country-wide dispersion and the necessity, in view of our extensive and ever-increasing literature, of obtaining publication funds on a large scale. Miss Jewell Hughes, in a speech replete with wit, depicted some of the lighter, even if logical, experiences of those mathematically inclined. Professor Hedrick, coming from the Pacific Coast, spoke as president of the Society, agreeing with Professor Young in the hope that sooner or later we may be free from the present burden of seeking adequate finances for the Society. He expanded his remarks made at the close of the Gibbs lecture by saying that we must not limit ourselves, in our definition of the scope of mathematics, to purely formal matters but must include all those fields which can be treated by the instrumentalities of mathematics, and we must include such men as Gibbs, who was primarily a physicist, in the list of those doing essentially mathematical work.

The Des Moines meetings are noteworthy by reason of the hospitality shown by the citizens of Des Moines and that region. A cordial welcome was evident on every hand, the Des Moines Club entertained groups of scientists each noonday, and the authorities of the State University of Iowa and of Iowa State College made it possible for the visitors to see the grounds, buildings and laboratory facilities of these institutions. These and similar services were recognized in a resolution, adopted at the joint dinner by a rising vote, expressing the appreciation of the visitors for the admirable efficiency of the arrangements and the generosity of the entertainment, in particular expressing to Mr. Hunter of the Equitable Life Insurance Company of Iowa, to President Morehouse and Professor Neff of Drake University, to Iowa State College and the State University of Iowa, our cordial thanks and the assurance that the meetings will long be a cherished memory.

The American Mathematical Society held sessions in Hotel Fort Des Moines on Monday and Tuesday mornings, partly in the form of simultaneous sections, for the reading of papers, one section Tuesday morning being a joint session with Section K (Economics and Social Sciences). It held a joint session Monday afternoon with Section A (Mathematics) of the American Association, at which Professor R. C. Archibald gave his retiring address as Vice-President of Section A on "Mathematics before the Greeks" and Professor O. D. Kellogg gave an address on "An unsolved problem of uniqueness in potential theory" on invitation of the Society and Section A. The Gibbs lecture for 1929 delivered by Professor Irving Fisher, on invitation of the Society, has already been mentioned.

The program of the Mathematical Association consisted of a joint session with the Society on Tuesday afternoon and two sessions on Wednesday. The program was prepared by a committee consisting of Professors Arnold Dresden, L. M. Graves, W. L. Hart, H. W. Kuhn, and E. W. Chittenden, Chairman.

Joint Session of the Association with the Society

"Linear inequalities and some related properties of functions" by Professor L. L. Dines, University of Saskatchewan, by invitation of the two organizations.

This interesting and instructive address will appear in the *Bulletin* of the American Mathematical Society, since it is somewhat more suited to that publication.

First Session of the Association

1. "The geometric representation of groups" by Professor R. P. Baker, University of Iowa.

2. "Aspects of the theory of linear associative algebras of infinite order" by Professor M. H. Ingraham, University of Wisconsin.

3. "The fourth Carus monograph" by Professor J. W. Young, Dartmouth College.

1. The paper gave a description of five classes of representations of a group with illustrations on lantern slides. These are (1) Schraffirte diagrams, (2) Cayley diagrams with independent generators, (3) General Cayley diagrams, (4) Geometric configurations with a group of transformations, (5) Representations of the substitution letters by geometric objects.

In field (2) Maschke (American Journal, vol. 18) enumerated the Cayley diagrams on the sphere; these are all half regular.

In carrying on this work to the projective plane and anchor ring it is practically necessary to postulate half-regularity to prevent being overwhelmed with examples mostly bizarre. For the projective plane only three such diagrams are possible, for $G_{12\cdot 4}$, $G_{24\cdot 4}$, $G_{60\cdot 5}$. For the anchor ring 68 sets exist. Four of these are unique groups of order 16. The others have one, two, or three arbitrary integers in the order. They are based on 7 nets of the half-regular type. Three other nets are possible but lead to an extra generator and the situation explored by Dyck for groups of genus one. These groups can be described as having an Abelian self conjugate subgroup with two independent generators. The quotient group, which is cyclic for groups of genus one, may be E, C_2, C_3, C_4, G_4, G_6, C_6, $G_{8\cdot 4}$, $G_{12\cdot 5}$.

Looking at the repeated development of the anchor ring in the plane as a homogeneous assemblage the elements may be called 'atoms' and the sets connected by generators of the quotient group 'molecules.'

The generators of the Abelian subgroup can be called 'translations' but 'rotations' is hardly applicable to the operations of the quotient group.

Incidentally the work led to a complete solution of the space-filling problem for the plane, ten types dual to the ten nets of this problem.

2. The usual postulates for a linear associative algebra may be generalized in several directions. Interesting results have been obtained by omitting the associative law. This paper deals with associative algebras which do not have a finite base in terms of which all elements can be expressed as linear combinations with scalar coëfficients. Wedderburn, in an article[1] entitled *Algebras which do not possess a finite base*, has developed a theory which considers not only finite sums but such sums as sums of series and integrals. J. W. Young has given a very interesting point of departure for the theory of algebras from the group standpoint in two papers *On the partitions of a group and the resultant classifications* (Bulletin American Mathematical Society, vol. 33, pp. 453–461), *A new foundation for general algebra* (Annals of Mathematics, vol. 29, pp. 147–60. This paper chiefly discusses generalizations which arise from assuming that aggregates can be well ordered, and in certain portions that the cardinal number of the algebra is an "aleph" number. The generalizations of the theory of linear sets, reducibility, matrix representations and nilpotent algebras are discussed.

[1] Transactions of the American Mathematical Society, vol. 26 (1924), pp. 395–426.

3. Dr. J. S. Georges who was to have given a paper on "Some social aspects of mathematics" was unable to be present because of illness in the family. Professor Young was therefore asked by the program committee to give a description of the forthcoming Carus monograph.

The subject matter of projective geometry can be studied with only a small amount of previous mathematical knowledge and hence this monograph can be thought of as adapted to gaining recruits for mathematics. Approximately the first half is of rather simple character and is purely synthetic. The development of projective geometry out of elementary geometry through metric considerations has its drawbacks, for it does not show how projective properties can be established without using metric geometry.

On the other hand such a monograph as this should show the relation of projective to metric geometry; hence after giving the general properties of conic sections, there is a study of the specializations necessary for exhibiting the various special geometries.

Professor Young has depended on an intuitive rather than a postulational basis; for example, he has tacitly assumed the postulates of alignment.

In the latter half he lays the foundation for an analytic treatment on a purely projective basis, and he brings out the definitions of the various geometries in terms of groups of transformations.

Professor Slaught emphasized the importance of having the Association members suggest in written form desirable subjects for future monographs and spoke of the ideals which have always been kept in the minds of the committee in carrying on the development of the monographs. He announced that the Young monograph was then in page proof and might be expected to be out late in March.

Second Session of the Association

1. "Appreciation of form as an index of mathematical ability" by Professor E. B. Lytle, University of Illinois.

2. "The laws of rolling motion and the complete elimination of sliding friction in roller bearings" by Professor W. H. Roever, Washington University.

3. "Mathematics and the problem of ore location" by Professor Warren Weaver, University of Wisconsin.

1. An analysis of reflective thinking shows at least three stages: first, there is a problem, a question, or a difficulty; secondly, there must be suggested possible answers, possible methods of procedure, or possible investigations to make; and thirdly, all suggestions produced must be carefully tested to determine the best one. Often the second stage is not sufficiently emphasized, for how to get suggestions when perplexed is a common difficulty in reflective thinking. When thinking with students or before students, teachers might be more helpful if they were more careful to exhibit what are the sources of their ideas, what are their cues to action, what suggests the transformation or operation they carry

out. One very fertile suggestor in mathematics is the notion of algebraic form. Transformations are suggested by the desire to get certain forms which are rich in meanings; such transformations generally seem to be mere symbol juggling to one unappreciative of forms. Ability to get information and cues to action from forms is so necessary in mathematical work that this ability may be taken as an index of mathematical ability.

2. The paper was introduced with a statement of the Poncelet theorem that the instantaneous state of velocities of a rigid body in motion is that due to a certain ruled surface of the body moving on a ruled surface fixed in space in such a way that the former surface is tangent to the latter all along a common generator, on which the body rolls and slides. It was then shown that two rigid bodies in contact with each other may slide, roll, or pivot at the contact. If, in particular, the sliding is nil, the rolling and pivoting are tangential and normal components of the instantaneous velocity vector, which now passes through the contact. Owing to the fact that sliding friction is greater than rolling friction, journal bearings have been replaced by roller (including ball) bearings. In early forms of the latter no provision was made for separating consecutive rollers, and therefore sliding occurred at the contacts of such rollers, thus counteracting to some extent the advantage of rolling at other points of the bearing. To obviate this defect, devices called "cages" were used to separate rollers, but these introduced sliding friction at the points where they came into contact with the rollers which they separated. Pictures were shown of several types of such devices. However, it was shown by simple apparatus that it is possible to separate rollers without having sliding motion at any of the contacts. Pictures were shown of a bearing in which use was made of this principle. Since balls must travel in channels to be properly guided, pivoting, as well as rolling, occurs where balls are used, but sliding friction can be entirely eliminated.

3. This paper opens with introductory remarks concerning earth physics, and the relations of this subject to atomic and astro-physics. It is then pointed out that any geo-physical prospecting method consists essentially of a sending device which dispatches messengers down into the earth, and a receiving device which decodes the messages when part of the messengers return to the surface. The magnetic, electrical, electro-magnetic, gravitational, and seismic methods are then very briefly discussed. The second main division of the paper considers the mathematical problems presented by these five methods. The central unity of the problems, from a mathematical point of view, is emphasized by pointing out that all of the methods are governed by the same general differential equation, and are subject to boundary conditions of the same type. Seven general reasons are advanced to indicate why the problems of geophysical prospecting are new problems, even though they involve the application of classical physical theories, and classical mathematical methods. Various problems which have been satisfactorily treated are mentioned, as well as a considerable number of problems which have not been solved. Indication is

given of some investigations of pure mathematics, whose successful treatment would contribute greatly to advances in this new art.

This paper will appear in full in the April issue of this MONTHLY.

MEETINGS OF THE BOARD OF TRUSTEES

Eleven members of the Board of Trustees were present at the sessions.

The following fourteen persons and one institution were elected to membership on applications duly certified:

To Individual Membership

F. R. BAMFORTH, PH.D. (Chicago). Asst. Prof., Cornell Univ., Ithaca, N. Y.

R. H. CAMERON, A.B. (Cornell). Instr., Cornell Univ., Ithaca, N. Y.

JOSÉ ISAAC CORRAL, Ing. de Minas. Director, Montes y Minas, Republic of Cuba, Havana, Cuba.

NAT EDMONSON, JR., PH.D. (Rice). Acting Asst. Prof., Texas Tech. Coll., Lubbock, Tex.

ARTHUR HAAS, A.B. (C.C.N.Y.), B.S. (C.U.), LL.B (N.Y.U). Chairman, Dept. of Math., Thomas Jefferson High School, Brooklyn, N. Y.

ELIZABETH L. HARRIS, A.B. (Washington Univ.). Computer, Southwestern Bell Telephone Co., Kirkwood, Mo.

F. T. H'DOUBLER, M.D., PH.D. (Wisconsin). Springfield, Mo.

G. H. MURRAY, B.S. in Bus. Admin. (Missouri). 2934 Victor St., Kansas City, Mo.

F. P. NASH, JR., A.M. (Columbia). Head of Dept., Groton School, Groton, Mass.

H. T. H. PIAGGIO, M.A., D.Sc. Prof., Head of Dept., University College, Nottingham, England.

H. A. POOL, A.M. (Minnesota). Instr., State School of Mines, Rapid City, S. D.

MARY B. RUMSEY, A.M. (Illinois). Instr., Knox Coll., Galesburg, Ill.

W. O. SHRINER, PH.D. (Michigan). Head of Dept., State Teachers Coll., Terre Haute, Ind.

R. F. WICK, B.S. in E.E. (A.&M. Coll. of Texas). Asst., E.E. Dept., A.&M. Coll. of Texas, College Station, Tex.

To Institutional Membership

ST. BONAVENTURE'S COLLEGE, St. Bonaventure, N. Y.

The financial statement for the year 1929 was presented by the Secretary-Treasurer. It was accepted and approved by the Trustees, subject to an inspection by a committee of the Trustees; this examination was made on Wednesday by Professors Archibald, Slaught and Young and the report approved.

It was voted to authorize the Finance Committee to invest further current funds for the General Endowment as the surplus funds warrant. (As a consequence of this action, three thousand dollars have been invested in trust funds in January 1930.)

It was voted to approve the printing of a catalog of the library of the Association; to thank Professor Slaught for his effective work in connection with the Carus monographs, and the Carus family for their continued generosity; to express our interest in a proposed summer school for those who teach mathematics to engineering students to be held under the auspices of the Society for the Promotion of Engineering Education in connection with the Minnesota Summer Meeting in 1931; to express our interest and promise

our cooperation in the reestablished International Commission on the Teaching of Mathematics; and to express our interest in the plans for the possible establishment of a journal in applied mathematics.

The Trustees ratified the action of President Young in appointing H. S. Vandiver and Mrs. Anna Pell Wheeler as representatives of the Association on the Editorial Board of the Annals of Mathematics for a three-year term beginning January 1930. Because the work of the standing Committee on Relations with the Annals had been completed, this committee was discontinued.

Expressions of hearty thanks to Doctor A. B. Chace for his generous gift of several hundred sets of the Rhind Mathematical Papyrus to the Association, and to Professor Archibald for his great service involved in his manifold work on the Papyrus were adopted, these expressions of appreciation to be presented to them personally in suitable form.

After extensive consideration by correspondence and at the meetings at Boulder and Des Moines, the Trustees voted to approve and to put into operation a plan for the nomination of officers whereby a nominating committee will suggest five names of persons eligible and suitable for President, ten for Vice-President, and twenty for Trustees, when the blanks for the preliminary ballots are sent out, it being understood that this is purely advisory and is given in the interest of avoiding a scattering of ineffective votes, and not to limit the free choice of members. This plan involves no change in the By-Laws.

The Chauvenet Prize

The committee on the second award of the Chauvenet Prize recommended that the award be made to Professor T. H. Hildebrandt of the University of Michigan for his paper on "The Borel theorem and its generalizations" published in the *Bulletin* of the American Mathematical Society, volume 32(1926), pages 423–474. Professors A. J. Kempner and D. R. Curtiss for the committee stated that "the paper presents in clear and elementary fashion, and with adequate references to literature, the development of a rather broad range of ideas and results connected with the Borel theorem. It gives the reader a compact picture not easily obtained by independent reading of the scattered literature on the subject." The Trustees adopted the recommendation and the award was announced at the annual business meeting. The prize, $100 in cash, has since been conferred.

Annual Business Meeting of the Association

The Secretary-Treasurer announced the names of those elected to membership. He reported also the deaths of the following members:

J. A. Ankebrandt, Professor of mathematics, Western State College, (May 25, 1929).

C. A. Epperson, Teacher of mathematics, Kansas City, Mo., (November 5, 1928).

Sister Blanche Marie Maskell, Teacher, College of St. Elizabeth, (December 1928).

R. M. MATHEWS, Associate professor of mathematics, West Virginia University, (October 20, 1929).

A. W. PRATER, Assistant, University of California at Los Angeles, (April 16, 1929).

FRED REUSSER, Head of department of mathematics, Buena Vista College, (October 11, 1929).

E. D. ROE, JR., Professor of mathematics, Syracuse University, (December 11, 1929).

W. H. SHERK, Professor of mathematics, University of Buffalo, (January 1929).

SARAH E. SMITH, Professor of mathematics, Mount Holyoke College, (November 18, 1929).

The election of officers for the year 1930 resulted in the following, as reported by the tellers, Professors U. G. Mitchell and W. J. Risley:

For Vice-Presidents: E. T. Bell, 348 votes; Tomlinson Fort, 195 votes; W. C. Graustein, 245 votes; E. B. Stouffer, 210 votes.

For additional members of the Board of Trustees, to serve until January 1933: A. A. Bennett, 340 votes; H. T. Davis, 179 votes; B. F. Finkel, 257 votes; W. L. Hart, 310 votes; Miss Jewell C. Hughes, 149 votes; D. N. Lehmer, 293 votes; C. C. MacDuffee, 225 votes; E. R. Smith, 180 votes.

The following were accordingly declared elected: Vice-Presidents: E. T. BELL, California Institute of Technology; W. C. GRAUSTEIN, Harvard University.

Additional members of the Board of Trustees: A. A. BENNETT, Brown University; B. F. FINKEL, Drury College; W. L. HART, University of Minnesota; D. N. LEHMER, University of California.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DEC. 16, 1929

RECEIPTS

Balance Dec. 17, 1928	$9,876.02
1928 indiv. dues	730.95
1928 instit. dues	23.10
1929 indiv. dues	6,826.45
1929 instit. dues	811.10
1929 subscriptions	943.84
Initiation fees	226.00
Advertising	583.32
Sale copies of MONTHLY	46.15
Sale First Carus Mon.	13.75
Sale Second Carus Mon.	17.50
Sale Third Carus Mon.	28.75
Sale Fourth Carus Mon.	51.25
Sale reprints	8.87
For *Annals* subscription	3.50
Sale Rhind Papyrus	151.00
Contributions A.B. Chace	4,000.00
Refund New York meeting expense	25.25
Int. Oberlin Savgs. Bk.	157.56

EXPENDITURES

Publisher's bills (Nov. '28–Oct. '29)		$5,030.47
President's office		34.09
Manager's office		31.42
Editor-in-Chief's office		453.49
Committee on Geometry		124.95
Register		115.00
Secretary-Treasurer's office:		
Postage	$309.56	
Bond	5.00	
Safety deposit	4.00	
Office supplies	125.17	
Express, tel., etc.	58.59	
Ins. back copies of MONTHLY	17.40	
Clerical work	851.34	
Printing	279.80	
Library expense	30.50	
Paid copies MONTHLY	97.40	
New York Meeting	95.00	
Boulder Meeting	100.00	

Int. Peoples Bkg. Co.	115.74		Refund 1929 dues	7.00	
Int. Liberty Bonds	85.00		Refund subscription	3.15	
Int. Hardy Fund	120.00		Refund Papyrus	15.00	
Int. certifs. of deposit	64.33		Forwarded for members'		
Int. from Genl. Endowment			*Annals*	5.00	
Fund Bonds	155.00		Paid members' *Annals*	12.00	
Int. Chace Fund	50.00		Forwarded for reprints	8.87	
Int. Chauvenet Fund	25.00				
Int. from investment of current funds	150.00				2,024.78
			Annals subvention		375.00
			Paid to sections from initiation fees		112.68
		15,413.41	Transferred to Genl. Endowment..		750.00
Total 1929 receipts		$25,289.43	Cost above par of new bonds for Genl. Endowment Fund		122.09
			Pd. Drury College int. Hardy Fund, credit B. F. Finkel		120.00
			Sustaining memb. in Amer. Math. Soc.		100.00
			Printing Rhind Papyrus		4,000.00
			Expense acct. *Bibl. Math.*		118.75
			Expense acct. Carus Mons.		13.78
			Bought bond for Chace Fund		1,000.00
			Transfer to Chace Fund		416.00
			Transfer to Carus Mon. Fund		122.05
Total expenditures		$15,064.55	Total expenditures		15,064.55
			Cash on hand		13.28
Balance to the end of 1929 business		$10,224.88	Checking account		140.64
			Oberlin Savgs. Bk. acct.		2,845.77
			Peoples Bkg. Co. acct.		2,981.37
			Liberty Bonds		1,000.00
			5% Bonds, Harris Trust Co.		3,000.00
Received on 1930 business		996.50	Certif. of deposit		1,240.32
Book balance Dec. 16, 1929		$11,221.38	Bank balance Dec. 16, 1929		$11,221.38

EXHIBIT OF THE FUNDS OF THE ASSOCIATION

Carus Monograph Fund

Balance Dec. 17, 1928		$4,292.19
Receipts: Sales	$111.25	
Interest	171.23	282.48
		$4,574.67
Expenditures		13.78
		$4,560.89
Certificates of deposit	$4,510.89	
Cash in bank	50.00	
Balance Dec. 16, 1929		$4,560.89

ARNOLD BUFFUM CHACE FUND

Balance Dec. 17, 1928		$1,230.00
Receipts: From advance sales	$151.00	
Interest	50.00	201.00
		$1,431.00
Refund		15.00
		$1,416.00
Iowa Rwy & Light Co. 5% Bond	$1,000.00	
Certificate of deposit	416.00	
Balance Dec. 16, 1929		$1,416.00

CHAUVENET PRIZE FUND

Balance Dec. 17, 1928		$560.00
Interest	$25.00	
Amount set aside for 1929	20.00	45.00
		$605.00
Iowa Rwy & Light Co. 5% Bond	$500.00	
Cash in bank	105.00	
Balance Dec. 16, 1929		$605.00

LIFE MEMBERSHIP FUND

Liability on life memberships Dec. 17, 1928	$472.04
To be transferred to current funds, surplus	12.84
Liability on life memberships as of Jan. 1, 1930	$459.20

GENERAL ENDOWMENT FUND

Balance Dec. 17, 1928		$4,250.00
Transferred from current funds in 1929		750.00
		$5,000.00
Liberty Bond	$1,000.00	
Land Trust Certificate	1,000.00	
Cleveland Trust Securities Co. Gold Bond	1,000.00	
Idaho Power Co. 5% Bonds	2,000.00	
Balance Dec. 16, 1929		$5,000.00

Of the funds on hand, indicated in the first division of the financial report, $50.00 belongs to the Carus Monograph Fund (not yet transferred); $105.00 belongs to the Chauvenet Prize Fund; and $459.20 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date Jan. 1, 1930. Aside from these amounts, the Carus Monograph Fund, to the amount of $4,510.89 is carried as a separate fund in the form of two certificates of deposit which bear 4%, compounded quarterly; $1,000.00 of the Arnold Buffum Chace Fund is carried in securities as listed above, and $416.00 more in the form of a certificate of deposit; the contribution of Professor W. B. Ford to the Chauvenet Fund is carried as a 5% Gold Bond; and the sum of $5,000.00

is held in reserve as the General Endowment Fund, in securities as listed above.

When the accounts were closed Dec. 16, 1929, there remained on the total business for the year 1929 the following items:

Bills Receivable (partly estimated)		Bills Payable (partly estimated)	
1929 individual dues	$150.00	Publisher's bills (Nov., Dec. 1929)	$1,150.00
Advertising	100.00	President's office	30.00
		Manager's office	40.00
	$250.00	Editor-in-chief's office	100.00
		Other editors' postage	50.00
		Register	500.00
		Secretary-Treasurer's office	300.00
		Annals subvention	150.00
		Initiation fees due to sections	750.00
		Carus Monograph Fund	50.00
		Chauvenet Prize Fund	105.00
		Life Membership Fund	459.20
		Typewriter	70.00
		Printing catalog of library	400.00
			$4,079.20

If to the balance on 1929 business shown in this report, $10,224.88, there be added the bills receivable, $250.00, and there be subtracted the estimated bills payable, $4,079.20, there results an estimated final balance on 1929 business of approximately $6,400, which represents the accumulated surplus in current funds, as compared with the corresponding figure of $5,190 of a year ago. This shows a healthy condition of the Association for which we may be gratified at a time when many banks and other organizations are in financial difficulties.

W. D. Cairns, *Secretary-Treasurer*

THE FOUNDATIONS OF THE IDEAL THEORY OF ZOLOTAREV[1]

By N. Tchebotarev

Let us recall some definitions and theorems on algebraic integers. Suppose that the algebraic number ω is a root of the irreducible equation

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_n = 0 \tag{1}$$

with rational coefficients.

Definition 1: If the coefficients in (1) are rational integers then ω is called an algebraic integer.

The following theorems follow directly. Proofs may be found in any book on algebraic numbers.

[1] Translated from the Russian by S. Skolnik; edited and revised by H. T. Engstrom. This version is not a literal translation, but has been adapted to English-speaking readers. E. T. Bell.

I. Every algebraic integer which is rational is a rational integer.

II. Every algebraic number satisfying an equation with integral rational coefficients and leading coefficient unity is an algebraic integer. (Theorem of Gauss).

III. Every algebraic number satisfying an equation with integral algebraic coefficients and leading coefficient unity is an algebraic integer.

IV. The integers of an algebraic field form a ring.

Definition 2: The product of the conjugates of an algebraic number ω is called its norm, $N(\omega)$; i.e.

$$N(\omega) = \omega\cdot\omega^{(1)}\cdots\omega^{(n-1)}. \tag{2}$$

It follows immediately that

$$N(\omega_1\cdot\omega_2) = N(\omega_1)N(\omega_2). \tag{3}$$

Furthermore if ω satisfies (1) and t is a rational number

$$N(t-\omega) = \pm f(t). \tag{4}$$

Definition 3: A system of n integers $\omega_0, \omega_1, \cdots, \omega_{n-1}$ of an algebraic field K of n-th degree is called an integral base of K if every integer of the field can be represented in the form

$$c_0\omega_0 + c_1\omega_1 + \cdots + c_{n-1}\omega_{n-1} \tag{5}$$

with rational integral coefficients $c_0, c_1, \cdots, c_{n-1}$.

V. There exists an integral base in every algebraic field.

2.

The theory of algebraic integers cannot be immediately developed as in rational number theory since the uniqueness of prime decomposition is not valid. To remedy this difficulty we introduce the notion of divisibility for a rational prime modulus p.

Definition 4: One algebraic integer, ω_1, is divisible by another, ω_2, for the modulus p if there exists a rational integer c, prime to p, such that $c\omega_1/\omega_2$ is an algebraic integer.

If $N(\omega)$ is prime to p, then all integers of the field are divisible (mod p) by ω. For, since $N(\omega)\,\omega_1/\omega$ is an integer, we may choose $c=N(\omega)$.

If ω_1 is divisible (mod p) by ω_2, the power of p contained in $N(\omega_1)$ is not less than the power contained in $N(\omega_2)$. For there exists a c such that $(c, p)=1$ and $c\omega_1/\omega_2$ is an integer. Hence its norm $c^nN(\omega_1)/N(\omega_2)$ is an integer, and the theorem follows.

If ω_1 and ω_2 are each divisible by the other (mod p) we shall call them associate (mod p) and we shall regard them as identical in questions of divisibility (mod p).

Definition 5: The algebraic integer π is a prime (mod p) if π can divide a product $\omega_1\omega_2$ only when it divides at least one of the factors ω_1, ω_2.

We shall determine the decomposition of p in the field in the form

$$c\cdot p = \pi_1\cdot\pi_2\cdots\pi_k, \tag{6}$$

where $(c, p)=1$ and $\pi_i(i=1, 2, \cdots, k)$ is prime (mod p). Let us consider the expression

$$c_0\omega_0 + c_1\omega_1 + \cdots + c_{n-1}\omega_{n-1}, \tag{7}$$

where $\omega_0, \omega_1, \cdots, \omega_{n-1}$ is an integral base of the field. By giving to each of the coordinates $c_0, c_1, \cdots, c_{n-1}$ the values $0, 1, 2, \cdots, p-1$ (mod p) we obtain $\sigma=p^n$ different numbers $\alpha_0\equiv 0, \alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$.

If $N(\alpha_i)$ is prime to p for all $i\neq 0$ then p is a prime in the field. To prove this let us note that every integer of the field can be written in the form $\alpha_i+p\omega'$, where ω' is an integer. Hence

$$N(\omega) = N(\alpha_i) + p\Omega, \tag{8}$$

where Ω is an integer, i.e.,

$$N(\omega) \equiv N(\alpha_i) \pmod{p}. \tag{9}$$

Therefore $N(\omega)$ is divisible by p only if $N(\alpha_i)$ is divisible by p, i.e., according to our hypothesis, if $\alpha_i\equiv 0$. That is, $N(\omega)$ is divisible by p only if ω is divisible by p. Suppose now that $\omega_1\,\omega_2$ is divisible (mod p) by p, i.e. $c\,\omega_1\omega_2/p$ is an integer where $(c, p)=1$. Then its norm $c^nN(\omega_1)N(\omega_2)/p^n$ is a rational integer and hence either $N(\omega_1)$ or $N(\omega_2)$ must be divisible by p, i.e., either ω_1 or ω_2 must be divisible by p. Hence p is a prime in the field.

As a preliminary to an important theorem of Zolotarev, let us choose an arbitrary integer ω and consider the integers $\omega+p^\nu\omega'$ where ω' runs through the integers of the field. Among these integers we choose one, $\Omega=\omega+p^\nu\Omega'$, whose norm contains the least power of p. This choice can be made from the finite set for which

$$\omega' = c_0'\omega_0 + c_1'\omega_1 + \cdots + c_{n-1}'\omega_{n-1}, \tag{10}$$

where each of the coordinates $c'_0, c'_1, \cdots, c'_{n-1}$ takes the values $0, 1, 2, \cdots p^{\lambda-\nu+1}-1$, and p^λ is the power of p contained in $N(\omega)$. For Ω' may be written in the form

$$\Omega' = \omega' + p^{\lambda-\nu+1}\omega'', \tag{11}$$

where ω' is one of the numbers (10), i.e.,

$$\therefore\quad \Omega = (\omega + p^\nu\omega') + p^{\lambda+1}\omega''.$$

Hence

$$N(\Omega) \equiv N(\omega + p^\nu\omega') \pmod{p^{\lambda+1}}. \tag{12}$$

It follows that $N(\Omega)$ and $N(\omega+p^\nu\omega')$ contain the same power of p. In other words, we may assume $\Omega=\omega+p^\nu\omega'$ where ω' is one of the set (10).

The generalized theorem of Zolotarev is the following:

Theorem: If among the norms $N(\omega+p^{\nu}\omega')$, where ω' runs over all integers of the field, $N(\omega)$ contains the least power of p then p^{ν} is divisible (mod p) by ω.

Suppose ω satisfies the conditions of the theorem and is a root of the equation

$$(13)\quad f(\omega) = \omega^n + p^{\lambda_{n-1}}\cdot c_1\omega^{n-1} + p^{\lambda_{n-2}}\cdot c_2\omega^{n-2} + \cdots + p^{\lambda_1}c_{n-1}\omega + p^{\lambda}c_n = 0,$$

where $c_1, c_2, \cdots, c_n$ are prime to p. Let $\mu = r/s$ be the greatest of the numbers

$$\frac{\lambda-\lambda_1}{1}, \frac{\lambda-\lambda_2}{2}, \cdots, \frac{\lambda-\lambda_{n-1}}{n-1}, \frac{\lambda-\lambda_n}{n},$$

and suppose $(r, s) = 1$.

We shall prove that $p^{\mu}c_n/\omega$ is an integer. Multiplying (13) by $p^{n\mu-\lambda}\cdot c_n^{n-1}/\omega^n$ and rewriting in the reverse order, we see that $p^{\mu}c_n/\omega$ satisfies the equation

$$(14)\quad \left(\frac{c_np^{\mu}}{\omega}\right)^n + p^{\lambda_1+\mu-\lambda}c_{n-1}\left(\frac{c_np^{\mu}}{\omega}\right)^{n-1} + \cdots + p^{\lambda_{n-1}+\mu(n-1)-\lambda}c_1\cdot c_n{}^{n-1}\left(\frac{c_np^{\mu}}{\omega}\right) + p^{\mu n-\lambda}c_n{}^{n-1} = 0,$$

where the coefficients are algebraic integers since, by definition, $\mu \geqq (\lambda-\lambda_h)/h$, or $\lambda_h+\mu h-\lambda \geqq 0$. [The coefficients in (14) may not be rational since μ may be fractional.] It follows that

$$(c_np^{\mu}/\omega)^s = c_n{}^sp^r/\omega^s$$

is also an integer of the field.

Since $N(\omega)$ contains p^{λ}, the norms of all numbers

$$(15)\qquad \zeta = \omega - p^{\nu}\left(\frac{c_n{}^sp^r}{\omega^s}\right)^i = \omega - \frac{p^{ri+\nu}c_n{}^{si}}{\omega^{si}} \qquad (i = 1, 2, \cdots)$$

will contain p to a degree not less than λ. But

$$N(\zeta) = N\left(\omega - \frac{p^{ri+\nu}c_n{}^{si}}{\omega^{si}}\right) = \frac{N(\omega^{si+1} - p^{ri+\nu}c_n{}^{si})}{[N(\omega)]^{si}}.$$

Hence it follows that $N(\omega^{si+1}-p^{ri+\nu}c_n^{si})$ is divisible by p to a degree at least equal to $\lambda(s_i+1)$.

If $\epsilon = e^{2\pi i/(si+1)}$ is a primitive root of unity, we have the identity

$$x^{si+1} - y^{si+1} = (x-y)(x-\epsilon y)\cdots(x-\epsilon^{si}y).$$

By applying this identity to $\omega^{si+1}-p^{ri+\nu}\cdot c_n^{si}$ and taking the norm we obtain

$$(16)\qquad N(\omega^{si+1} - p^{ri+\nu}c_n{}^{si}) = \prod_{j=0}^{si} N(\omega - \epsilon^jp^{(ri+\nu)/(si+1)}c_n{}^{si/(si+1)}),$$

where in taking the norms, the quantities ϵ, $p^{(ri+\nu)/(si+1)}$, $c_n{}^{si/(si+1)}$ must be treated as rational numbers. Applying formula (4), we have

$$N(\omega - \epsilon^i p^{(ri+\nu)/(si+1)} \cdot c_n^{si/(si+1)}) = \pm f(\epsilon^i \cdot p^{(ri+\nu)/(si+1)} c_n^{si/(si+1)}). \tag{17}$$

Performing the substitutions on the right, the successive terms from (13) will contain p with degrees

$$n \cdot \frac{ri+\nu}{si+1}, \quad \lambda_{n-1} + (n-1)\frac{ri+\nu}{si+1}, \cdots, \lambda_1 + \frac{ri+\nu}{si+1}, \lambda, \tag{18}$$

respectively. If we choose i in (15) so that no two of (18) are equal, then the degree of p contained in the norm (17) will be the least of the numbers (18). We may choose i so that no two are equal except in case the equality

$$h\frac{ri+\nu}{si+1} + \lambda_h = k\frac{ri+\nu}{si+1} + \lambda_k, h \neq k,$$

is satisfied for every i. That is

$$(h-k)(ri+\nu) = (\lambda_k - \lambda_h)(si+1),$$

or

$$(h-k)r = (\lambda_k - \lambda_h)s \quad \text{and} \quad (h-k)\nu = (\lambda_k - \lambda_h).$$

Hence i may be properly chosen unless $\nu = (r/s) = \mu$. We shall exclude this case for the present.

We are now able to show that $\mu \leqq \nu$. For suppose $\mu > \nu$, i.e., $r > s\nu$. Adding rsi to both sides of the inequality we have

$$r(si+1) > s(ri+\nu) \quad \text{or} \quad \mu = \frac{r}{s} > \frac{ri+\nu}{si+1}. \tag{19}$$

On the other hand, from the definition of μ, there exists a subscript f for which

$$\mu = (\lambda - \lambda_f)/f \quad \text{or} \quad \lambda = \lambda_f + \mu f.$$

Hence, from (19),

$$\lambda > \lambda_f + f(ri+\nu)/(si+1). \tag{20}$$

Since the expression on the right is one of (18) we conclude that (17), for each j, contains p to a degree less than λ. That is, the product (16) contains p to a degree less than $\lambda(si+1)$. This contradicts the statement made above. Hence $\mu \leqq \nu$. The excluded case, $\mu = \nu$, is included in the conclusion.

It follows immediately that $(p^\nu \cdot c_n/\omega) = p^{\nu-\mu} \cdot (p^\mu \cdot c_n/\omega)$ is an algebraic integer, i.e., p^ν is divisible by ω (mod p). Q. E. D.

Applying this theorem to the case $\nu = 1$, it is seen that $\alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$ may be chosen so that p is divisible (mod p) by each of them.

We shall now prove the existence of the greatest common divisor (mod p) of two integers ω_1 and ω_2. Let ω_3 be an integer of the form $\omega_1 + \omega_2\omega'$ whose norm contains the least power of p. As above, it is sufficient to consider ω' of the

form $c_0'\omega_0+c_1'\omega_1+\cdots+c_{n-1}'\omega_{n-1}$ where each c' takes on the values $0, 1, 2, \cdots$, $p^\nu-1$ and p^ν is the highest power of p dividing $N(\omega_1)$ or $N(\omega_2)$.

Let p^ν be the least power of p divisible (mod p) by ω_2. It exists, since if $N(\omega)=c\cdot p^{\nu'}$ then $p^{\nu'}$ is divisible (mod p) by ω, and it may be determined in a finite number of steps. By definition of ω_3, among the norms $N(\omega_3+\omega_2\omega')$ the norm $N(\omega_3)$ contains the least power of p. Multiplying by the integer $\omega_4=(cp^\nu/\omega_2)$, it follows that of all norms of the type

$$N[\omega_4(\omega_3+\omega_2\omega')] = N(\omega_4\omega_3+p^\nu c\omega')$$

the norm of $\omega_4\omega_3$ contains the least power of p. But ω' is equivalent to $c'\omega''$, where ω'' runs through all integers of the field and

$$c\cdot c' \equiv 1 \pmod{p^\mu}$$

for sufficiently large μ. Hence, of all norms $N(\omega_4\cdot\omega_3+p^\nu\cdot c\cdot\omega')$, the norm of $\omega_4\omega_3$ contains the least power of p, and, by the generalized theorem of Zolotarev, it follows that p^ν is divisible (mod p) by $\omega_4\cdot\omega_3$. But $cp^\nu=\omega_2\cdot\omega_4$, and hence, for some c'',

$$c''\cdot c\cdot\omega_2\cdot\omega_4/(\omega_4\cdot\omega_3) = c''\cdot c\cdot\omega_2/\omega_3$$

is an integer i.e., ω_2 is divisible (mod p) by ω_3. It follows directly that $\omega_1=\omega_3-\omega_2\cdot\omega'$ is divisible (mod p) by w_3.

The form of $\omega_3=\omega_1+\omega_2\omega'$ indicates that every divisor of ω_1 and ω_2 is also a divisor of ω_3, and therefore the term "greatest common divisor" is justified. It is readily proved that an interchange of ω_1 and ω_2 will lead to an associate of ω_3.

If ω' can be chosen so that $N(\omega_1+\omega_2\omega')$ is not divisible by p, then ω_1 and ω_2 will be called relatively prime (mod p).

We shall prove the following lemmas concerning divisibility (mod p).

Lemma 1: If $\alpha\beta$ is divisible by γ, and α is prime to γ, then β is divisible by γ.

In fact, if α and β are relatively prime there exists an integer $\epsilon=\alpha+\gamma\omega'$ such that $N(\epsilon)$ is not divisible by p. Multiplying by the integer $\epsilon'\beta=(c/\epsilon)\beta$, we obtain

$$\beta\cdot c = \alpha\beta\cdot\epsilon' + \gamma\cdot\epsilon'\beta\cdot\omega'.$$

But, by hypothesis, $c'\alpha\beta/\gamma$ is an integer. Hence

$$(\beta/\gamma)c\cdot c' = (\alpha\beta/\gamma)\epsilon\cdot c' + \epsilon'\beta\cdot\omega' c'$$

is an integer, i.e., β is divisible (mod p) by γ.

Lemma 2. If α and β are both prime to γ then $\alpha\beta$ is prime to γ.

For we may choose $\epsilon'=\alpha+\gamma\omega'$ and $\epsilon''=\beta+\gamma\omega''$ so that $N(\epsilon)$ and $N(\epsilon')$ are not divisible by p. Multiplying, we obtain

$$\epsilon\cdot\epsilon' = \alpha\beta + \gamma(\alpha\omega'' + \beta\omega' + \gamma\omega'\cdot\omega'')$$

where $N(\epsilon\cdot\epsilon')$ is not divisible by p, i.e., $\alpha\beta$ is prime to γ.

Lemma 3: If α is divisible by two relatively prime integers β and γ then it is divisible by $\beta\cdot\gamma$.

For, if α is divisible by β, then, for some c prime to p, $c\alpha/\beta$ is an integer. But $c\alpha=(c\alpha/\beta)\beta$ is divisible by γ. Hence, since β is prime to γ, it follows that $c\alpha/\beta$ is divisible by γ. That is, for some c' prime to p, $c'c\alpha/(\beta\gamma)$ is an integer. Hence α is divisible by $\beta\gamma$.

If π is a prime (mod p) then every integer of the field is either divisible by π or prime to π. For if ω is not divisible by π, by Definition 5, it follows that π and ω have no common divisor (mod p).

We are now prepared to factor p into its prime factors (mod p). The factors may be found in the system $\alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$. For every integer of the field can be written in the form $\alpha_i+p\omega'$ where p is divisible by α_i and hence $\alpha_i+p\omega'$ is divisible by α_i (mod p).

If the norms of $\alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$ are prime to p, it follows directly that p is a prime (mod p) in the field.

Suppose that some of the norms $N(\alpha_1), N(\alpha_2), \cdots, N(\alpha_{\sigma-1})$ are divisible by p. Then if π_1 is that α whose norm contains the least power of p, say p^{f_1}, then π_1 is a prime. To prove this, let us show that any integer ω of the field is either divisible by π_1 or prime to it. If there exists a number of the type $\pi_1+\omega\cdot\omega'$ whose norm does not contain p, then π_1 is prime to ω. On the other hand if $N(\pi_1+\omega\cdot\omega')$, for arbitrary ω', is divisible by p it must be divisible by p^{f_1}, since any such integer can be expressed as $\alpha_i+p\omega''$. Hence among all $N(\pi_1+\omega\cdot\omega')$ the norm $N(\pi_1)$ contains the least power of p, i.e., ω is divisible (mod p) by π_1.

Let $\pi_1^{e_1}$ be the greatest power of π_1 dividing p (mod p). The integer $c\cdot p/\pi_1^{e_1}$ will not be divisible by π_1 and consequently will be prime to it.

Let us now choose among those of $\alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$ which are prime to π_1 an integer π_2 whose norm contains the least power of p, say p^{f_2}. Then π_2 is a prime (mod p). For let ω be an arbitrary integer of the field and consider all integers $\pi_2+\pi_1\omega\cdot\omega'$. If there exists an integer $\pi_2+\pi_1\omega\cdot\omega'$ whose norm does not contain p then π_2 is prime to ω. If not, $N(\pi_2+\pi_1\omega\omega')$ will be divisible by at least p^{f_2} since we may write

$$\pi_2+\pi_1\omega\cdot\omega' = \alpha_i + p\omega'',$$

where $\pi_1\omega\omega'$ and $p\omega''$ are divisible by π_1, and π_2 is prime to π_1, i.e., where α_i is prime to π_1. Hence $\pi_1\omega$ is divisible by π_2 and, by Lemma 1, π_2 divides ω.

If $\pi_2^{e_2}$ is the highest power of π_2 dividing p (mod p) then, by Lemma 3, p must be divisible by $\pi_1^{e_1}\cdot\pi_2^{e_2}$ and the quotient $cp/(\pi_1^{e_1}\cdot\pi_2^{e_2})$ will be prime to both π_1 and π_2.

By selecting from among those of $\alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$ which are prime to $\pi_1\cdot\pi_2$ the integer π_3 whose norm contains the least power of p we obtain another prime factor of p (mod p). Since each π_i is a distinct member of the set $\alpha_1, \alpha_2, \cdots, \alpha_{\sigma-1}$, the complete decomposition may be found in a finite number of steps. We have then

$$cp = \epsilon\cdot\pi_1^{e_1}\cdot\pi_2^{e_2}\cdots\pi_k^{e_k} \tag{21}$$

where c is prime to p. It is readily shown that this decomposition is unique in the sense that if we also have

$$c'p = \epsilon' \cdot \pi_1'^{e_1'} \pi_2'^{e_2'} \cdots \pi_{k'}'^{e_{k'}'} \tag{22}$$

then the factors of (21) and (22) must be associate in pairs. For we have

$$c'\epsilon \cdot \pi_1^{e_1} \cdot \pi_2^{e_2} \cdots \pi_k^{e_k} = c\epsilon' \cdot \pi_1'^{e_1'} \cdot \pi_2'^{e_2'} \cdots \pi_{k'}'^{e_{k'}'}. \tag{23}$$

Hence the left of (23) is divisible (mod p) by π_1' and one of the factors, say π_1, must be divisible by π_1'. But since both π_1 and π_1' are prime, each is divisible by the other and hence they are associate. Dividing (23) by π_1' and repeating the process it follows that $e_1 = e_1'$. A similar conclusion follows for the other factors.

We shall now prove that $\pi_1, \pi_2, \cdots, \pi_k$ are the only primes (mod p) in the field. Let ω be an arbitrary integer of the field divisible (mod p) by $\pi_1^{m_1}, \pi_2^{m_2}, \cdots, \pi_k^{m_k}$ and therefore by their product. The quotient

$$\bar{\omega} = \frac{c' \cdot \omega}{\pi_1^{m_1} \cdot \pi_2^{m_2} \cdots \pi_k^{m_k}}$$

is prime with each π_i and consequently with $c \cdot p = \pi_1^{e_1} \cdot \pi_2^{e_2} \cdots \pi_k^{e_k}$, i.e., it is possible to choose an ω' so that $N(\bar{\omega} + p\omega')$ is not divisible by p. By (9), it follows that $\bar{c} = N(\bar{\omega})$ is prime to p. Multiplying $\bar{\omega}$ by $\bar{\bar{\omega}} = (\bar{c}/\bar{\omega})$ we have

$$\bar{c} = \frac{c' \cdot \omega \cdot \bar{\bar{\omega}}}{\pi_1^{m_1} \cdot \pi_2^{m_2} \cdots \pi_k^{m_k}},$$

i.e.,

$$\frac{\bar{c} \cdot \pi_1^{m_1} \cdot \pi_2^{m_2} \cdots \pi_k^{m_k}}{\omega} = c \cdot \bar{\bar{\omega}}.$$

Hence the product $\pi_1^{m_1} \cdot \pi_2^{m_2} \cdots \pi_k^{m_k}$ is divisible (mod p) by ω. Hence $\pi_1^{m_1} \cdot \pi_2^{m_2} \cdots \pi_k^{m_k}$ is the decomposition of ω into prime (mod p) factors.

Taking the norm of (21), we have

$$c^n \cdot p^n = [N(\pi_1)]^{e_1} [N(\pi_2)]^{e_2} [N(\pi_3)]^{e_4} \cdots [N(\pi_k)]^{e_k}. \tag{24}$$

It follows immediately from (24) that

$$n = e_1 f_1 + e_2 f_2 + \cdots + e_k f_k. \tag{25}$$

The number f_i is called the degree of π_i.

In the case where the discriminant of the equation $f(x) = 0$ which defines the field is not divisible by p the decomposition of p may be found very simply. Let

$$f(x) \equiv f_1(x) f_2(x) f_3(x) \cdots f_k(x) \pmod{p}, \tag{26}$$

where $f_i(x)$, $i=1, 2, \cdots, k$, is irreducible (mod p). We may write

$$f(x) = f_1(x)f_2(x) \cdots f_k(x) + p\cdot\phi(x), \tag{27}$$

where $\phi(x)$ is not divisible by $f_i(x)$. Hence, if ω is a root of $f(x)=0$,

$$-p\phi(\omega) = f_1(\omega)f_2(\omega) \cdots f_k(\omega). \tag{28}$$

The numbers $\pi_i = f_i(\omega)$, $i=1, 2, \cdots, k$, are primes (mod p). For every integer α of the field can be written in the form

$$c\alpha = a_0 + a_1\omega + \cdots + a_{n-1}\omega^{n-1} = \psi(\omega),$$

where a and c are rational integers and $(c, p)=1$. The polynomial $\psi(x)$ is either divisible by $f_i(x)$ (mod p) or prime to it. In the first case α is divisible (mod p) by π_i, in the second case it is prime to π_i.

Let us note that the degree of the prime π_i will be equal to the degree of the corresponding polynomial. The method is also valid for some primes which divide the discriminant. The prime must not divide the index of ω, (i.e., it must not be an "ausserwesentlicher Diskriminantenteiler").

3.

Let us consider the problem of determining the divisors of an integer ω of the field for all rational prime moduli. We need consider as moduli only the rational primes dividing $N(\omega)$. (For other moduli the integer ω obviously plays the role of unit.)

Definition: To each prime divisor π (mod p) we associate a symbol $\mathfrak{P}$, called a prime divisor, and say that ω contains the divisor $\mathfrak{P}^k$ when ω is divisible (mod p) by π^k. The norm of $\mathfrak{P}$ is defined as the degree of p dividing $N(\pi)$. The product of several prime divisors we call a divisor.

Let us represent ω symbolically as the product of its divisors $\mathfrak{P}$ for all rational prime divisors of $N(\omega)$ as moduli, and define the norm of a product of prime divisors as the product of the norms. We see that $N(\omega)$ is equal to the product of the norms of the divisors of ω.

We shall make use of the following lemmas:

Lemma 1: If α is divisible (mod p) by β we may determine an integer $c\alpha/\beta$ where $(c, p)=1$ and c divides $N(\beta)$.

For let $\lambda=(c\alpha/\beta)$ be an integer. Then $\mu=(N(\beta)\alpha/\beta)$ is also an integer. The equation $c\cdot X-N(\beta)\cdot Y=d$, where d is the greatest common divisor of c and $N(\beta)$ has an integral rational solution X, Y. Then $\lambda X-\mu Y=(d\alpha/\beta)$ is an integer and $(d, p)=1$ since d divides c. Furthermore d divides $N(\beta)$.

Lemma 2: If α is divisible by β for all rational prime divisors of $N(\beta)$ as moduli, then α is divisible by β algebraically.

Let $N(\beta)=p_1^{w_1}p_2^{w_2}\cdots p_k^{w_k}$. By Lemma 1 there exist integers of the field

$$\lambda_1 = \frac{M_1\alpha}{\beta},\ \lambda_2 = \frac{M_2\alpha}{\beta},\ \cdots \lambda_k = \frac{M_k\alpha}{\beta},$$

where M_i, divides $N(\beta)/p_i^{w_i}$, $i=1, 2, \cdots, k$. Hence $M_1, M_2, \cdots, M_k$ have no common divisor and

$$M_1X_1 + M_2X_2 + \cdots + M_kX_k = 1$$

has a solution in rational integers. Then

$$\lambda_1X_1 + \lambda_2X_2 + \cdots + \lambda_kX_k = \frac{\alpha}{\beta}$$

is an integer, i.e., α is algebraically divisible by β.

Lemma 3: If α and β are relatively prime for all rational primes as moduli which divide both $N(\alpha)$ and $N(\beta)$, then they are relatively prime algebraically i.e., there exist integers λ and μ such that

$$\alpha\lambda + \beta\mu = 1. \tag{29}$$

For let $p_1, p_2, \cdots, p_k$ be the primes dividing both $N(\alpha)$ and $N(\beta)$. By hypothesis there exists an integer ω_i for each i such that

$$\alpha + \beta\omega_i = \epsilon_i,$$

where $N(\epsilon_i)$ is prime to p_i. Multiplying by $\epsilon_i' = (N(\epsilon_i)/\epsilon_i)$ we have

$$\alpha\rho_i + \beta\sigma_i = N(\epsilon_i), \tag{30}$$

where ρ_i and σ_i are integers of the field. We also have

$$\begin{aligned} \alpha\cdot\alpha' &= N(\alpha), \\ \beta\cdot\beta' &= N(\beta), \end{aligned} \tag{31}$$

where α' and β' are integers. Since $N(\epsilon_1), N(\epsilon_2), \cdots, N(\epsilon_k), N(\alpha), N(\beta)$ have no common divisor, the equation

$$N(\epsilon_1)X_1 + N(\epsilon_2)X_2 + \cdots + N(\epsilon_k)X_k + N(\alpha)Y_1 + N(\beta)Y_2 = 1$$

has an integral rational solution. Hence, from (30) and (31)

$$\begin{aligned} &\alpha(\rho_1X_1 + \rho_2X_2 + \cdots + \rho_kX_k + \alpha'Y_1) \\ &\qquad + \beta(\sigma_1X_1 + \sigma_2X_2 + \cdots + \sigma_kX_k + \beta'Y_2) = 1, \end{aligned} \tag{32}$$

i.e., we have an equality of type (29).

Let us now prove the theorem:

Theorem: The algebraic integer ω is defined by its divisors to within an algebraic unit.

For if two integers ω and ω' have the same divisors each must be divisible by the other for all rational prime moduli which divide $N(\omega) = \pm N(\omega')$ and hence, by Lemma 2, each must divide the other algebraically, i.e., their quotient is an algebraic unit.

The following theorem establishes the independence of the divisors.

Theorem: Given a divisor $\mathfrak{M}$ and a rational integer M, an algebraic integer μ may be determined so that $\mu/\mathfrak{M}$ is prime to M, (i.e., its norm, $N(\mu)/N(\mathfrak{M})$, is prime to M.)

Let us first prove the theorem for a prime divisor $\mathfrak{P}$ dividing $\mathfrak{M}$. Let π be the prime (mod p) corresponding to $\mathfrak{P}$ and let $N(\mathfrak{P})=p^k$, $N(\pi)=c\cdot p^k$ where $(c, p)=1$. Suppose $M=M'p^m$ where $(M', p)=1$. Then $\pi'=M'\pi+p^{k+1}$ will satisfy the condition of the theorem since

$$N(\pi') \equiv N(p^{k+1}) = p^{n(k+1)} \pmod{M'},$$
$$N(\pi') \equiv M'^n\cdot c\cdot p^k \pmod{p^{k+1}}.$$

To determine a μ for the divisor $\mathfrak{M}$, construct such integers for each of its prime divisors and multiply them.

Let us compare the divisor concept with Dedekind's ideal numbers. Dedekind's definition of an ideal is the following:

A set of integers of a field is called an ideal if: (a) The sum of any two integers of the set belongs to the set. (b) The product of any integer of the set with an arbitrary integer of the field belongs to the set.

We shall prove that the concept of an ideal is equivalent to the concept of the set of all integers of the field divisible by a given divisor. It is obvious that the set of all integers divisible by a given divisor is an ideal. Let us prove the converse. We choose an ideal $\overline{\mathfrak{M}}$ by means of its base $\mu_1, \mu_2, \cdots, \mu_n$ and determine the common divisor $\mathfrak{M}$ of $\mu_1, \mu_2, \cdots, \mu_n$. It will be necessary to consider the rational primes $p_1, p_2, \cdots$ which divide the greatest common divisor D of $N(\mu_1), N(\mu_2), \cdots, N(\mu_n)$. Let

$$\mathfrak{M} = (\mathfrak{P}_{11}^{w_{11}}\cdot\mathfrak{P}_{12}^{w_{12}}\cdots\mathfrak{P}_{1k_1}^{w_{1k_1}})(\mathfrak{P}_{21}^{w_{21}}\mathfrak{P}_{22}^{w_{22}}\cdots\mathfrak{P}_{2k_2}^{w_{2k_2}})\cdots \tag{33}$$

where $\mathfrak{P}_{ji}$ corresponds to prime divisors for the modulus p_j. Clearly each number in $\overline{\mathfrak{M}}$ will be divisible by the divisor $\mathfrak{M}$. We must show that if an integer ω of the field is divisible by $\mathfrak{M}$, then ω belongs to $\overline{\mathfrak{M}}$. Let us consider separately each rational prime modulus in $N(\mathfrak{M})$. Suppose $\lambda=b_1\mu_1+b_2\mu_2+\cdots+b_n\mu_n$ is an integer in $\overline{\mathfrak{M}}$ divisible by $\mathfrak{P}_{11}^{w_{11}}\mathfrak{P}_{12}^{w_{12}}\cdots\mathfrak{P}_{1k_1}^{w_{1k_1}}$ and having no other divisors (mod p_1). Such an integer must exist by definition of $\mathfrak{M}$. Then ω is divisible (mod p_1) by λ, i.e.,

$$c_1\omega = c_{11}\mu_1 + c_{12}\mu_2 + \cdots + c_{1n}\mu_n. \tag{34}$$

where $(c_1, p_1)=1$. Similarly

$$\begin{aligned} c_2\omega &= c_{21}\mu_1 + c_{22}\mu_2 + \cdots + c_{2n}\mu_n, \\ c_3\omega &= c_{31}\mu_1 + c_{32}\mu_2 + \cdots + c_{3n}\mu_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{34}$$

where $(c_i, p_i)=1$. The integers $N(\mu_1), N(\mu_2), \cdots, N(\mu_n)$ and hence their greatest common divisor D belong to the ideal $\overline{\mathfrak{M}}$. But D contains only the

prime divisors of $N(\mathfrak{M})$. Hence $D, c_1, c_2, \cdots$, have no common divisor and the equation

$$DY + c_1X_1 + c_2X_2 + \cdots = 1 \tag{35}$$

has an integral rational solution. But $D\omega$ belongs to $\overline{\mathfrak{M}}$ with D, i.e.,

$$D\omega = d_1\mu_1 + d_2\mu_2 + \cdots + d_n\mu_n. \tag{36}$$

Multiplying (36) by Y and (34) by $X_1, X_2, \cdots$, respectively, we obtain

$$\omega = (d_1Y + c_{11}X_1 + c_{21}X_2 + \cdots)\mu_1 + (d_2Y + c_{12}X_1 + c_{22}X_2 + \cdots)\mu_2 + \cdots + (d_nY + c_{1n}X_1 + c_{2n}X_2 + \cdots)\mu_n.$$

Hence ω belongs to $\overline{\mathfrak{M}}$. Q. E. D.

The theory of congruences for an ideal modulus may be carried through very simply by this method.

AN EXAMPLE OF THE IDEAL THEORY OF ZOLOTAREV

By H. T. ENGSTROM[1]

In the following note the prime ideal decomposition of a common index divisor (gemeinsamer ausserwesentlicher Diskriminantenteiler) is determined to illustrate the method of Zolotarev. The index divisors are the exception in Dedekind's theorem for the determination of the prime ideal decomposition of a rational prime p from the decomposition (mod p) of an equation defining the field.

Let us consider the field K generated by a root θ of the irreducible equation $f(x)=x^3-x^2-2x-8=0$. This field is given by Bachmann[2] as an example of a field having a common index divisor. The discriminant d_θ of θ is $-2^2\cdot 503$, and since $f(x)=x^2(x-1)+2(x-4)$, where $x-4\equiv 0 \pmod{2,\, x}$ it follows that 2 divides the index of θ. We shall determine the ideal decomposition of 2.

The integers $1, \theta, \eta=\frac{1}{2}\theta(\theta-1)-1$ form a base of K. Furthermore if $\alpha=a+b\theta+c\eta$, then

$$N(\alpha) = \begin{vmatrix} a & b & c \\ 2b+4c & a+b & 2b \\ 4b-2c & 2c & a-c \end{vmatrix}.$$

From $N(\alpha)$ we determine an integer in each congruence class (mod 2) whose norm contains the least power of 2 and obtain the set,

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[1] National Research Fellow, California Institute of Technology.

[2] P. Bachmann, *Allgemeine Arithmetik der Zahlenkörper*, 1926, pp. 280–284.

$$\alpha_0 = 0,\ \alpha_1 = 1,\ \alpha_2 = \theta + 2\eta,\ \alpha_3 = 2\theta + \eta,\ \alpha_4 = -1 + \theta,$$
$$\alpha_5 = 1 + \eta,\ \alpha_6 = 2 + \theta + \eta,\ \alpha_7 = 1 - \theta + \eta,$$

where

$$N(\alpha_0) = 0,\ N(\alpha_1) = 1,\ N(\alpha_2) = 100,\ N(\alpha_3) = 68,$$
$$N(\alpha_4) = 10,\ N(\alpha_5) = 10,\ N(\alpha_6) = 2,\ N(\alpha_7) = -4.$$

Since $N(\alpha_4)$ contains exactly 2^1 it follows from the general theory that 2 is divisible (mod 2) by the prime divisor $\alpha_4 = -1+\theta$. Furthermore $-1+\theta$ and $1+\eta$ are relatively prime (mod 2) since by choosing $a=0$, $b=1$, $c=3$ in the expression

$$\beta = 1 + \eta + (-1 + \theta)(a + b\theta + c\eta)$$

we obtain $\beta = 11$; i.e., the greatest common divisor (mod 2) of $-1+\theta$ and $1+\eta$ is a unit. Hence 2 is also divisible by the prime $1+\eta$. Similarly it follows that $2+\theta+\eta$ is prime (mod 2) to both $-1+\theta$ and $1+\eta$ and divides 2. Since $N(2) = 2^3$, there must exist a rational integer d prime to 2 and a unit ϵ (mod 2) of K such that

$$d \cdot 2 = \epsilon(2 + \theta + \eta)(-1 + \theta)(1 + \eta).$$

By writing $\epsilon = a + b\theta + c\eta$ and multiplying out it is seen that we may choose $\epsilon = 5 + 2\theta - 6\eta$ and $d = 25$. Hence in the prime ideal divisors of Zolotarev,

$$2 = \mathfrak{P}_1 \cdot \mathfrak{P}_2 \cdot \mathfrak{P}_3,\quad N(\mathfrak{P}_i) = 2,$$

where $\mathfrak{P}_1$ corresponds to the divisor $2+\theta+\eta$, $\mathfrak{P}_2$ to $-1+\theta$, and $\mathfrak{P}_3$ to $1+\eta$.

The decomposition of other rational primes may be determined from the decomposition of $f(x)$ in prime functions (mod 2) as in the theorem of Dedekind.

The decomposition of 2 in Dedekind ideals obtained by Bachmann (loc. cit.) is the following:

$$[2] = \mathfrak{p}_1 \cdot \mathfrak{p}_2 \cdot \mathfrak{p}_3,\quad N\mathfrak{p}_i = 2,$$

where

$$\mathfrak{p}_1 = [2,\theta,\eta],\qquad \mathfrak{p}_2 = [2,\theta + 1,\eta],\qquad \mathfrak{p}_3 = [2,\theta,\eta + 1].$$

It may be readily shown that the ideal $\mathfrak{p}_1$ consists of all integers of K divisible (mod 2) by $2+\theta+\eta$, $\mathfrak{p}_2$ of all integers divisible (mod 2) by $-1+\theta$, and $\mathfrak{p}_3$ of all integers divisible (mod 2) by $1+\eta$.

THE ORTHOPOLE LOCI OF SOME ONE-PARAMETER SYSTEMS OF LINES REFERRED TO A FIXED TRIANGLE

By O. J. RAMLER, The Catholic University of America

An orthopole is defined by J. Neuberg[1] as follows: Let there be a triangle ABC and a line L in the plane of the triangle. From A, B, and C draw perpendiculars to L cutting it in P, Q, and R, respectively. From P draw a perendicular to BC, from Q a perpendicular to AC, and from R a perpendicular to AB. These three perpendiculars meet in a point S which is called the orthopole of L with respect to the triangle ABC.

It is the purpose of this paper to determine and study the loci of orthopoles of some singly infinite systems of lines. A statement of some of the more important theorems to be found in the literature bearing on the subject is first given.

Theorem I[2]: *The orthocenter of a triangle is the orthopole of each side of the triangle referred to itself.*

Theorem II[2]: *The orthopole of a line perpendicular to a side of a triangle is the foot of that perpendicular.*

Theorems I and II follow immediately from the definition.

Theorem III: *The orthopoles of a system of parallel lines lie on a line perpendicular to that system.*

Theorem IV: (a) *The locus of orthopoles of a pencil of lines on a point P is a conic whose center is the midpoint of PH where H is the orthocenter of the reference triangle.*

(b) *If the point P is the circumcenter*[3] *of the reference triangle, the locus of orthopoles of the pencil of lines is the nine-point circle of the triangle.*

(c) *If P is on the circumcircle of ABC, and p is its pedal or Simson line referred to ABC, then the conic is the line p taken twice.*

Theorem V[4]: *The Simson lines of the extremities of any chord TT' of the circle ABC pass through the orthopole of TT'.*

Theorem VI[2]: *The locus of orthopoles of tangents to a curve of class m is a curve of order $2m$; the locus of lines whose orthopoles move on a curve of order n is a curve of class $2n$.*

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[1] J. Neuberg, *Sur une transformation des figures*, Nouvelle Correspondance Mathématique, vol. 4 (1878), p. 379.

[2] J. Neuberg, *Die Verwandtschaft zwischen einer Geraden und ihrem Lotpunkt in Bezug auf ein Dreieck*, Archiv der Mathematik und Physik, (3), vol. 3 (1902).

[3] M. Soons, Mathesis (1896), p. 57. K. Cwojdzinski, *Der Lotpunkt, ein neuer merkwürdiger Punkt des Dreiecks*, Archiv der Mathematik und Physik, (3), vol. 1 (1901), p. 175.

[4] Gallatly, *The Modern Geometry of the Triangle*, 2nd edition, Chapter VI, p. 49.

Neuberg has shown that the relation between a line m and its orthopole M is three to one; that is, a line m has a definite orthopole M, but a point M may be the orthopole of three distinct lines m_1, m_2, m_3. Three lines so related are said to be "associated." After showing that the vertices of the triangle $m_1m_2m_3$ (at least one of whose sides and the vertex opposite to it are real) lie on the circumcircle of the fundamental triangle ABC, Neuberg gives the following construction to find m_2 and m_3 when m_1 is known: Determine M as the orthopole of m_1 with respect to ABC; join the orthocenter H of ABC to M and produce it to H_m making $MH_m = HM$. Let m_1 meet the circumcircle of ABC in N_2 and N_3. Construct the triangle $N_1N_2N_3$ having H_m as orthocenter and ABC as circumcircle. This is done by drawing through N_2 a perpendicular to N_3H_m which meets the circle ABC again at N_1. Then N_1N_2 and N_1N_3 are the required lines m_3 and m_2, respectively. If m_1 does not meet the circle ABC, then the sides N_1N_2 and N_1N_3 are imaginary, meeting at the real vertex N_1, which is found by drawing N_1H_m perpendicular to m_1.

The theorems mentioned above and a number of additional theorems may be readily obtained by the use of "absolute" or "circular" coordinates.[5] Taking the vertices of our fundamental triangle ABC on a circle of unit radius and denoting the unit vectors from O, the circumcenter, to the vertices as α, β, γ, we call their symmetric functions $\sigma_1, \sigma_2, \sigma_3$. Let x be the vector to any moving point and y its conjugate. Then if the reflection of the origin O in any line m is u, we have as the conjugate equation of the line m

$$u^{-1}x + v^{-1}y = 1, \tag{a}$$

where v is the conjugate of u. The orthopole M of m with respect to ABC is then given by

$$x = \tfrac{1}{2}\sigma_1 + \tfrac{1}{2}u + \tfrac{1}{2}u^{-1}v\sigma_3. \tag{1}$$

To prove Theorem IV (b) we must put $u = v = 0$. The product vu^{-1} is a unit vector, say t. The vector to the orthopole of a circumdiameter is therefore given by the equation

$$x = \tfrac{1}{2}(\sigma_1 + t),$$

which is the map equation of the nine-point circle.

Theorem I is proved by putting $u = \frac{1}{2}(\beta+\gamma)$, $v = (\beta+\gamma)/2\beta\gamma$ in (1), whence $x = \sigma_1$, the vector to the orthocenter of ABC. Theorem III is proved by putting $u^{-1}v = \text{constant} = k$. Equation (1) becomes $2x = k + u$, where u runs along the normal from O to the system of parallel lines. The equation $2x = k + u$ says that the locus of orthopoles is homothetic to this normal with respect to the fixed point given by k. Hence the locus is a straight line.

Considering the line m as the line through the points t_1 and t_2 on the circum-

[5] Winger, *Projective Geometry*, p. 324. F. Morley, Transactions of the American Mathematical Society, vol. 1 (1900), pp. 97–115; vol. 4 (1903), p. 1; vol. 8 (1907), pp. 14–24.

circle of ABC we obtain another and sometimes more useful expression for the vector to the orthopole, namely

$$x = \tfrac{1}{2}\sigma_1 + \tfrac{1}{2}(t_1 + t_2) + \tfrac{1}{2}\sigma_3 t_1^{-1} t_2^{-1}. \tag{2}$$

Now associate a third point t_3 on the unit circle with t_1 and t_2, and let s_1, s_2, and s_3 be their symmetric functions; equation (2) may then be written

$$x = \tfrac{1}{2}(\sigma_1 + s_1) - \tfrac{1}{2}t_3 + \tfrac{1}{2}\sigma_3 t_3 s_3^{-1}. \tag{3}$$

This gives the orthopole of t_1t_2. Now write the equation

$$x = \tfrac{1}{2}(\sigma_1 + s_1) - \tfrac{1}{2}(1 - \sigma_3 s_3^{-1})t, \tag{4}$$

where t is any point on the circumcircle of ABC. Giving t the values t_3, t_2, t_1, we get in turn the orthopoles of lines t_1t_2, t_3t_1, and t_2t_3, respectively. Equation (4) is the map equation of a circle, the circumcircle of the three orthopoles of the sides of the triangle $t_1t_2t_3$, referred to ABC. The center of this circle is at $\frac{1}{2}(\sigma_1+s_1)$, the midpoint of the line segment joining the orthocenter of ABC to that of $t_1t_2t_3$. The radius is one-half the absolute value of $1-\sigma_3 s_3^{-1}$. The symmetry of these results suggests

Theorem VII: *If the product of the vectors to the vertices of one triangle inscribed to the unit circle is equal to the product of the vectors to the vertices of a second inscribed triangle, the midpoint of the line joining their orthocenters is the orthopole of any side of either triangle referred to the other.*

The Deltoid

In 1857 Steiner[6] stated the theorem that the envelope of the Simson lines referred to a triangle is a curve of class three and order four. Later Cremona[7] showed that this curve is a hypocycloid of three cusps inscribed to the triangle. Morley[8] has named this curve the *deltoid.* We shall refer to it as the Steiner deltoid of the triangle in which it is inscribed.

Since the altitudes and sides of a triangle are Simson lines of the corresponding vertices and their reflections in the circumcenter, respectively, we have as a consequence of theorems I and II the following:

Theorem VIII: *The orthopole locus of the tangents to the Steiner deltoid of a triangle ABC is a quartic curve having a triple point at the orthocenter, and passing through the vertices of the pedal triangle. The three tangents at the triple point are parallel to the sides of the triangle.*

According to Theorem VI the locus should properly be a sextic, but since

[6] J. Steiner, *Über eine besondere Curve dritter Klasse (und vierten Grades)*, Crelle's Journal vol. 53 (1857), p. 231.

[7] M. Cremona, *Sur l'hypocycloide à trois rebroussements*, Crelle's Journal, vol. 64 (1865), p. 101.

[8] F. Morley, *Orthocentric properties of the plane N-line*, Transactions of the American Mathematical Society, vol. 4 (1903), p. 1.

the deltoid has the line at infinity as a double tangent, the locus degenerates into a quartic and the line at infinity taken twice.

The pedal or Simson line of a point $P(t)$ of the circumference ABC has an equation

$$2tx - 2\sigma_3 y + \sigma_3 t^{-1} + \sigma_2 - t^2 - \sigma_1 t = 0, \tag{5}$$

where t is the vector to the point P. Differentiating with respect to t we obtain the map equation of its envelope,

$$x = \tfrac{1}{2}\sigma_1 + t + \tfrac{1}{2}\sigma_3 t^{-2}. \tag{6}$$

Equation (6) is then the map equation of the Steiner deltoid of ABC. Now write the equation,

$$x = \tfrac{1}{2}\sigma_1 + \tfrac{1}{2}(t_1 + t) + \tfrac{1}{2}\sigma_3 t^{-1} t_1^{-1}.$$

It is the equation of a two cusped hypocycloid, i.e., a line segment of length equal to that of the circumdiameter of ABC. Moreover this line segment meets the deltoid (6) where $t = t_1$ and has at that point the same direction as the deltoid. The segment is therefore tangent to the deltoid. If we put $t = t_2$ in (7) we again obtain equation (2) and x is then the vector to the point of intersection M of the tangents to the deltoid given by $t = t_1$ and $t = t_2$. Since these tangents are the Simson lines of ABC for the points t_1 and t_2 we have a proof of Theorem V. The deltoid being of class three, three tangents may be drawn to it from M. Let t_3 be the parameter giving the third tangent from M. Then triangle $t_1t_2t_3$ is such that its sides are the three lines each of which has M as its orthopole with respect to ABC. Whence we have:

Theorem IX: *The three lines that have a given point M as their orthopole with respect to a triangle ABC are real when M lies inside the Steiner deltoid of ABC; two of the three lines become coincident when M lies on the deltoid and but one line is real when M lies outside the deltoid. The coincidence of two of the three lines also requires the coincidence of two of the three parameters $t_1t_2t_3$.* Hence we have the:

Corollary: *The locus of the orthopoles of the tangents to the circumcircle of ABC is the Steiner deltoid of ABC.*

This result could easily have been obtained from equation (1) by putting $u = 2t$, $v = 2/t$. We merely remark here that the triangle $t_1t_2t_3$ also has a Steiner deltoid whose size and orientation are the same as those of the Steiner deltoid of ABC, and which is obtained by merely translating the latter curve by an amount equal to and in the direction of the vector from H to M, where H is the orthocenter of ABC.

We turn our attention next to Theorem IV (a) which was first stated by Neuberg.[9] Cwojdzinski,[10] without proof, states the following:

[9] Archiv der Mathematik und Physik, (3), vol. 3 (1902), p. 89.

[10] Archiv der Mathematik und Physik, (3), vol. 1 (1901), p. 178.

Theorem X: *If the lengths of the perpendiculars dropped from any point P upon the sides of ABC are laid off from the orthocenter on the upper segments of the altitudes, then the three points thus obtained together with the feet of the perpendiculars from P will lie on an ellipse whose center bisects the line segment joining the orthocenter of ABC to P.*

We have no way of knowing from Cwojdzinski's article that he was aware of the facts as stated in Theorem IV (a), particularly since Neuberg did not publish his theorem until the following year. Neuberg merely states that the locus of orthopoles of lines on a point P is a conic, but he does not give the type of conic. We now prove:

Theorem XI: *The locus of orthopoles of the lines on a fixed point with respect to a fixed triangle is an ellipse.* This includes Cwojdzinski's theorem.

Letting the vector to the fixed point P be z, and z' its conjugate, the equation of any line through P will be

$$x - z = t(y - z') ,$$

whence

$$u = z - tz', \quad v = (tz' - z)/t, \quad v/u = -1/t.$$

Hence the orthopole of any line of the pencil given by parameter t is

$$x = \tfrac{1}{2}(\sigma_1 + z) - \tfrac{1}{2}(tz' + \sigma_3 t^{-1}) \tag{8}$$

which is the map equation of an ellipse, because the elimination of t from (8) and its conjugate yields an equation of the second degree in x and y, and for no value of t as it moves on the unit circle can x become infinite. The major and minor axes of the ellipse are $1+|z'|$ and $|(1-|z'|)|$, respectively; that is if r is the circumradius of ABC, the major and minor axes are $r \pm OP$. The center is at $\frac{1}{2}(\sigma_1+z)$, that is, at the midpoint of HP, where H is the orthocenter of ABC. Putting $t=\beta\gamma$ in (8), we obtain the foot of the perpendicular from P upon BC. This point is therefore on the ellipse. Putting $t=-\beta\gamma$ we readily verify the second part of Theorem X. Again putting $z=t_1$ we obtain the locus of orthopoles of lines on a point of the circumcircle of ABC, namely,

$$x = \tfrac{1}{2}(\sigma_1 + t_1) - \tfrac{1}{2}(tt_1^{-1} + \sigma_3 t^{-1}),$$

which is a two-cusped hypocycloid, or since $1-|t_1|=0$, an ellipse with zero minor axis; we have thus proved Theorem IV (c). The foci of the ellipse (8) are given by $dx/dt=0$, namely by $t=\pm(\sigma_3/z')^{1/2}$. The vectors to the foci are then

$$\tfrac{1}{2}(\sigma_1 + z) \pm (\sigma_3 z')^{1/2}.$$

Interpreting the above analysis geometrically we have the following scheme for the *determination of the directions of the axes of the ellipse locus*:

As axis of reals choose some line as OA. Then $\sigma_3^{1/2}$ is the midpoint of arc BC; call it X. Bisect angle AOP, letting the bisector cut the circumcircle at Q. Locate Y on the circumcircle so that angle $AOY = AOX - AOQ$. Then OY is parallel to the major axis.

THE INSCRIBED ELLIPSE

The ellipse (8) intersects the side BC in points whose parameters are $t = \beta\gamma$ and $t = \alpha(z+\alpha)/(1+\alpha z')$, the former giving the foot of the perpendicular from P upon BC, the latter giving the second intersection which may be determined as the orthopole of a line through P parallel to OM, where M is the midpoint of AP. Equating the expressions for the two intersections we shall have a locus for P which will give ellipses tangent to BC. The locus is found to be the straight line,

$$z - \beta\gamma z' = \beta\gamma\alpha^{-1} - \alpha, \tag{9}$$

whose equation is satisfied by $z = -\sigma_1$ and $z = -\alpha$. Moreover the line (9) is perpendicular to BC; in fact it is the homothetic with respect to the circumcenter of the altitude upon BC. From the symmetry of the result, $z = -\sigma_1$, we have

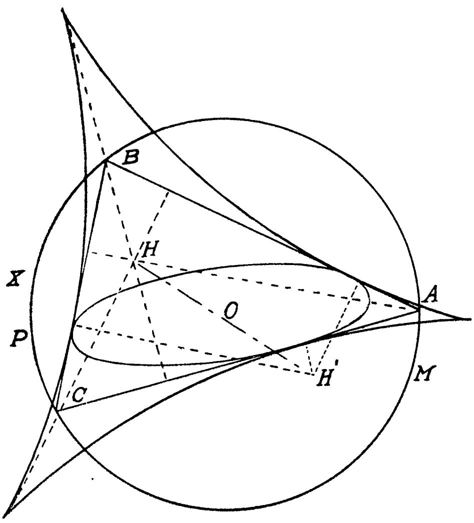

FIG. 1

Theorem XII: *The locus of orthopoles, with respect to a triangle, of lines on the point symmetric to the orthocenter with respect to the circumcenter is an ellipse tangent to the three sides of the triangle and concentric with the circumcircle.*

If the triangle is acute angled the ellipse will be inscribed; if the triangle is a right triangle, the ellipse is merely the hypotenuse taken twice; if the triangle is

obtuse, the ellipse is "escribed," lying within the obtuse angle. The points of tangency are the same as the points of tangency of the Steiner deltoid with the triangle, and are in fact the feet of the perpendiculars dropped from H' upon the sides, where H' is the symmetric of the orthocenter with respect to the circumcenter (Fig. 1).

For the sake of brevity some results are now given that can easily be determined by further investigation using methods similar to those used above.

The loci of orthopoles, referred to ABC, of pencils of lines on the four equicenters $I_i(i=1, 2, 3, 4)$ of ABC are four mutually tangent ellipses, whose centers E_i are members of an orthocentric set of four points, and whose com-

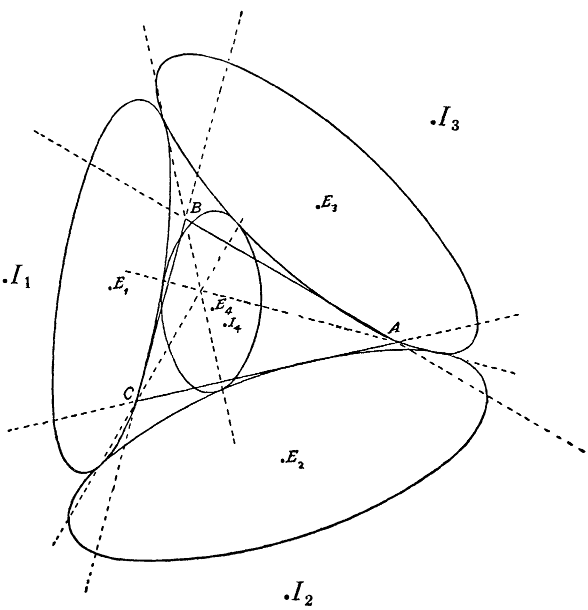

FIG. 2

mon tangents meet by threes in four points of an orthocentric set T_i. The four triangles contained in the set T_i and the reference triangle ABC have the same nine-point circle. The medial triangle of ABC is the pedal triangle to each of the four triangles contained in the set T_i. The six points of common tangency of the four ellipses lie in pairs on the altitudes of ABC. The six common tangent lines of the four ellipses are likewise tangent lines to the Steiner deltoid of ABC, the points of tangency being identical with the points of tangency of the six lines with the four ellipses.

MAXIMUM NUMBERS ASSOCIATED WITH THE DIOPHANTINE EQUATION $\Sigma(1/x_1x_2\cdots x_{n-1})=b/[(m+1)b-1]$

By H. A. SIMMONS, Northwestern University

1. *Introduction.* The case $r=n-1$ of the cyclo-symmetric equation which the author has recently studied[1] may be written in the form

$$(1)\qquad \sum\frac{1}{x_1x_2\cdots x_{n-1}}=\frac{1}{(m+1)b-1},$$

where Σ is the elementary symmetric function of the n variables $x_1, x_2, \cdots, x_n$, taken $(n-1)$ at a time, and where b, m are positive integers. The principal results obtained in the present paper are that the number v_n in the solution $(x_1, x_2, \cdots, x_n)\equiv(v_1, v_2, \cdots, v_n)$ of equation (1), where

$$(2)\quad v_i=1(i=1,2,\cdots,n-2),\quad v_{n-1}=m+1,\quad v_n=[(m+1)b-1](m+n-1)$$

is the largest number that exists in any solution in positive integers of this equation and that $v_1+v_2+\cdots+v_n$, $v_1v_2\cdots v_n$ are the largest sum and product, respectively, of the numbers in any solution of (1). These results constitute a partial solution of problems 1, 2, 3 which were stated at the end of the paper referred to above. Unfortunately the method which we use to identify v_n as the maximum number does not seem to apply except when $r=n-1$.

In §5 we exhibit for the equation $\Sigma(1/x_1x_2\cdots x_r)=b/[(m+1)b-1]$, in which r is a positive integer $<n$, a solution in positive integers which includes solution (2) of equation (1).

Throughout this paper $x_1, x_2, \cdots, x_{n-1}$ stand for positive integers, conveniently taken in the order $x_1\leqq x_2\leqq\cdots\leqq x_{n-1}$, while x_n, $(\geqq x_{n-1})$, in §§2, 3, 4, is not restricted to be an integer; in these sections the term *solution* means a set $(x_1, x_2, \cdots, x_n)$, occasionally referred to as *set* x, which satisfies equation (1). In each of the three problems which we solve, x_n assumes an integral value.

2. *An auxiliary theorem.* In making the proof proposed above, we shall need Theorem 1 below, in which the variables $u_1, u_2, \cdots, u_n$ are positive integers, taken in the order $u_1\leqq u_2\leqq\cdots\leqq u_n$.

Theorem 1a: $u_1+u_2+\cdots+u_n\leqq u_1u_2\cdots u_n+n-1$.

Theorem 1b: *The equality sign in* 1a *holds if, and only if,* $u_i=1(i=1, 2, \cdots, n-1)$.

Proof of 1a: If $n=1$, Theorem 1a is obviously true. If $n=2$, the inequality also holds since $u_1+u_2\leqq u_1u_2+1$ is equivalent to $(u_1-1)(u_2-1)\geqq 0$. Now if

$$u_1+u_2+\cdots+u_k\leqq u_1u_2\cdots u_k+k-1,$$

then

[1] This Monthly, vol. 36 (1929), pp. 148–155.

$$u_1 + u_2 + \cdots + u_k + u_{k+1} \leqq u_1u_2 \cdots u_k + u_{k+1} + k - 1 \leqq u_1u_2 \cdots u_{k+1} + k,$$

or
$$u_1 + u_2 + \cdots + u_{k+1} \leqq u_1u_2 \cdots u_{k+1} + k\,;$$

hence Theorem 1a is true for all positive integral values of n.

Proof of 1b: Suppose $u_i = 1 (i = 1, 2, \cdots, n-1)$. Then the equality sign in Theorem 1a holds whatever be the value of u_n. We wish to prove that the inequality sign in Theorem 1a holds if two or more of the u's exceed 1, so that certainly $u_{n-1} > 1$, $u_n > 1$. With this hypothesis, $(u_n - 1)(u_{n-1} - 1) > 0$, so that $u_n + u_{n-1} < u_nu_{n-1} + 1$. Then by induction of the type used above, we find that $u_1 + u_2 + \cdots + u_n < u_1u_2 \cdots u_n + n - 1$; hence Theorem 1b is true.

Incidentally we state for Theorem 1 a generalization which can be obtained by induction.

Theorem 2a: $\Sigma u_1u_2 \cdots u_r \leqq {}_{n-1}C_{r-1} \cdot u_1u_2 \cdots u_n + {}_{n-1}C_r$.

Theorem 2b: *The equality sign in Theorem* 2a *holds if, and only if,* $u_i = 1 (i = 1, 2, \cdots, n-1)$.

Putting $u_i = 1 (i = 1, 2, \cdots, n)$ in Theorem 2a, we obtain as a corollary the well known relation:

$${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r.$$

3. *Proof that the v_n of* (2) *is the maximum number in a solution of* (1). It is convenient to consider (1) in the form

$$\frac{1}{x_1x_2 \cdots x_{n-1}} + \frac{1}{x_n} \cdot \frac{x_1 + x_2 + \cdots + x_{n-1}}{x_1x_2 \cdots x_{n-1}} = \frac{b}{(m+1)b - 1}. \tag{3}$$

Let $X_1, X_2, \cdots, X_n$ denote a solution in which X_n is the sought maximum number. On solving (3) for x_n and introducing X in place of x, we obtain

$$X_n = \frac{[(m+1)b - 1](X_1 + X_2 + \cdots + X_{n-1})}{bX_1X_2 \cdots X_{n-1} - (m+1)b + 1},$$

in which the denominator is a positive integer. Consequently,

$$X_n \leqq [(m+1)b - 1](X_1 + X_2 \cdots + X_{n-1}).$$

Application of Theorem 1a gives

$$X_n \leqq [(m+1)b - 1](X_1X_2 \cdots X_{n-1} + n - 2). \tag{4}$$

From (2) we also have

$$X_n \geqq v_n. \tag{5}$$

From (3), (5), and Theorem 1a, we now observe that

$$\frac{1}{X_1X_2 \cdots X_{n-1}} + \frac{X_1X_2 \cdots X_{n-1} + n - 2}{v_n \cdot X_1X_2 \cdots X_{n-1}} \geqq \frac{b}{(m+1)b - 1},$$

from which

(6) $$X_1X_2\cdots X_{n-1} \leqq \frac{v_n+n-2}{b(m+n-1)-1}$$
$$\equiv \frac{[(m+1)b-1](m+n-1)+n-2}{b(m+n-1)-1} = m+1.$$

From (3) it is obvious that $X_1X_2\cdots X_{n-1} \geqq m+1$. Then, on account of (6),

(7) $$X_1X_2\cdots X_{n-1} = m+1.$$

From (4) and (7), therefore, $x_n \leqq v_n$, which, in connection with (5), proves that $x_n = v_n$.

From Theorem 1b and the fact that $X_1+X_2+\cdots+X_{n-1} = X_1X_2\cdots X_{n-1}+n-2 = m+n-1$, it is evident that the maximum number v_n appears in no solution of (1) except (2).

Remark: Solution (2) is not necessarily the only solution of (1). For example, if $n=4$, $b=1$, $m=10$, equation (3) becomes

$$1/x_1x_2x_3 + (x_1+x_2+x_3)/x_1x_2x_3x_4 = 1/10,$$

which has (perhaps among others) the solutions (1, 1, 12, 70), (1, 1, 15, 34), (1, 1, 20, 22) in addition to solution (2), which is (1, 1, 11, 130). There may also be more than one solution when $b>1$. For example, if $n=5$, $b=2$, $m=2$, equation (3) becomes

$$1/x_1x_2x_3x_4 + (x_1+x_2+x_3+x_4)/x_1x_2x_3x_4x_5 = 2/5,$$

which has (perhaps among others) the solutions (1, 1, 2, 3, 5), (1, 1, 1, 5, 8), (1, 1, 2, 2, 10) in addition to solution (2), which is (1, 1, 1, 3, 30).

We summarize our result in

Theorem 3: *The maximum number that exists in a solution of the equation* $\Sigma(1/x_1x_2\cdots x_{n-1}) = b/[(m+1)b-1]$, *in which* b, m *are positive integers is* $[(m+1)b-1](m+n-1)$. *This number appears in but one solution, namely solution* (2): $v_i = 1 (i=1, 2, \cdots, n-2), v_{n-1} = m+1, v_n = [(m+1)b-1](m+n-1)$.

4. *The maximum sum and product of the numbers in a solution of* (1). From (3) it is obvious that the maximum sum and the maximum product are simultaneously attained. We shall solve both problems by finding the solution $(x_1, x_2, \cdots, x_n) \equiv (Y_1, Y_2, \cdots, Y_n)$ of maximum sum. Our proof follows from the set of lemmas below, in which S stands for the maximum sum, $Y_1+Y_2+\cdots+Y_n$.

Lemma 1: $S \geqq b(m+1)(m+n-1)$.

This follows from the definition of S and the fact that the sum of the numbers in solution (2) equals $b(m+1)(m+n-1)$.

Lemma 2: $Y_{n-1} \leqq m+1$.

Proof: Suppose $Y_{n-1} > m+1$, say $Y_{n-1} = m+1+\delta$, where δ is a positive in-

teger, and $Y_{n-1} \leqq Y_n$ since the set Y is a set x. We shall reach a contradiction by showing that if α is a positive integer $\leqq \delta$, so that

$$b/[(m+1)b-1] > 1/Y_1Y_2 \cdots Y_{n-2}(Y_{n-1}-\alpha)$$

whatever be the (positive integral) value of $Y_1Y_2 \cdots Y_{n-2}$, and if α, Y_n' are such that the set $Y_1, Y_2, \cdots, Y_{n-2}, Y_{n-1}' \equiv Y_{n-1}-\alpha, Y_n'$ satisfies (1), then $Y_{n-1}'+Y_n' > Y_{n-1}+Y_n$, and the set Y does not have the largest sum possessed by any solution of the type that we consider.

Letting $\pi = Y_1Y_2 \cdots Y_{n-2}$ and $\sigma = Y_1+Y_2+ \cdots +Y_{n-2}$, we have, by hypothesis [see (3)],

$$(8_1) \qquad \frac{1}{Y_{n-1}} + \frac{1}{Y_n} + \frac{\sigma}{Y_nY_{n-1}} = \frac{b\pi}{a},$$

$$(8_2) \qquad \frac{1}{Y_{n-1}'} + \frac{1}{Y_n'} + \frac{\sigma}{Y_n'Y_{n-1}'} = \frac{b\pi}{a}.$$

The desired result will now follow from (8_2) and the definitions of Y_n', Y_{n-1}' if we can show that

$$(9) \qquad \frac{1}{Y_{n-1}-\alpha} + \frac{1}{Y_n+\alpha} + \frac{\sigma}{(Y_{n-1}-\alpha)(Y_n+\alpha)} > \frac{b\pi}{a}.$$

That (9) follows from (8_1) is a consequence of the two equivalent statements,

$$\frac{1}{Y_{n-1}-\alpha} + \frac{1}{Y_n+\alpha} > \frac{1}{Y_{n-1}} + \frac{1}{Y_n} \quad \text{and} \quad \frac{\sigma}{(Y_{n-1}-\alpha)(Y_n+\alpha)} > \frac{\sigma}{Y_nY_{n-1}}.$$

Hence Lemma 2 is true.

Lemma 3: $Y_{n-1} \geqq m+1$.

Suppose $Y_{n-1} < m+1$, say $Y_{n-1} = m+1-\delta$, where δ is a positive integer, $\leqq m-1$ since $Y_{n-1} \geqq 2$. Then $Y_n < a(m+n-1)$, [see (2) and Theorem 3]. Therefore $Y_1+Y_2+ \cdots +Y_{n-1} \geqq m+n$ by Lemma 1. To make the desired proof now, we only need to establish two facts: (p) if $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ are positive integers such that $\alpha_1+\alpha_2+ \cdots +\alpha_{n-1} \geqq m+n-1$, then $\alpha_1\alpha_2 \cdots \alpha_{n-1} \geqq m+1$; ($q$) if a set $\beta_1, \beta_2, \cdots, \beta_n$ is a set x for which $\beta_1+\beta_2+ \cdots +\beta_{n-1} \geqq m+n$, then the set β is not the set Y.

Proof of (p): This is an immediate consequence of Theorem 1a.

Proof of (q): Consider the set $\beta_1, \beta_2, \cdots, \beta_{n-2}, \beta_{n-1}' \equiv \beta_{n-1}-1, \beta_n'$, in which β_n' is so selected that these n numbers form a solution of (1). When the first $n-1$ elements of this set are suitably arranged, a set x results for which $\beta_1\beta_2 \cdots \beta_{n-2}\beta_{n-1}' \geqq m+1$, by ($p$); and such that $\beta_{n-1}'+\beta_n' > \beta_{n-1}+\beta_n$, as can be shown by the type of argument that was used in proving Lemma 2. Hence (q), and consequently Lemma 3, is true.

From the last two lemmas, $Y_{n-1} = m+1$. By the method used in proving

Lemma 2, one can now readily show that $Y_i=1 (i=1, 2, \cdots, n-2)$. Hence the set Y is identical with solution (2).

We state the results of this section in:

Theorem 4. *The sum and the product of the numbers in solution* (2) *are greater than the sum and product, respectively, of the numbers in any other solution of equation* (1).

5. *A solution of* $\Sigma(1/x_1x_2\cdots x_r)=b/[(m+1)b-1]$. By such an inductive attack as was employed to obtain a solution of the cyclo-symmetric equation mentioned above, we have found for the equation

$$\sum\frac{1}{x_1x_2\cdots x_r}=\frac{b}{(m+1)b-1}, \tag{10}$$

a solution which includes solution (2) of equation (1). In order to exhibit this solution, we define the symbol $\Sigma_{i,j}$ as the elementary symmetric function of $x_1, x_2, \cdots, x_i, (i<n)$, taken $j\leqq i$ at a time. In terms of this symbol it is not difficult to see that (10) can be written in the form

$$\begin{aligned}\frac{1}{x_1x_2\cdots x_r}+\frac{1}{x_{r+1}}\cdot\frac{\Sigma_{r,1}}{x_1x_2\cdots x_r}+\frac{1}{x_{r+2}}\cdot\frac{\Sigma_{r+1,2}}{x_1x_2\cdots x_{r+1}}\\ +\cdots+\frac{1}{x_n}\cdot\frac{\Sigma_{n-1\ n-r}}{x_1x_2\cdots x_{n-1}}=\frac{b}{a},\quad a\equiv(m+1)b-1.\end{aligned} \tag{11}$$

When $n=r$, (11) does not always have a solution in positive integers. For example, if $n=r=3$, $b=2$, $m=2$, equation (11) becomes $1/x_1x_2x_3=2/5$, which has no such solution. When $n>r$, a solution of (11) is given by the numbers $(x_1, x_2, \cdots, x_n)=(w_1, w_2, \cdots, w_n)$, where

$$(12)\quad\begin{cases}\text{(12a)}\quad w_k=1\ (k=1,2,\cdots,r-1);\\ \text{(12b)}\quad w_r=m+1;\\ \text{(12c)}\quad w_{k+1}=a\sum_{k,k-r+1}+1\ (k=r,r+1,\cdots,n-2);\\ \text{(12d)}\quad w_n=a\sum_{n-1,n-r}.\end{cases}$$

To see that (12) is a solution of (11), substitute into (11) for the x's their values from (12); multiply the resulting equation through by a; in the result add 1 to, and subtract 1 from each numerator after the first; then collect the first two terms, using $(m+1)b-1$ in place of a, and get $b-1/(m+1)(a\Sigma_{r,1}+1)$; next, to the result just obtained, add the third term and get

$$b-1/(m+1)(a\Sigma_{r,1}+1)(a\Sigma_{r+1,2}+1);$$

to this result add the fourth term and get

$$b-1/(m+1)(a\Sigma_{r,1}+1)(a\Sigma_{r+1,2}+1)(a\Sigma_{r+2,3}+1);\text{ etc.}$$

By the methods used in connection with the cyclo-symmetric equation, we have proved for solution (12) properties that are analogous to those which were found for our solution of the cyclo-symmetric equation.

We therefore wish to propose to the reader problems which are analogous to those that were proposed in connection with the cyclo-symmetric equation.

Problem 1: If $n>r$, is the w_n of (12) the largest number that exists in a solution in positive integers of equation (10)?

Problem 2: If $n>r$, are $w_1+w_2+\cdots+w_n$ and $w_1w_2\cdots w_n$ the maximum sum and product, respectively, of any set of n positive integers which constitute a solution of equation (10)?

ON A POINT-TO-PLANE TRANSFORMATION IN FOUR-SPACE AND AN ∞^4-SYSTEM OF TETRAHEDRAL COMPLEXES IN ANY THREE-SPACE CONTAINED IN THE FOUR-SPACE

By B. C. WONG, University of California at Berkeley

Consider two hyperquadric surfaces Q and Q' in 4-space whose equations referred to their common self-polar simplex are, respectively,

$$\sum_{i=0}^{4} a_i x_i = 0, \qquad \sum_{i=0}^{4} a_i' x_i = 0.$$

The polar hyperplanes of a given point P (y) with respect to Q and Q' intersect in a plane π given by the equations,

$$\sum a_i y_i x_i = 0, \qquad \sum a_i' y_i x_i = 0.$$

The plane π is said to correspond to the point P. This paper proposes to describe some of the features of this point-to-plane transformation and also to obtain a system of ∞^4 tetrahedral complexes contained in any 3-space of S_4 in connection with the transformation. It will be found that a general line determines a quadric hypercone of planes, a general plane a cubic surface which is peculiarly related to it, and a general hyperplane a normal quartic curve which in turn determines a unique twisted cubic in the hyperplane. We shall also find that a general hyperplane section of all the planes obtained in this transformation consists of all the ∞^4 lines of the hyperplane forming an ∞^4-system of tetrahedral complexes.

From the equations of transformation above or otherwise we see that the vertices of the self-polar simplex go into the opposite hyperplanes or faces, any point on an edge into the opposite plane, any point in a plane into a plane through the opposite edge, and any point in a face into a plane through the opposite vertex.

Let a point $P(y)$ describe a line l given by the intersection of three hyperplanes $S_3(u)$, $S_3'(u')$, $S_3''(u'')$. The corresponding plane π describes a quadric hypercone V_3^2 whose equation is

$$\Delta \equiv \begin{vmatrix} a_0x_0 & a_1x_1 & a_2x_2 & a_3x_3 & a_4x_4 \\ a_0'x_0 & a_1'x_2 & a_2'x_2 & a_3'x_3 & a_4'x_4 \\ u_0 & u_1 & u_2 & u_3 & u_4 \\ u_0' & u_1' & u_2' & u_3' & u_4' \\ u_0'' & u_1'' & u_2'' & u_3'' & u_4'' \end{vmatrix} = 0.$$

Consider for a moment the hyperplane $S_3(u)$. Its points transform into the ∞^3 trisecant planes of a definite 4-space quartic curve C^4. The equations of C^4 can be obtained by equating to zero the matrix of the first three rows of Δ or can be written parametrically

$$\rho x_i = u_i/(a_i - \lambda a_i') \qquad [i = 0,1, \cdots, 4].$$

To a given S_3 there is a definite C^4 corresponding and since there are ∞^4 hyperplanes in S_4, there are ∞^4 such quartic curves. They all have in common the five vertices of the self-polar simplex. The quadric hypercone V_3^2, and there are ∞^6 such corresponding to the ∞^6 lines in S_4, has on it ∞^2 quartics of the type C^4 and through each C^4 pass ∞^4 hypercones of the type V_3^2.

Each C^4 is also the locus of points whose corresponding planes all lie in the S_3 determining C^4. The planes are such that through a general point P of S_3 three of them pass. These three planes come from the three points on C^4 determining the plane π corresponding to P. Hence the planes in question envelop a twisted cubic curve γ^3 whose points go into the osculating planes of C^4. S_3 meets C^4 in four points forming a tetrahedron. The cubic γ^3 osculates the four faces of this tetrahedron and also the five planes in which S_3 intersects the five hyperplanes of the self-polar simplex.

The curve C^4 is also the locus of the poles of S_3 with respect to all the hyperquadric surfaces of the pencil determined by Q and Q'.

Now let P describe a plane ϕ common to $S_3(u)$ and $S_3'(u')$. The locus of points whose corresponding planes meet ϕ in lines is a ruled cubic surface in S_4 and has for equations the matrix of the first four rows of Δ equated to zero. This cubic surface, F^3, is also the locus of the vertices of the ∞^2 hypercones corresponding to the ∞^2 lines in ϕ. If ϕ is a plane π' corresponding to a point P', that is, an osculating plane of γ^3 in S_3, F^3 becomes a point coincident with P' which is on C^4. It is not difficult to see that a point P in ϕ goes into a plane π meeting F^3 in a conic, for a 3-space through P meets F^3 in a twisted cubic curve and π in a line bisecant to the curve.[1] There are ∞^2 conics on F^3 corresponding

[1] The line is said to correspond to the point P in a point-to-line transformation by means of two quadric surfaces in the 3-space. See Bing Chin Wong, *A study and classification of ruled quartic surfaces by means of a point-to-line transformation*, University of California Publications in Mathematics, vol. 1, No. 17, pp. 371–387.

to the ∞^2 points in ϕ. If P describes a line l in ϕ, there is an ∞^1-system of conics on F^3 all having a point L in common and their planes are the planes of the V_3^2 corresponding to l and having L for vertex.

While through a given C^4 pass ∞^3 F^3 corresponding to the ∞^3 planes in S_3, on each F^3 lie ∞^1 C^4 corresponding to the ∞^1 hyperplanes through ϕ. F^3 may be said to be generated by this ∞^1-system of C^4. If we consider the points of F^3 and the lines of ϕ as corresponding elements, we have in ϕ an ∞^1-family of curves of class 3 and order 4, each curve being the intersection of ϕ with the developable to the γ^3 contained in an S_3 through ϕ. The tangents to such a curve correspond to the points of the C^4 on F^3 determined by S_3. All the curves of the family have five common tangents which are the lines of intersection of ϕ with the faces of the self-polar simplex.

Consider ϕ as a plane in a fixed S_3. Any hyperplane S_3', which may be S_3 itself, meets the F^3 corresponding to ϕ in a cubic curve C^3 and determines a quartic curve C'^4. Let S_3 meet C'^4 in the four points A, B, C, D. The points of C^3 go into lines enveloping a conic κ^2 in ϕ. If C^3 is made up of a line g and a conic C^2, κ^2 degenerates into two pencils of lines. In this case, one of the four points common to S_3 and C'^4, say A, is in ϕ and the vertex of one of the pencils of lines in ϕ is at A, the vertex of the other is on the line in which ϕ meets the plane BCD. Now if C^3 is made up of three lines of which one, g, meets the other two, g', g'', in distinct points, two of the four points common to S_3 and C'^4, say A and B, are in ϕ and they are the vertices of the two pencils making up the degenerate κ^2. Corresponding to the points on g which must be the line CD is the same line AB. But if the components g, g', g'' of C^3 are concurrent at a point P' of F^3, three of the four points A, B, C, D, say A, B, C, are in ϕ which is now coincident with an osculating plane π' of the γ^3 in S_3 and the point P' is no other than the point D itself.

Note that there is another correspondence between the points of a plane θ, which may be coincident with ϕ, and the planes of the conics of F^3. This correspondence arises from associating with a given point P' of θ that plane π which passes through P' and meets F^3 in a conic. As π is the transform of a point P in ϕ, we have a quadratic birational transformation between the points of ϕ and those of θ. If P describes a line l in ϕ, P' describes a conic which is the intersection of θ with with the V_3^2 corresponding to l; and if P describes a conic in ϕ, the corresponding plane π describes a V_3^4 containing F^3 as double surface, thus giving rise to a trinodal quartic curve in θ as the locus of P'. But if P' describes a line in θ, π describes a V_3^4 containing F^3 doubly and hence P in ϕ describes a conic. This correspondence between the points of θ and ϕ is, therefore, birational and quadratic.

Now we come to the consideration of the tetrahedral complexes. It is known that the ∞^3 trisecant planes of a normal quartic curve in S_4 meet any S_3 in the lines of a tetrahedral complex, the fundamental tetrahedron being formed by

the points in which S_3 meets the curve.[2] Since the points of every hyperplane transform into the trisecant planes of a quartic curve which that hyperplane determines, there are ∞^4 tetrahedral complexes in a given S_3 obtained in this way. Each of the ∞^4 tetrahedra has for faces four of the osculating planes of the unique cubic γ^3 contained in S_3. Any four osculating planes of γ^3 form such a tetrahedron. Among the complexes there are many degenerate ones. The nature of the degeneracy of such a complex depends upon the relation which the corresponding hyperplane bears to the simplex self-polar to Q and Q', such as passing through one, two, three, or even four of its vertices. We shall not go into this.

Any two complexes Γ', Γ'' whose lines lie in the planes coming from the points of two hyperplanes S_3', S_3'' respectively have in common a congruence of lines bisecant to a cubic curve C^3. The lines of this congruence lie in the planes which are trisecant to the two quartic curves C'^4, C''^4 determined by S_3', S_3'' respectively. C^3 is the intersection of S_3 with the F^3 determined by the plane ϕ common to S_3' and S_3'', and it passes through the vertices of the two tetrahedra of Γ' and Γ'' and indeed through all the vertices of the ∞^1 tetrahedra whose complexes correspond to the ∞^1 hyperplanes through ϕ. There are ∞^6 such cubics giving rise to as many congruences corresponding to the ∞^6 planes in S_4. Any two congruences if they belong to the same complex have a quadratic regulus in common and if they belong to different complexes have just a line in common. Any three complexes always have in common a regulus which may be a proper quadric surface, or a quadric cone, or a pair of pencils of planes.

RECENT PUBLICATIONS

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All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

REVIEWS

Einführung in die mathematische Behandlung naturwissenschaftlicher Fragen. Erster Teil: Funktion und Graphische Darstellung. Differential- und Integral Rechnung. By Alwin Walther. Julius Springer, Berlin, 1928. viii+220 pages.

The author informs us in his preface that this volume, entitled "Introduction to the mathematical treatment of questions in natural science," is a result of a course which he gave in the years 1924–1926 at Göttingen, primarily for physicists, chemists and biologists. The author's objective, he informs us, was not to load the reader with formulae nor to give him worthless skill in mere

[2] Carrone, *Sopra un nuovo modo di generazione del complesso tetraedrale.* Napoli Rendiconti (3), No. 7, pp. 57–66.

manipulation, but rather to give a clear understanding of mathematical methods in their application to questions of natural science.

The volume is divided into two sections. Section I, pages 1–67, is entitled "Function and Graphical Representation." Section II, pages 67–202, is entitled "Differential and Integral Calculus."

The first 67 pages are devoted to a treatment of functions and graphs. While the principal consideration is of continuous, one valued functions and of functions whose corresponding graphs have only a finite number of oscillations within a finite interval, attention is directed to the "natural" existence of discontinuous functions of various kinds as well as to functions whose corresponding graphs have an infinite number of oscillations within a finite interval. The treatment of functions and graphs, though very brief, introduces rectangular coordinates, polar coordinates, logarithmic, semi-logarithmic and triangulation graphing and their uses as a means of adequate mathematical formulation from empirical data, in the various fields of natural science. Even the uses of nomographic scales for representation of various functions of several variables is given a brief treatment. If we note that the author includes a treatment of interpolation, deriving Newton's as well as Stirling's formulae, it is evident that these 67 pages are concentrated.

The next 135 pages are devoted to what constitutes the basis of the usual first course in differential and integral calculus in most of our colleges. The following topical heads will indicate, to some extent, the author's treatment of the subject:

Pages 67–79: *Derivative and indefinite integral.* Integration treated as an inverse operation, differentiation of functions of several variables, partial differentiation; applications of integration.

Pages 79–91: *Applications of the first and higher derivatives.*

Pages 91–117: *The differential and increment of a function.* Infinitesimals with applications; theorem of the mean; Taylor's series with remainder; Taylor's infinite series; brief mention of Fourier series and their uses.

Pages 117–138: *The definite integral.* Integration as a limit of a sum of infinitesimals; mechanical integration (planimeter); graphical integration.

Pages 138–165: *Derivation of formulae for differentiation and integration.* Additional applications.

Pages 165–202: *The natural logarithm and exponential function.* The differential and integral calculus of such functions in problems that arise in natural science.

The volume has 174 figures, which indicates the author's emphasis on the need and value of visualizing the significance of the corresponding analysis (granting that often in special considerations the analysis transcends such possibility—as in continuous functions without derivatives at any point).

This volume is not a treatise nor is it a text book. It does not contain drill exercises. The greatest value of this volume is not in the advancement of the content of mathematics but in advancing the appreciation of mathematics as

a marvelous tool in the heads and hands of the natural scientists. The book should prove a stimulus to the scientist whose work has emerged from the empirical to the analytical stage, but he will need a different text to obtain any facility in the use of the tool for which his interest has been stimulated by this introduction. Such books in this country as well as abroad, confirm my observations that mathematics is more and more coming to be recognized as the desirable, yea, the necessary language of science, not only to physicists, chemists, and engineers, but to entomologists, horticulturists, zoologists, etc.

Perhaps Bertrand Russell's definition of mathematics as "the science in which we never know what we are talking about, nor what we say is true," in being restricted to entities in natural science rather than the realm of generalizations (abstractions) will lead to more and more true statements of facts. The author evidently prefers Whitehead's conception of mathematics as "the intellectual instrument which has been created for rendering clear the aspects of the world quantitatively."

H. L. SLOBIN

Begriff und Anwendungen des Differentials. By Alwin Walther. B. G. Teubner, Berlin, 1929. iv+96 pages.

This pamphlet, "The Concept and Applications of the Differential," is also the outgrowth of a series of lectures delivered by the author at Göttingen, in 1926. Pages 1–11 are in the nature of an introduction and give an historical and analytical setting of the development of the differential and increment of functions. Pages 11–17 deal with continuity and infinitesimals. Pages 17–45 deal with the derivative and differential. Pages 45–86 deal with the theorem of the mean, Taylor's formula, and the definite integral.

The pamphlet is in some respects a portion of the volume reviewed above and in other respects a further development of that portion. It really is a bit of the theory of functions, primarily the part that pertains to the differential calculus, and belongs in an introduction to a second course in calculus. Though the author again addresses himself to teachers of the natural sciences, in the secondary schools, this work, unlike the volume above, has primarily the point of view of the mathematician. It is probably of some value in bridging the gap between a first course of calculus and function theory.

H. L. SLOBIN

College Algebra. By N. R. Wilson and L. A. H. Warren. Oxford University Press, American Branch, New York, 1928. $2.08.

Many teachers who were reluctantly forced to the conclusion that "Hall and Knight" would not do for American colleges will be glad to consider this college algebra by two Canadians. In it an effort has been made to adapt the traditional English course to the needs of the American student.

Great pains have been taken in the presentation, particularly in the first part of the book. This has in some cases resulted in explanations so long as to

repel the student for whose benefit they were intended. Examples might be cited from the eleven rules of the first chapter, or the four page explanation of the characteristic in the sixth chapter. Space saved here and from the 27 pages of ratio and variation might have been used to supplement the very inadequate review that precedes the first chapter. In the later chapters the style is marred by a tendency to use the equality sign of an equation as the principal verb of the sentence. As a whole, however, the book is well written; the treatment of such topics as the progressions, permutations and combinations, the binomial theorem and mathematical induction compares very favorably with that of most of our texts.

Apart from its attention to presentation, the Americanization of this book takes such forms as the appearance of an index, a greater variety in the printing, the use of American money (even such sums as \$mP); and, most important, the treatment of limits and convergence is made to agree with the usual practice in this country.

JOHN M. STETSON

A Simplified Presentation of Einstein's Unified Field Equations. By Tullio Levi-Civita. Translated by John Dougall. Blackie and Son, London, 1929. 22 pages.

This exposition can be recommended for those who have time to read only one paper on this subject. The formulas are given in such detail that the only prerequisites are an acquaintance with Maxwell's equations and a slight knowledge of general relativity. Only the mathematics of the theory is given. There is no discussion of the physical ideas, or of the probable place of the theory in physics.

K. W. LAMSON

Poetry and Mathematics. By Scott Buchanan. The John Day Co., New York, 1929. 197 pages.

The opening chapter gives the purpose of the book as being to show by use of Plato's dialectic method "what mathematics and poetry are and what they are not." The author sees the two as more or less successful attempts to deal with ideas. His denunciation of the professional mathematician or "prover of propositions," who through elaborate manipulation of formulae and figures succeeds in mystifying the student, is no more bitter than that of the professional in the poetry workshop who lays too great a stress on versification and prosody. In both cases the means or "vehicle" by which an idea is to be reached has been confused with the idea itself.

The first vehicle discussed is the mathematical figure as a symbol. There is a diverting section at this point in which the author, quoting the dialogue between the caterpillar and Alice in Lewis Carroll's "Alice in Wonderland," pictures the caterpillar as a typical mathematics teacher and Alice as both the

typical pupil and the figure to be understood. This he contends is a lesson in projective geometry. Another more serious example taken from Euclid's "Optics" is that of the figure in a moving projective plane with the point of convergence at the eye. These demonstrations show the figure in transformation just as a poem reveals the character in action. This comparison is more thoroughly developed further on in the book. The author gives at length a demonstration of the Pythagorean theorem and shows how the mathematical technique has there been used to define an abstract idea.

Next he turns to the field of numbers and discusses the two theories: that of operations where the entire number system is built up by starting with unity and applying different operators; and that of postulates, where the numbers are the elements in a system, and different relations being assigned to hold between them, the numbers are differentiated as reflexive, symmetric and transitive, or their opposites. Similar to the ordinal and cardinal numbers are the ordinal and cardinal elements in a story, the first where a series of events happen to a character and are reflected by him and the latter where the events come from the character and are transfigured by his habits and wishes.

From the discussion of space and number the author proceeds to the ratio, pointing out the linguistic bridge between poetry and mathematics built by the Greeks and Romans in their words "logos" and "ratio" or reason. This he now calls analogical thought and he traces the expansion of the analogy into the allegory, giving examples from old and modern literature. A corresponding mathematical example is Archimedes' principle of the lever developed in the fields of mechanics and astronomy. These ratios in combination give the algebraic equation. Trigonometry is a science built up on ratios and having these equations for its laws. Descartes developed his "geometrico-algebraic" allegory by the introduction of loci and equations. Just as the ratio varies in the algebraic equation, the dramatic character varies in the dramatic equation acting under conditions which determine the plot of the play.

Taking the ratio and letting the second term approach zero, we get the operator $\Delta y/\Delta x$ which, when applied to the algebraic equation gives the differential or fluxion. Here the idea of functionality is introduced, a new aspect of the mathematical object. Up to this point the similarity between poetry and mathematics has been stressed. The author now states that mathematics sees and deals with relations and has for its symbol the function, while poetry deals with qualities and the symbol it operates with is the adjective. The method of mathematics is analytic and that of poetry synthetic.

The relation between the esthetic object or symbol with which the two subjects work and the intellectual object or idea is that the former finds interpretation and comprehension of the latter. Symbols or metaphors are the analogies which upon expansion into the allegorical form and introduction of more adequate symbols approach the idea.

With the shift of attention from ratio and proportion to function the author sees a turning from mechanics to new routes to the absolute in which mathe-

matics and poetry are more closely bound. Modern Pythagoreans with their new analogies allow everything, numbers, forces, functions, etc., to vary, thus making mathematics correspond to the scientific observations. He gives an interesting picture of mathematics and poetry each developing as a series of concentric spheres, each expansion including the preceding; and says that as soon as relations can be found to hold between qualities a new sphere will be built including both series and we shall have an allegory which is mathematical and poetic. The tragedy is an example of such a sphere but ends in the inevitable destruction of the sphere. The comedy on the other hand uses the method of substitution applied in functional work such as series or equations in mathematics. Viewing the popularizer of science as the stage manager and ourselves as the actors, the author deplores the missing spirit of the play and the scarcity of clear and distinct ideas today.

MARGARET M. YOUNG

MATHEMATICS CLUBS

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CLUB TOPICS

1930 AS A CENTENNIAL YEAR IN THE HISTORY OF MATHEMATICS

By WALTER CROSBY EELLS, Stanford University, California

In continuation of previously published lists[1] of centennial dates in the history of mathematics, the following list of important 1930 centennial dates is presented.

B. C. 570. Birth of Anaximenes, prominent pupil of Thales, of Ionian school. Greek mathematician and astronomer.

470. Birth of Hippocrates of Chios, one of the greatest Greek geometers, celebrated for his studies of the quadrature of lunes.

370. Death of Democritus of Abdera, Thracian geometer.

A. D. 130. Appearance of Ptolemy's *Almagest*, standard work on mathematical astronomy for over one thousand years, until superseded by work of Copernicus and Kepler.

330. Death of Iamblicus, Alexandrian mathematician who advanced the Greek theory of numbers.

430. Death of the church father, St. Augustine, who discussed Zeno's celebrated paradoxes and arrived at a conception of the "actually infinite" as a constant.

530. Computation by the Hindoos of a table of sines for every $3\,{}^3/_4{}^\circ$ of the quadrant.

1530. Probable date of birth of Bombelli, Italian mathematician, author of "the most teachable and the most systematic treatment of Algebra that had appeared in Italy up to that time" (Smith, D. E.: *Source Book in Mathematics*, p. 80) which was noteworthy for its treatment of cubic and biquadratic equations.

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[1] This Monthly, vol. 36 (1929) pp. 99-100 for a list of 1929 events and for references to previous volumes for corresponding lists from 1925 to 1928.

1530. Rudolff used a symbol of a vertical bar as a decimal point in a compound interest table.

1630. Birth of Barrow, English mathematician whose method of tangents was one of the forerunners of the differential calculus.

1630. Galileo first called attention to the cycloid and suggested that arches of bridges should be built in the form of this graceful curve.

1630. Galileo in his *Dialog über die beiden hauptsächlichtsen Weltsysteme* considers questions later recognized as belonging to the field of the calculus of variations.

1630. Publication of *Grammelogia, or the Mathematicall Ring*, by Delamain, mathematical teacher of London, earliest publication describing a slide rule. For facsimile of title page, with illustration of circular slide rule, see Smith, D. E., *Source Book in Mathematics*, p. 157.

1630. Death of Kepler, celebrated geometrician and astronomer, whose three laws form the basis of modern astronomy.

1730. Birth of Bezout, French mathematician, whose *Théorie générale des Equationes Algébriques* contained his theory of elimination and symmetric functions of the roots of an algebraic equation.

1730. Results of "Maclaurin's Theorem" first given by Stirling in his *Methodus Differentialis.*

1730. Publication of De Moivre's *Miscellanea Analytica*, which brought about his election to the Berlin Academy.

1730. Publication of English translation of Venema's *Coffer Konst*, in New York, second oldest arithmetic printed in America.

1830. Birth of Cremona, Italian geometer, discoverer of the "Cremona transformation," of fundamental importance in modern geometry.

1830. Publication of Legendre's monumental *Théorie des nombres* in two large quarto volumes (third, final, enlarged edition), still a standard work on number theory.

1830. Word "group" first used in the technical, mathematical sense by Galois.

1830. Death of Fourier, French physicist and mathematician, discoverer of "Fourier series" by which any arbitrary function of a real variable may be represented by a trigonometric series.

1830. Publication at Hartford of Barnard's *A Treatise on Arithmetic*, first American arithmetic to contain pictures to aid beginners in mastering the subject.

DISCORD IN MATHEMATICS LAND

By MARIE WHELAN, Hunter College of the City of New York

INTRODUCTORY SCENE: Two strips of white tape are laid out parallel on the floor. Math Student enters and walks along one of the lines saying slowly, wearily and rhythmically:
Math Student: "And *on* and *on* and *on* and *on* and *on*—(out of the room)."

SCENE I:

$f(x)$: My children, we are gathered here in court assembled to consider matters grave.
From trusted sources it has reached mine ears
That discord rears its head in this our land.
The order that we cherish is assailed
And harmony is threatened in this its very home,
That harmony the like of which has ne'er been equalled in the world
Outside the borders of our realm.
And shall we tamely sit and watch it go?
It shall not be! On that I am resolved.
And so I've called you here today

To air your grievances in open court.
If any one has plaint against his brother
Let him speak now, or henceforth hold his peace.
Big D of x, take you your stand upon my right,
And differentiate between the various claimants for the floor,
And likewise calculate for us the rate of change from case to case,
That all may move with speed and order
As doth become this court.
And Integral of mine, do you be seated on my left
And lend attentive ear to every case,
So at the end to make summation for the court
Of all that does transpire.
Big D, proceed!

D_x: My Lord, Addition claims the floor.
He has a charge to make.

Addition: My Lord, I am a thrifty fellow, as you know,
And all my life have saved and saved and saved.
I have struggled hard and piled up wealth on wealth
And oft have been a multimillionaire.
I would not have a care or fear
For aught the future has in store
Save for the depredations of this thieving rogue, Subtraction.
Throughout the years, he's robbed me
Right and left, small sums and large
And often at one blow
Has stolen all my hard earned wealth away
And left me penniless.
Patiently have I endured till now
And countless times have struggled up
From want, to wealth exceeding great.
With what results? This rascal comes along
And steals it all away.
I am growing old and weary in the game.
I will not work with naught to show for it
Unless the court will guarantee my rights
And place restraint upon this thieving knave.
I have concluded.

$f(x)$: Subtraction, take the floor
And justify your actions if you can!

Subtraction: I've naught to say but this.
I claim the right to make a living
By means that nature has endowed me with
By taking here and there, and when and where
I can. A man must live, my Lord.

$f(x)$: My judgment I'll reserve upon this case.
Stand down! Big D, proceed!

D_x: Multiplication has the floor. I present him to the court.

Multiplication: Your Honor, and my Brothers, I've a charge against Division.
His malice brings to nothing all my toil.

My duty 'tis, and pleasure too,
To bring together diverse folk in this our land
And mould them into one,
And many happy unions have I formed
Which stood intact from all assaults save one
And that, Division's. He takes delight
In breaking every bond I make.
I'll cite a case in point.
I took ab and formed a union strong
With ρ' and with ρ.
The result was very happy. I was pleased
And they were pleased. And then
Division came along and spoiled my handiwork
And parted ab from his ρ's.
This instance is but one
Of thousands like to this.
His malice is unbounded, I maintain
My Lord will find he is the root and cause
Of all the discord that is rife today.
His very name proclaims it.
My grievances are many at his hands!

$f(x)$: Division! What say you to this charge?

Division: My Lord, his charge is foolish. Would you blame
The *light*, that it is seen,
Or *sound*, that it is heard,
Or *black*, that it is black,
Or *white*, that it is white?
His mission is to join.
My mission is to part.
And sometimes mine the greater blessing is.

$f(x)$: I cannot give decision on this case.
Who seeks the floor?

D_x: The Logarithm claims the floor.
I recognize his right.

Logarithm: My right and privilege it is
To go out walking by the side of x
And many happy hours have I spent
In this, my normal way of living.
And none there are that can contest my claim
Save only one, this base e, the Exponential.
Every time, with mischievous delight
He takes x from me. He has spoiled my life.
I ask the court to guarantee my rights
As prior unto his.

$f(x)$: Well, e, do you admit this claim?

e: Admit this claim? Why should I?
How does he rate a higher claim than mine?
When, as you know, without me

He never had been born. He has no rights,
Save those that I allow. I've been
Too generous with the fellow, that I see.
A fault I'll mend in future.

$f(x)$: It seems to me this is a case
Where x himself might have a word to say.
Let him be called before the court
To state his preference.

x: Which do I prefer? Why neither one, or both,
It all depends, my Lord.
You see, I like to go out walking with the Log
But, sometimes, I grow weary, and am quite content
To have the exponential rescue me.
For although a very little fellow, e is strong,
His custom is to bear me on his shoulders.
I find that very restful when I'm tired.

$f(x)$: Why then, it seems, 'tis e that you prefer.

x: Oh no, my Lord, not always.
I weary too of being carried thus
And signal to the Log to take me down.
I needs must have my exercise
Or soon would I grow old and stiff
And lose my famed agility,
And every stupid schoolboy catch me out
In all my tricks.
'T would never do, my Lord!

$f(x)$: Well, x, stand down! You do not help us much.
I cannot give my judgment, yet awhile.
We will hear the next case.

D_x: The Sine desires to speak.
The Sine, my Lord.

Sine: My Lord, I wish to bring complaint
Against the Inverse Sine.
The patient labor of my hands
He tears down, every time.
I speak not only for myself, but by request
For all my clan, the cosine and the rest.
They have the same complaint to make
Against the whole *arc*-tribe.
We've reached the limit——

(Excitedly, angrily, the Arc Sine interrupts.)

Arc Sine: He's reached the limit, I protest——

$f(x)$: (thundering) Silence! Who dares to speak
Without permission here?
I judge him guilty of contempt of court.
Let him forthwith be ejected,

To infinity projected,
By order of the Chair.

D_x: α, β, γ, δ, perform your office
And project the Arc Sine to infinity.

(α, β, γ, δ seize the Arc Sine and heave him out of the room)

$f(x)$: Well, Sine, you may continue now
Without fear of interruption.

Sine: I have nothing else to say, my Lord,
I simply wished to put our case before the court.
I am sure we shall have justice at your hands.

$f(x)$; Who speaks for the defendants in this case?

D_x: I recognize the Arc Cosine.

Arc Cosine: We are taken at a serious disadvantage
Our natural spokesman has been exiled
From the court—for cause, we do admit,
And must apologize that one of us
So far forgot the honor due the Chair.
But, truly, 'twas enough to overwhelm us all
To hear the very charges we prepared
Flung in our teeth. I, as impromptu spokesman for my group,
Do hurl them back again, where they belong.

D_x: The last case has been heard, my Lord.
There are no other claimants for the floor.

$f(x)$: Now, Integral of $f(x)$, do you perform
Your office for the court,
And sum the cases you have heard.
Be brief and to the point.

Integral of $f(x)$: Never have I had an easier summation
To perform. 'Tis briefly this:
That every citizen in our land
Is up in arms against his inverse neighbor.
My task is easy, yours I fear is hard.
No evidence was produced to fix the blame
On one side or the other.
It seems a very mutual affair.

$f(x)$: My task is difficult, I confess
I do not see my way, for every plaintiff
Seems as guilty or as free of blame
As the accused. It was my hope
To bring a common cause to view,
The root of all the trouble.
It has not so transpired. I need a light.

D_x: I think, perhaps, that I can help,
My Lord, if you will give me leave to speak.

$f(x)$: Speak up! Big D, if you have light to shed,
It never was more welcome.

D_x: My Lord, I am a discriminating fellow, as you know,
And I've been putting two and two together, all the day,
And now, at last, I have the answer—*four.*
Deep down into my own experience
I have dug, and hence derived
The common cause, the root of all our war,
The guilty culprit!
For every time that I attacked the work
Of my inverse ('tis true he did not charge me),
One there was who put me up to it.
I think, if you'll inquire, that you will find
The same is true of all my brothers.
Who is that one? Well known is she
To all of you—the Math Student!

(Excited cries from the court—several speaking at once)

Why yes, 'tis true! She did! She is the one!
'Tis true, 'tis true! She is to blame!

$f(x)$: Ha! It seems we have the truth at last.
Now, now I see my way.
Let the Math Student be summoned before me.

Math Student, it has been clearly shown
That you, and you alone
Have undermined the harmony of Math Land.
You are not one of us, but an intruder,
A harmless one, we thought,
Whom we have suffered to come and go
Amongst us. And how have you repaid us?
Nearly have you brought my realm about my ears
In civil strife. For justice' sake you must be punished.
Prepare yourself to hear the sentence of the court.
At your feet you see two tracks upon the ground.
Start walking now upon the right hand one,
And at the rate of two steps to the second—
Go on and on and on!
You may come back on the parallel track—
But don't turn round 'till they meet.
March on! You'll trouble us no more.

(The court files out of the room leaving Math Student who toes the tape line saying slowly. wearily, rythmically)

Math Student: And *on* and *on*, and *on* and *on*
And *on* and *on* I sped,
And *on* and *on*, and *on* and *on*
I'd still be going ahead,
But—the alarm went off (smiling and bowing)
At half past six
And woke me out of bed.

$$\frac{7\sum n^6 + 5(p+1)\sum n^4 + p\sum n^2}{7\sum n^6 - 5(p-1)\sum n^4 - p\sum n^2} = \frac{n^2+n+p}{n^2+n-p}.$$

where

$$\sum n^6 = (6n^7 + 21n^6 + 21n^5 - 7n^3 + n)/42\ ;$$
$$\sum n^4 = (6n^5 + 15n^4 + 10n^3 - n)/30\ ;$$

and

$$\sum n^2 = (2n^3 + 3n^2 + n)/6.$$

SOLUTIONS

3028 [1923, 275]. *Proposed by Norman Anning, University of Michigan.*

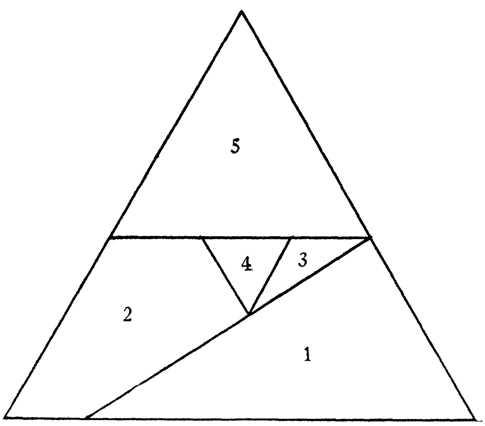
Equilateral triangles are described on the sides of a right triangle. Dissect the triangles on the legs and reassemble the parts to form the triangle on the hypotenuse.

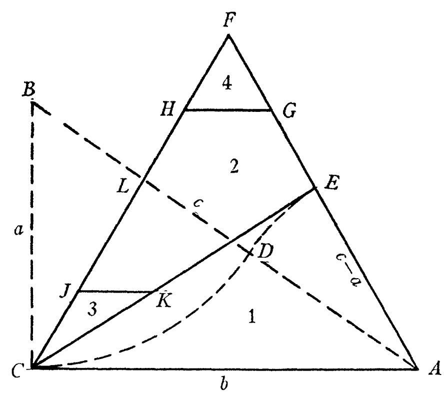
3048 [1923, 450]. *Proposed by H. C. Bradley, Massachusetts Institute of Technology.*

Cut two equilateral triangles of any relative proportions into not more than five pieces which can be assembled to form a single equilateral triangle.

Solution by H. C. Bradley, Massachusetts Institute of Technology.

The joker in this problem lies in the fact that one of the given triangles is not cut at all. Let a and b be the sides of the given two equilateral triangles,





$b \geqq a$. Form the right angled triangle ABC with the right angle at C, $a=BC$, $b=CA$, and $AB=c$. On BA lay off $BD=BC=a$. Let F be the third vertex of the equilateral triangle FCA, and lay off on AF the length $AE=AD=c-a$. Bisect EF in G and lay off on CF the lengths CJ and HF, each equal to FG. Draw CE; also draw JK parallel to CA cutting CE in K, which will lie within the segment CE. Then draw HG. The triangle FCA may be cut into four pieces, CAE, CKJ, $JKEGH$, HGF, which may be assembled to form a trape-

zoid with the equal slant sides AE, JH, and with parallel sides of lengths $c=JK+CA$ and $a=2HG+JK$. Upon this trapezoid the equilateral triangle of side a may be placed so as to complete the equilateral triangle of side c.

No piece is turned over, and only one piece need be rotated. This construction is always possible leaving intact the smaller, or equal, triangle of side a. It is also possible to leave the triangle of side b intact and to cut up the other with the side a in the same manner, provided $a>\frac{3}{4}b$. For the special case $a=\frac{3}{4}b$ there are also two solutions making only four pieces. We may divide a by two straight line cuts parallel to the base into three pieces having equal altitudes; or we may divide b into three pieces by two straight line cuts parallel to the sides, one of length $\frac{1}{2}b$ and the other of length $\frac{1}{4}b$.

The proof of the construction is simple. It may be shown that $JK=c-b$ by comparing the triangle CKJ with the similar triangle having a vertex at C and a base through E parallel to JK.

There are other special cases, one of which is given by $3^{1/2}a=b$. Here each triangle may be split into two equal right triangles, and these four pieces may be reassembled to form an equilateral triangle. Or a may be left intact, while b is divided first into two equal right triangles, and then one of these parts is divided into an isosceles triangle of base b and a right triangle of altitude $\frac{1}{2}b$.

3373 [1929, 232]. *Proposed by Paul Capron, U. S. Naval Academy.*

A plane cuts the lateral edges of a quadrangular pyramid so that the ratios of the edges of the whole pyramid to the corresponding edges of the smaller pyramid cut off are x_1, x_2, x_3, x_4 and the ratio of the smaller volume to the larger is r. Either diagonal of the base of the larger pyramid divides this base into triangles whose areas bear the ratio r_i, r_{i+2} to the whole area of the base, each of these ratios being numbered to correspond with the lateral edge opposite its triangle, so that $r_1+r_3=1=r_2+r_4$. Show that

$$x_1x_2x_3x_4r = x_1r_1 + x_3r_3 = x_2r_2 + x_4r_4.$$

Solution by J. H. Neelley, Carnegie Institute of Technology.

Let E_i and e_i, $i=1, 2, 3, 4$ be the edges of the large and small pyramids respectively. Then the plane through E_2 and E_4 divides the large pyramid into two triangular pyramids of volumes P_1, P_3 and the small pyramid into two triangular pyramids of volumes p_1, p_3. Now $P_1\cdot p_1^{-1}=x_2x_3x_4$ and $P_3\cdot p_3^{-1}=x_1x_2x_4$, being pairs of triangular pyramids with a triedral angle equal. Therefore

$$p_1 + p_3 = P_1(x_2x_3x_4)^{-1} + P_3(x_1x_2x_4)^{-1} ;$$

and dividing this equation by P, the whole pyramid, and using

$$(p_1 + p_3)\cdot P^{-1} = r, \quad P_1\cdot P^{-1} = r_1, \quad P_3\cdot P^{-1} = r_3,$$

we have when we clear of fractions:

$$x_1x_2x_3x_4r = x_1r_1 + x_3r_3.$$

Similarly, using the plane through E_1 and E_3, we get the second relation.

Also solved by A. Pelletier and B. D. Roberts.

3374 [1929, 233]. *Proposed by J. Rosenbaum, Milford, Conn.*

Given two triangles, one within the other, to construct a third triangle which shall be inscribed in the outer and circumscribed about the inner triangle. Also prove that if the two given triangles are equilateral and concentric the third triangle is equilateral.

Solution by Philip Franklin, Massachusetts Institute of Technology.

Let $A'B'C'$ be a triangle inside ABC. Take P any point on AB, draw straight line $PA'Q$ to cut BC in Q; $QB'R$ to cut CA in R and $RC'S$ to cut AB in S. As P varies, its range is perspective to that of Q, and hence projective to it, similarly through B' to the range of R and finally through C' to that of S. If three points P are taken, and the corresponding S points are found, we may find the double points of this projectivity by the standard construction (cf. Veblen and Young, Projective geometry, vol. I., p. 246) and so locate the two points which are the vertices of triangles of the kind wanted. With a given order, $A'B'C'$, there are two solutions, one solution, or no solution according to the number of real distinct double points; with this order not given, there are not more than twelve solutions. Of course the vertices are on the sides produced in some of these cases. (cf. Veblen and Young, l. c. problem 4, p. 250, and the reference there).

When both triangles are equilateral and concentric, and so lettered that a rotation of 120° about the center taking ABC into BCA takes $A'B'C'$ into $B'C'A'$, the two triangles found above will be equilateral, and concentric with the original pair. For, let their sides be pqr, and $p'q'r'$. Suppose that a rotation of 120° carries p, q, r into q', r', p'; then the same rotation must carry p', q', r' into q, r, p. A further rotation of 120° in the same sense must then carry q', r', p' into r, p, q; and a third rotation of 120° carries r, p, q into p', q', r'. Since a rotation of 360° makes p, q, r fall on p, q, r; p, q, r and p', q', r' coincide and there is only one solution, and it must be equilateral. If there are two distinct solutions, a rotation of 120° must carry p, q, r into q, r, p.

Also solved by the proposer.

3377 [1929, 285]. *Proposed by T. S. Peterson, The Ohio State University.*

Evaluate the summation

$$\sum_{i=0}^{n} \frac{(2i+1)!(2n-2i+1)!}{i!(i+1)!(n-i)!(n-i+2)!}.$$

Solution by B. P. Gill, College of the City of New York.

Writing the above expression in the form

$$f(n) = \sum_{i=0}^{n} \binom{2i+1}{i} \binom{2n-2i+1}{n-i} \cdot \frac{1}{n-i+2}, \tag{1}$$

we prove by induction that

$$f(n) = \frac{1}{2}\binom{2n+3}{n}. \tag{2}$$

Changing i in (1) to $n-i$, and adding the result to (1), we get

$$2f(n) = \sum_{i=0}^{n}\binom{2i+1}{i}\binom{2n-2i+1}{n-i}\cdot\frac{n+4}{(n-i+2)(i+2)}. \tag{3}$$

From (1),

$$f(n+1) = \sum_{i=0}^{n+1}\binom{2i+1}{i}\binom{2n-2i+3}{n-i+1}\cdot\frac{1}{n-i+3}$$

$$= \frac{1}{n+3}\binom{2n+3}{n+1} + \sum_{i=1}^{n+1}\binom{2i+1}{i}\binom{2n-2i+3}{n-i+1}\cdot\frac{1}{n-i+3}.$$

In the second term on the right, we note that

$$\binom{2i+1}{i} = \binom{2i-1}{i-1}\cdot\frac{2i(2i+1)}{i(i+1)} = \binom{2i-1}{i-1}\left(4-\frac{2}{i+1}\right).$$

Then changing i to $i+1$, and using (1) and (3), we get

$$f(n+1) = \frac{1}{n+3}\binom{2n+3}{n+1} + 4f(n) - \frac{4f(n)}{n+4}$$

$$= \frac{1}{n+3}\binom{2n+3}{n+1} + \frac{4(n+3)f(n)}{n+4}.$$

With the initial values $f(0)=1/2$, $f(1)=5/2$, the induction is completed by substituting in this recurrence relation from (2) and carrying out some easy algebraic reductions.

3378 [1929, 285]. *Proposed by Paul Wernicke, Washington, D. C.*

If a tetrahedron $A=A_1A_2A_3A_4$ has altitudes concurrent (in an orthocenter), then, denoting by (ij) the angle at A_j of the triangular face a_i (opposite A_i), the following products are equal: tan (41)·tan (32)·tan (23)· tan (14) = tan (31)·tan (42)·tan (13)·tan (24) = tan (21)·tan (12)·tan (43)·tan (34). The factors of each product are tangents of the four plane angles made by the edges which form a tetragram.

Solution by A. Pelletier, Montreal, Canada.

Let O be the orthocenter. The three planes passing through A_iO and the three edges concurring in A_i determine the three altitudes of the face a_i. Calling P_{jk} the intersection of the plane through A_iA_l with A_jA_k, we have:

$$\tan(ij) = \frac{A_l P_{jk}}{A_j P_{jk}} = \frac{A_k P_{jl}}{A_j P_{jl}}\cdot$$

Hence

$$\begin{aligned}\tan(31)\tan(42)\tan(13)\tan(24) &= \frac{A_2P_{14}}{A_1P_{14}}\,\frac{A_1P_{23}}{A_2P_{23}}\,\frac{A_4P_{23}}{A_3P_{23}}\,\frac{A_3P_{14}}{A_4P_{14}},\\ &= \frac{A_3P_{14}}{A_1P_{14}}\,\frac{A_4P_{23}}{A_2P_{23}}\,\frac{A_1P_{23}}{A_3P_{23}}\,\frac{A_2P_{14}}{A_4P_{14}},\\ &= \tan(21)\tan(12)\tan(43)\tan(34).\end{aligned}$$

Interchanging 2 and 4 in the above, we obtain:

$$\tan(31)\tan(24)\tan(13)\tan(42) = \tan(41)\tan(14)\tan(23)\tan(32).$$

3379 [1929,285] *Proposed by J. H. Neelley and T. L. Smith, Carnegie Institute of Technology.*

Two men own jointly x cows which they sell for x dollars per head and with the returns buy sheep at $12 per head. As their income from the cows is not divisible by 12 they purchase a lamb with the remainder. Later they divided the flock so that each had the same number of animals. How much money was due the man with the lamb by the other man?

Solution by P. S. Ganesa Sastri, Trichinopoly, S. India.

The price of x cows at x dollars a head is x^2 dollars. Since x^2 is not exactly divisible by 12, x cannot be 6 or any multiple of 6. Hence it should be $12n \pm y$, where n is any integer and y is an integer ranging from 1 to 5.

Now $(12n \pm y)^2$ when divided by 12 should leave a remainder and have an *odd* integer for its quotient.

$$\begin{aligned}(12n \pm y)^2 \div 12 &= (144n^2 \pm 24ny + y^2) \div 12\\ &= 12n^2 \pm 2ny + \frac{y^2}{12}\cdot\end{aligned}$$

But $12n^2 \pm 2ny$ is always even; therefore $y^2/12$ should leave a remainder and have an *odd* integer for its quotient. Since y is an integer ranging from 1 to 5, y must be 4. Since the remainder when 4^2 is divided by 12 is 4, the cost of the lamb is 4 dollars. Therefore the man who has one sheep more should pay the man having the lamb 4 dollars, the number of cows being $12n \pm 4$.

Also solved by W. E. Buker, P. S. Dwyer, C. H. Davis, Marie M. Johnson, H. S. Kaltenborn, A. Pelletier, E. A. Whiteman, and Margaret M. Young.

3381 [1929, 286]. *Proposed by E. B. Escott, Oak Park, Ill.*

The following construction is taken from a book on mechanical drawing: *Problem.* To bisect the angle between two given lines whose point of intersection is beyond the limits of the drawing.

Construction. Let the given lines be CD and AB. With an arbitrary point E on CD as center and any radius strike an arc cutting CD in I and AB in G. With G as center and the same radius strike an arc cutting AB in L and CD in E. Draw the common chord of the two circles, HF. With I and L as centers and the same radius strike arcs cutting in J and K. Draw JK. The intersection of HF and JK is O, a point on the bisector.

To find a second point, with O as center, strike an arc cutting AB and CD in two points M and N. With M and N as centers and the same radius strike arcs cutting in P. Draw OP, the required bisector.

Give the proof of the construction.

Solution by Margaret M. Young, Hunter College of the City of New York.

Since O lies on the common chord of the equal circles with centers E and G, $OE=OG$. O also lies on the common chord of the equal circles with centers I and L and hence $OI=OL$. By construction $EI=LG$ and it follows that the triangles EOI and OGL are congruent. Hence their attitudes are equal and O lies on the bisector of one of the angles between AB and CD. Let V be the intersection of AB and CD. Then, since VO bisects the angle under consideration and $ON=OM$, we have $VN=VM$, if M and N have been properly chosen. Hence VO is the perpendicular bisector of MN; and, since $NP=MP$, P must also lie on VO produced.

Note by the Editors. The essentials of this construction are as follows: Two equal segments EI and LG are laid off on the lines CD and AB, respectively. Then the perpendicular bisectors of EG and IL meet on the bisector of one of the angles between CD and AB, while the perpendicular bisectors of EL and IG meet on the bisector of the other angle.

Also solved by A. Pelletier.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Third International Congress of Applied Mathematics will be held at Stockholm, August 24–29, 1930.

On the occasion of the opening of its new chemical laboratory, Princeton University conferred honorary doctorates on Irving Langmuir, of the General Electric Company, and Jean Baptiste Perrin, director of the laboratory of physical chemistry at the University of Paris.

The division of mathematics at Harvard University has established a new mathematical society called the Harvard Mathematical Colloquium. The meetings occur fortnightly, and are open to all mathematicians in Cambridge and its vicinity. Among the speakers of the year are the following: Professors

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the Secretary-Treasurer, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio, before the 10th of each month. Otherwise the extensive business of the Association requires that members be held responsible for magazines sent to their former address; no duplicate copies will be sent except on payment of 45 cents for each copy.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.
INDIANA, Earlham College, May 2-3.
IOWA, May 2-3.
KANSAS, February 15.
KENTUCKY, Lexington, Ky., April 5.
LOUISIANA-MISSISSIPPI, Cleveland, Miss., March 7-8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.
MICHIGAN, Ann Arbor, Mich., March 22.
MINNESOTA, Carleton College, May 17.
MISSOURI.
NEBRASKA, Peru, Neb., May 9.
OHIO, Columbus, Ohio, April 3.
PHILADELPHIA, Philadelphia, Pa., November 29.
ROCKY MOUNTAIN, Denver, Colo., April 11-12.
SOUTHEASTERN.
SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, Calif., March 8.
TEXAS.

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THE AMERICAN MATHEMATICAL MONTHLY

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DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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MATHEMATICS AND THE PROBLEM OF ORE LOCATION[1]

By WARREN WEAVER, University of Wisconsin

A. Introduction

1. *Earth physics.* A large part of modern science consists of seeing the unseen. Indeed it is the essence of science to penetrate behind those external phenomena which appeal directly to our senses, and discover those inner relationships which are visible only to the eye of the mind. Thus today we probe into the atom and the cell, and we reach far out to the extra-galactic nebulae, always coordinating and interpreting the unseen and unseeable in terms of the seen. Recent popular expositions have, to a considerable extent, acquainted the public mind with the two most dramatic aspects of modern physics,—atomic theory, as it reveals to us the structure and behavior of the unimaginably small, and astro-physics, as it reveals to us the constitution and history of the unimaginably vast.

About midway between these two, as regards the scale of dimensions involved, is a third important division—that of earth physics. As old or older than either atomic or astro-physics, it has perhaps been less vigorously developed of late since it is not much concerned with the quantum and relativity theories. Earth physics, in fact, is concerned with the application of the older theoretical subjects such as classical electromagnetic theory, potential theory, and the theory of the propagation of waves in continuous elastic media.

There are, of course, very many branches and sub-divisions of the great science of earth physics. In its broadest aspect it articulates with astro-physics. In its most fundamental aspect it articulates with molecular physics. Between these two limits is a vast range of subject matter. Thus earth physics is concerned with the great problems of the origin and age of the earth; with the thermal history of the earth's surface and interior; with land and sea tides and their friction effect upon evolution of form and motion; with the theory of isostasy and its application to surface features; with the hydrodynamics of underground and surface waters; with the mechanics of plastic flow and elastic deformation as these furnish powerful investigative tools in structural geology; with meteorology in all of its manifold aspects; with terrestrial magnetism; w th atmospheric electricity; and with many, more specialized, subjects. I have taken the opportunity to mention these challenging and significant fields of research, even though we have no concern here with them, in order to register thereby a mild protest against the notion, apparently held by some, that the only worth while fields of modern physical research are those which concern either an atom or the cosmos. Here, as in all matters, there is a middle ground.

It has been noted above that a great purpose of science is to see with the aid of experiment and of the mind's eye what is otherwise unseen; for it is only

[1] Read at the Annual meeting of the Mathematical Association of America at Des Moines, Iowa on Jan. 1, 1930.

thus that we discover those hidden relationships and correlations whose systematization is the central core of a successful science. It is our purpose to consider some of those branches of earth physics which enable us to see below the surface of the ground, and obtain information concerning the hidden substance and structure. More particularly, it is our purpose to consider some of the interesting fields of mathematical research which are connected with this new experimental art.

2. *The possibility of sub-surface exploration.* As we stand on the surface of the earth and observe its opacity with our eye and its dense solidity with our foot, it does indeed seem strange that there is any way whereby we can penetrate that solid mass save by the actual brute method of diamond drill or dynamite. Yet when one stamps his foot on the ground to demonstrate its solidity and his impotence to penetrate it, this very act sends tiny vibrational waves, similar to sound waves, down to great depth. And as regards the apparent opaqueness, we must remember that the human eye is sensitive over a very narrow range of color or frequency,—actually only one octave; while we have instruments to produce or receive light or electromagnetic waves varying from hard X-rays of wave length 10^{-9} cm. to long radio waves of wavelength 10,000 meters or more—a range of roughly 50 octaves. We remember, moreover, that the penetrating power of any such light wave into a given medium depends on the frequency, and by departing from the frequency of visible light, we can find waves which penetrate with ease such dense materials as rock or earth. Thus, both sound and light do pass easily through the rocks and earth if we but choose the proper sort of sound and light, and we may hear the echoed message and see the otherwise hidden picture if we but use the proper sort of ear and eye.

One has reason to hope, therefore, that he may explore below the surface of the ground. The process reduced to its simplest terms consists of a sending device which produces messengers which are dispatched down into the earth; and a receiving device which decodes the report when part of these messengers find their way back to us. We shout down questions and the ore whispers back its answer. Sometimes the whispers are too faint for us to hear or too confused for us to understand; but, at other times, we get a clear reply. In some cases, moreover, we get the answers without even asking the questions. Some ore bodies are the oldest of advertisers, in that they have been broadcasting for ages. To discover them, we need only a receiver suitable for understanding what they are trying to tell us.

Of the messengers which we may send, the principal ones are electrical currents, electromagnetic fields of various sorts, or mechanical vibrations. Of the messengers which are always present whether we will or no, the principal ones are gravitational attractions; the earth's own magnetic field and the fields it automatically induces in magnetic material; and the natural earth currents due to the battery-like action of certain sulphides. Corresponding to these various messengers, we have five principal methods used in the geophysical

search for ore or oil; namely, 1) Magnetic, 2) Electrical, 3) Electro-magnetic, 4) Gravitational, 5) Seismic. Of these five methods, two, viz., the magnetic and gravitational, are pure receiving methods in that nature always furnishes the messengers and we need only listen. Two, viz., the electromagnetic and the seismic, are sending-receiving methods, the messengers being furnished by us. The electrical method makes use of both types of messengers.

B. The Methods of Geophysical Exploration for Ore and Oil

1. *The magnetic method.* In order to appreciate the significance of the mathematical problems, it is necessary to have at least a general idea of these five geophysical methods. Let us consider, first, the magnetic method, which, although it is particularly simple, is in several regards typical of all the methods which make use of electrical effects. The magnetic method of surveying is based on two simple facts; viz., the fact that a magnet attracts or repels a compass needle; and the fact that certain substances, when placed in a magnetic field, automatically become magnetized by that field and hence become magnets. When various magnetizable substances are located in the same magnetic field, the intensities of the magnetism induced in the various substances are different, and the ones that magnetize more strongly are said to possess the greater magnetic susceptibility. Granted the two effects mentioned just above, it is only necessary to have some external magnetic field present in order that the various rock or mineral masses become magnetized in amounts depending upon their various susceptibilities. We can then observe the action on a compass needle of the induced magnets thus formed, and can draw inferences as to the location, size, and constitution of the active masses. The necessary external magnetic field is conveniently omnipresent, for the earth's own magnetic field exists not only in the air, but at all depths below the surface, and plays this role very satisfactorily. This inducing external magnetic field furnishes the messengers, penetrating to the unknown regions, which stimulate or excite these unknown regions to send to us further messengers (the induced magnetic field) whose reports we receive in our instruments and interpret through our theory.

Now the vast majority of rocks and minerals have very low susceptibilities. That is to say, they are magnetized exceedingly feebly when placed in a magnetic field. Certain of the iron compounds, on the other hand, have abnormally high values for their susceptibilities. The most striking and important example is magnetite, which has a susceptibility ranging from a hundred to a thousand or more times the minimum detectable value. Besides magnetite, ferriferous minerals such as franklenite, ilmenite, pyrrhotite, and, to a lesser degree, specular hematite, are all strongly magnetic. It is not correct, however, to assume that the magnetic method is restricted to the search for iron ore. In a copper region, for example, it may be of the greatest importance to be able to locate and trace the contact between the sedimentary rock and an igneous intrusion; and there is often sufficient difference in susceptibility to permit

this. Magnetic methods have, in fact, been used to locate inclusions of iron formation in an intrusive gabbro; to locate faults in iron formations; to map distributions of lavas and intrusives; to trace copper bearing lodes; to trace

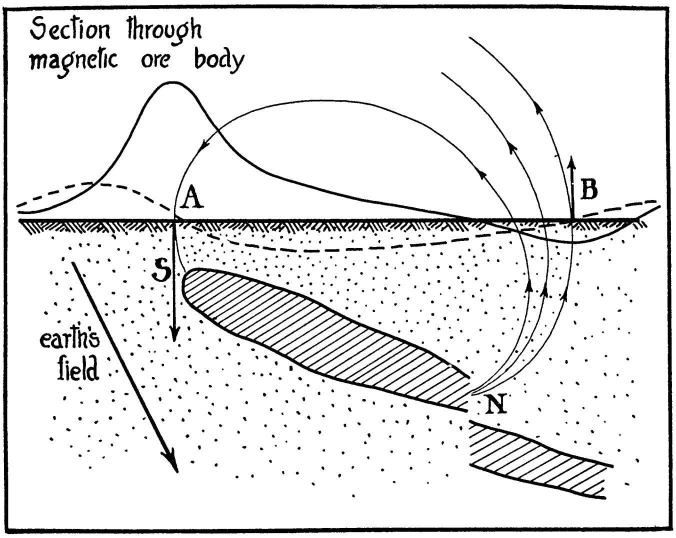

FIG. 1

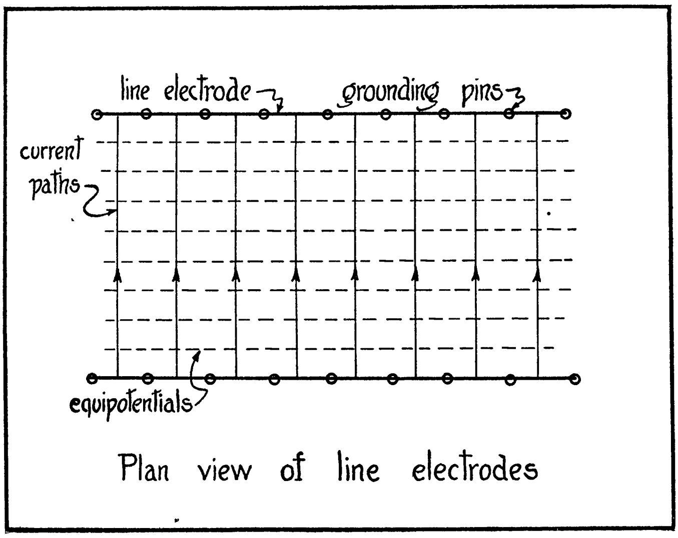

FIG. 2

igneous contacts and dikes; to trace the outcrop of a sedimentary bed; to locate gold placer deposits; and in many other connections as well.

We may now trace in a little more detail how the magnetic method pro-

[2] Figures 1 and 7 are modifications of figures occurring in Eve and Keyes *Applied Geophysics*.

ceeds. Suppose the cross hatched region of figure[2] 1 represents a magnetic ore body while the surrounded dotted material is relatively inert. Due to the earth's magnetic field, indicated by the arrow, the ore body becomes magnetized. If the ore body were absent, the magnetic field of the earth would be sensibly uniform over the region in question. Due to the ore body, the field is distorted. Any instrument measures the entire field, composed of the normal earth's field plus the field due to the ore body, If, however, one subtracts from the total measured field the normal earth's field, he obtains the portion due to the ore body itself. This abnormal part of the field, due to the disturbing presence of the body, is called the magnetic anomaly; and since it is a force, one may resolve it into a horizontal anomaly and a vertical anomaly. The solid and dotted curves, drawn with the ground line as the axis of abscissas, show the variation of the vertical and horizontal magnetic anomalies. The light curved lines passing from the north to the south poles of the ore magnet roughly indicate the field of magnetic force due to the ore magnet. For example, at a point A approximately above the south pole of the ore magnet, the field of the ore magnet is directly downward, so that the horizontal anomaly is zero, while the vertical anomaly is large. A similar situation exists at B over the north pole except that the vertical anomaly is reversed in sign. The fact that the faulted end N of the ore body is more deeply buried is revealed by the fact that the vertical anomaly at B is considerably less than that at A. If the indicated fault were not present so that the north pole of the ore magnet were very deeply buried, the negative value of the vertical anomaly would be entirely absent. It is clear that the quantitative interpretation of such data involves a large amount of theoretical information concerning the exact way in which bodies of various shapes magnetize, and concerning the fields which they then cause.

The actual measurement of the intensity and direction of the magnetic field at any point involves the use of instruments such as the dip needle, the magnetometer, the magnetic variometer, the earth inductor etc. It is outside our purpose to describe these instruments or explain their use. As a concluding remark concerning the magnetic method, it should be emphasized that this method reveals discontinuity of substance or structure only when these discontinuities are characterized by different values of the magnetic susceptibility. Only materials which have reasonably differing susceptibilities can be distinguished.

2. *The electrical method.* When a source of electric current exists in any three-dimensional region, the spatial distribution of the resulting currents depends on the geometry of the region, the location and shape of the source, and upon the electrical conductivities of the various parts of the region. If the region be homogeneous as regards conductivity, one may call the accompanying current distribution a *normal* one. If there be any departures from a uniform conductivity, the current distribution is abnormal. Accompanying this abnormal distribution of current is an abnormal distribution, throughout the interior and on the surface of the region, of the electrical potential. The electrical

methods of surveying measure the potential at points on the surface of the ground, and deduce from abnormalities in these potentials the sub-surface variation in electrical conductivities. As a source of electric current, one uses

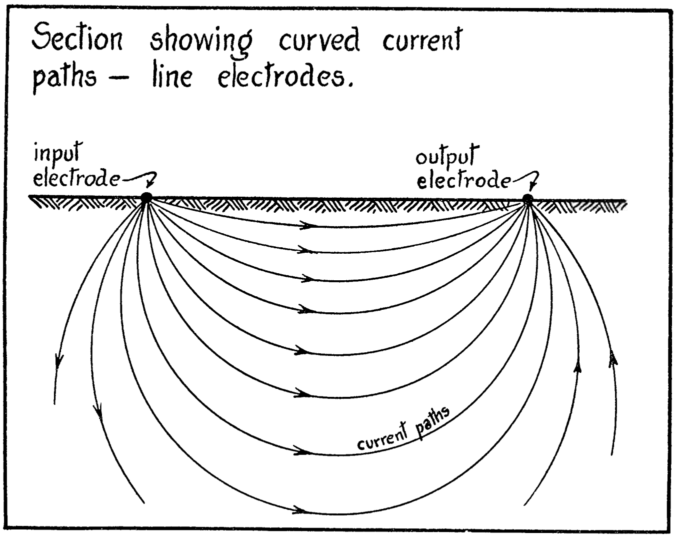

FIG. 3

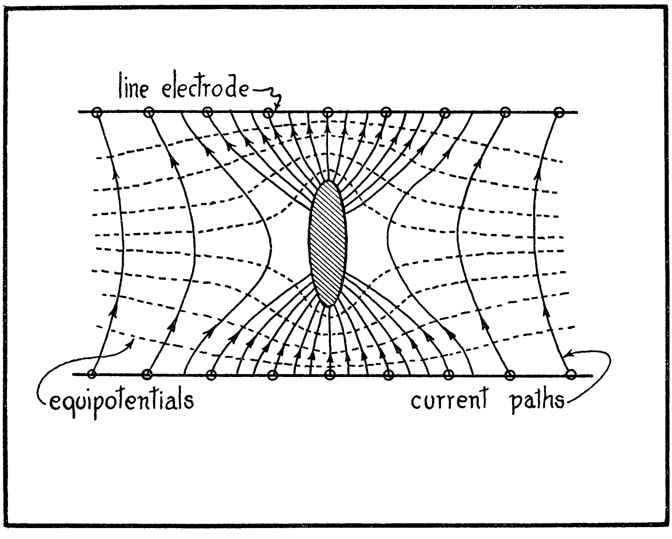

FIG. 4

either point or line electrodes, maintained at desired potentials by connecting them to a battery or generator. (See Fig. 2.) Long line electrodes are usually approximated by connecting together a series of metal stakes or grounding pins spaced at regular intervals. In the case of a homogeneous earth, the shallow

portion of the current which stays near the surface of the ground passes on straight lines from one electrode to the other, the equi-potentials being straight lines parallel to the line electrodes. It is only the surface current that passes along the straight line shown. In general, the current passes along the curved lines shown in Fig. 3. If there is in the region between the two line electrodes a mass of higher conductivity, then the whole current-distribution is altered. Fig. 4 shows the current paths and the accompanying equi-potentials, when a highly conducting lens is located between the electrodes. The distortion of the normally straight and parallel equi-potentials reveals the presence of the mass of higher conductivity.

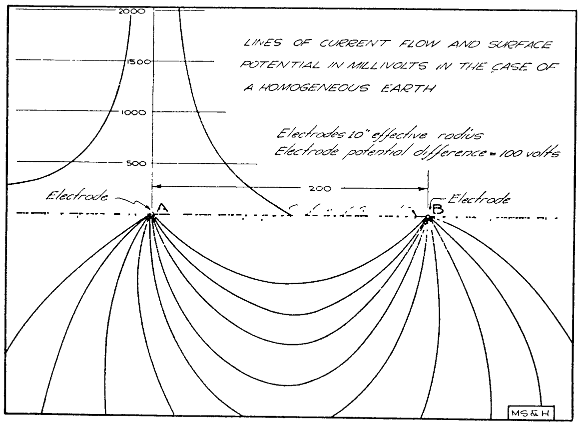

FIG. 5

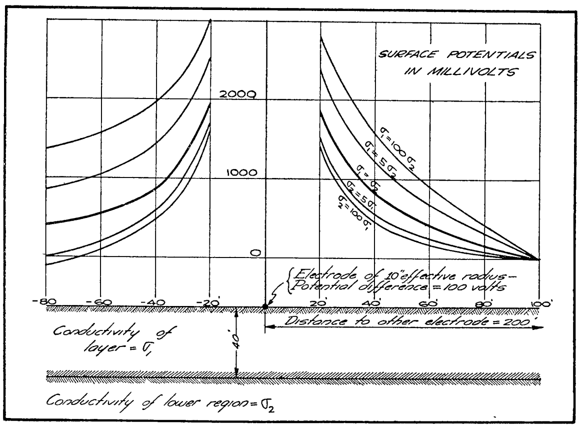

FIG. 6

The point electrode method differs from the line electrode method principally in the fact that the current enters and leaves the earth at points, rather than along lines. Fig.[3] 5 shows in section the surface of the ground; the input

[3] Figures 5 and 6 are the property of Mason, Slichter, and Gauld, Madison, Wisconsin, and are used with their permission.

and output electrodes at A and B; the deep-lying current paths; and, plotted above the surface of the ground, the potential at the surface of the ground and along the line joining the two electrodes. Fig. 6 illustrates the point electrode method as applied to the location of a layer of depth 40 feet and one conductivity σ_1, underlaid by material of conductivity σ_2. The heavy curve $\sigma_1 = \sigma_2$ is the normal potential found when the layer is absent. If the underlying region is a good conductor, the potential curve falls below the normal curve; while if the layer is much the better conductor, the curve lies above the normal curve.

3. *The electromagnetic method.* One should note that the magnetic and the electric methods utilize different physical characteristics of rock or mineral. The success of one method depends on differences in magnetic susceptibilities; the other on differences in electrical conductivity. When one shouts down to the ore, he must be careful to ask questions that are suited to the particular ore being sought. One characteristic of the electromagnetic method (which we will now consider briefly) is that its questions and answers do not depend on one single electrical characteristic but rather on several. This is both an advantage and a drawback. When one shouts out a question in five languages, there is, to be sure, a good chance that some one will understand and reply; but there is also a good chance that one will get several simultaneous and hence confused replies.

The electromagnetic method is not one that lends itself well to simple explanation. The sending device is usually a coil or loop antennae, a good deal like those sometimes used in radio. The receiving device is similar to a radio receiving set. Through the sending coil is passed an alternating current. This alternating current sets up, for very considerable distances from it, both an electric and a magnetic field. These fields penetrate the earth, the effective depth of penetration depending on the strength of the source, the frequency of the alternating current, and on the electrical and magnetic properties of the earth and rocks. Due to these penetrating fields there are induced, in the earth and in all bodies present, electric currents, magnetizations, and electric polarizations. Thus the total effect at any point is due, in various parts, to the electrical conductivities, the magnetic susceptibilities, and the dielectric properties of the earth and rocks. There are, also, very many things one can and should measure with his receiving instruments before he can hope to understand the messages the hidden region is trying to send. One can measure the magnitude and direction of the electric field and of the magnetic field; he can measure the phase difference, between these two fields, at various points; and he can observe the dependence of these quantities on variation of the frequency of the alternating current. One obviously has, here, a much more complicated situation than existed in the magnetic or electric cases, where only one quantity was available for measurement.

4. *The gravitational method.* All of the methods so far considered make use of the fact that various rocks and minerals have varying electrical properties. The gravitational method, as one might expect, has nothing to do with electrical effects, and depends solely upon density. The gravitational method makes use of almost unbelievably delicate instruments, called torsion balances. In essence, such an instrument is very simple. Two equal masses (Fig. 7) m and m' are swung, at different levels, by a very fine wire and aluminum rod system. A heavy mass below this torsion balance and on the side towards the reader attracts the lower bob more than the upper (simply because it is nearer the lower). This difference in attraction results in a torque which turns the rod

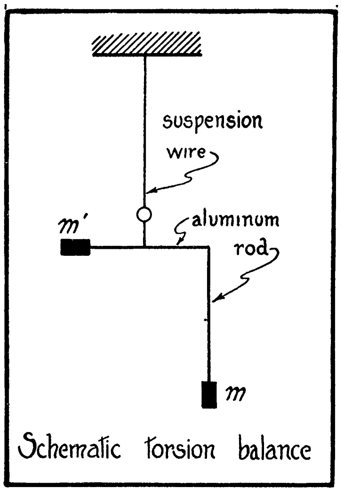

FIG. 7

and mass system until a counter-torque is developed in the wire suspension. In this way regions of sub or super-normal density below and to one side of the balance cause deflections of the suspended system.

5. *The seismic method.* We turn now to the last of the five methods, the seismic. In this method a powerful source of vibrational waves is obtained by exploding a buried charge of TNT. These waves are somewhat like the electric currents flowing between power electrodes, in that the vibrational waves also travel on curved paths, penetrate to very considerable depths, and eventually emerge at the surface. These waves are received by seismographs, entirely similar to those used to record earthquakes. These wave messengers are thus sent down, and return to give us their messages. And we draw our inferences chiefly from the time at which the messengers return to us; for the speeds with which they travel depend on the densities and elastic properties of the materials through which they have passed. The seismic method has been used chiefly in oil work, although it has also been used, in Sweden, to determine depth of overburden. The seismic method depends on several mechanical properties

of the rocks, rather than merely on density, as does the gravitation. Unfortunately we have meager data concerning the Young's modulus, bulk modulus, shear modulus, and Poisson's ratio for earth, rocks, and minerals, and the theoretical aspects of this method are by no means complete.[4]

C. The Mathematical Problems

Unsatisfactory as so brief a resume must necessarily be, we turn now to a consideration of the mathematical problems connected with the geophysical methods of prospecting. Many of these problems are new problems; not new in the sense that they involve recent theories, but new in that they involve, for the most part, the application of classical theories to hitherto uninteresting situations.

Until the advent of radio, the whole great development of the electrical art concerned the use of electricity *in wires*. A wire is essentially a one-dimensional object, its cross sectional dimensions being negligible as compared to its length, so that the theory of electricity in wires is essentially a one-dimensional theory. The problem of the spreading of radio signals was one of the first really practical electrical problems which involved more than one dimension. The theoretical problems of geophysical prospecting are practically all either two or three dimensional problems. A one-dimensional problem means, to the mathematician, an ordinary differential equation to solve; two or three-dimensional problems mean partial differential equations. We are all well aware of the enormous increase of difficulty that results, in boundary value problems, when the number of dimensions is increased from one to two, or from two to three. A startlingly small number of three-dimensional problems have been solved in exact form; and of the exact solutions actually obtained, few are usable in a practical computative sense.

The unity, from the point of view of mathematical theory, of all of the methods of geophysical prospecting becomes evident when we consider the fundamental equations which govern the various phenomena. One may write the general wave equation,

$$\nabla^2\Psi - \frac{1}{\kappa^2}\frac{\partial^2\Psi}{\partial t^2} = -\Theta,$$

and the equation suitable for any one of the five methods may then be obtained by specializing the quantities Ψ, κ, and Θ. In general, Ψ is the quantity whose behavior is sought; κ is the velocity with which values of Ψ are propagated,

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[4] The reader who wishes to supplement these exceedingly brief remarks concerning the magnetic, electrical, electromagnetic, gravitational, and seismic methods is referred to the volume, *Geophysical Prospecting*, published in 1929 by the American Institute of Mining and Metallurgical Engineers, and in particular to the article *Geophysical Prospecting For Ores*, by Dr. Max Mason (also published in the Bulletin of the Mining and Metallurgical Society of America, vol. 20 (1927), pp. 93–133); also to the bibliography by C. A. Heiland, published in the *Annotated Bibliography for Economic Geology* for 1928.

and Θ is a sort of forcing function or source function which generates non-vanishing values of Ψ. Thus, for the magnetic method:

Ψ is the magnetic vector potential $\boldsymbol{A}$;
κ^2 is infinite so that $(1/\kappa^2)\ \partial^2\boldsymbol{A}/\partial t^2$ is zero;
Θ is zero;

and the equation becomes Laplace's equation

$$\nabla^2\boldsymbol{A} = 0.$$

For the electric method:

Ψ is the electric potential Φ;
κ^2 is infinite;
Θ is zero;

and the equation becomes Laplace's equation

$$\nabla^2\Phi = 0.$$

For the electro magnetic method:

Ψ is the electric vector $\boldsymbol{E}$,
or the magnetic vector $\boldsymbol{B}$;
κ^2 is $\mu\epsilon/c^2$;
Θ is $-(\nabla\rho/\epsilon)-(\mu/c^2)\ \partial(\rho\boldsymbol{u})/\partial t$,
or μ curl $(\rho\boldsymbol{u})/c$,

where μ is the magnetic permeability, ϵ the dielectric constant, c the velocity of light in free space, ρ the density of charge, and $\boldsymbol{u}$ the vector velocity of charge. The general equation thus gives the two forms,

$$\nabla^2\boldsymbol{E} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2} = \frac{1}{\epsilon}\nabla\rho + \frac{\mu}{c^2}\frac{\partial(\rho\boldsymbol{u})}{\partial t},$$

$$\nabla^2\boldsymbol{B} - \frac{\mu\epsilon}{c^2}\frac{\partial^2\boldsymbol{B}}{\partial t^2} = -\frac{\mu}{c}\operatorname{curl}(\rho\boldsymbol{u}).$$

For the gravitational method:

Ψ is the gravitational potential Φ;
κ^2 is infinite;
Θ is the volume density of matter ρ;

and the equation becomes Poisson's equation

$$\nabla^2\Phi = -\rho.$$

For the seismic method:

Ψ is the dilatation s;
κ^2 is $(\lambda+2\mu)/\rho$;
Θ is zero;

where λ and μ are the standard elastic constants, and ρ is the density. The general wave equation then becomes

$$\nabla^2 s - \frac{\rho}{\lambda+2\mu}\frac{\partial^2 s}{\partial t^2} = 0.$$

One is not, of course, surprised to learn that the same general differential equation governs all these methods. The wave equation, and as special cases of it, the equations of Laplace and Poisson, express to the mathematician the very essence of the manner in which natural phenomena proceed; this observation having increased significance now that we have a wave theory basis for particle and quantum dynamics. The observation that there is a central unifying structure behind all these methods may often assist one in adapting one mathematical solution to several of the methods. One should remember that the formal equivalence of two such problems, however, involves equivalent boundary conditions, as well as the same differential equation. Let us, then, note the types of boundary conditions which arise, restricting our further attention to the three methods which make use of electrical effects. In the magnetic method:

A is continuous everywhere.
A vanishes at infinity as $1/R$.
The tangential derivative of A is continuous.
The normal derivative of A is discontinuous.

$$\frac{\partial A}{\partial n_1} + \frac{\partial A}{\partial n_2} = \left(1 - \frac{1}{\mu}\right)[\boldsymbol{n}, \text{ curl } A].$$

For the electrical methods:

Φ is continuous everywhere.
Φ vanishes at infinity as $1/R$.

$$\sigma_1\frac{\partial \Phi}{\partial n_1} + \sigma_2\frac{\partial \Phi}{\partial n_2} = 0.$$

For the electromagnetic methods:

$\boldsymbol{E}$ and $\boldsymbol{B}$ vanish at infinity as $1/R^2$.
The tangential components of $\boldsymbol{E}$ are continuous.
The normal component of $\boldsymbol{B}$ is continuous.

$$\mu_2 B_{t_1} + \mu_1 B_{t_2} = 0.$$

$$\epsilon_1 E_{n_1} + \epsilon_2 E_{n_2} = 0.$$

Thus in all these cases, the boundary equations which hold across a surface separating two media are linear in the dependent variable or its normal derivatives, the coefficients of these linear relationships characterizing the electrical properties of the two media.

Such boundary value problems have received the attention of mathematicians for a century and more. But the problems of geophysics differ from many classical problems in several respects. First of all, as mentioned above,

they are two or three-dimensional problems. Secondly, there are practically always three important regions of space involved, each with its own characteristic electrical properties,—the air, the normal homogeneous earth, and the ore mass. The older problems seldom involved more than two distinguishable regions.[5] Thirdly, it is necessary to have exact knowledge near the sources of disturbance, rather than far away, as in the classical problem of radio signals. Fourthly, it is useful, in the electromagnetic case, to obtain solutions when the wave length is comparable with the dimensions of the diffracting ore mass, rather than small, as in the classical optical case. This consideration makes invalid many of the approximation methods used in earlier work. Fifthly, it is necessary, in general, to assign finite but different conductivities to the various regions under consideration. This is a point of great importance, and the source of much difficulty. Sixthly, many regions of geometrical shapes, which were previously of only academic interest, now become of great practical interest. And, lastly, it is essential now that the solutions be usable from a computative point of view.

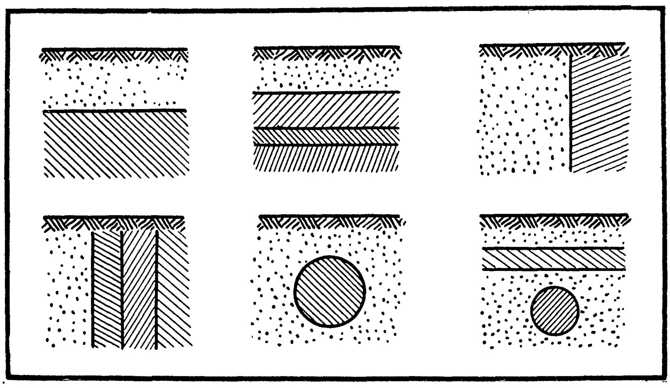


FIG. 8

The actual problems of nature are impossibly complicated by minor nonhomogeneities, by uneven surface topography, by irregular moisture content, by the possible presence of several disturbing masses, and by many other things

[5] One usually avoids the necessity of satisfying the boundary conditions on the surface of the ground between the earth and air by reflecting the entire configuration in the surface of the ground. Thus rather than dealing with a problem which involves the air, a semi-infinite earth, and a buried sphere, one solves the problem of an infinite earth containing two spheres. The plane which is perpendicular, at its mid-point, to the line joining the centers of the two spheres is a plane of symmetry. If the current source be located on this plane, no current will cross this plane. Thus one can "throw away" the upper half of the solution, and retain the correct solution for a semi-infinite earth and a buried sphere. This reflection method does not, however, afford an essential simplification of the three-region difficulty mentioned above: for there are still three regions,—the infinite earth, the first sphere, and the second sphere. Taking into account the interaction between the two spheres, is exactly equivalent mathematically to taking account of the boundary conditions between the earth and the air.

as well. Just as a child must begin with simple tunes if he is ever to understand a symphony, so the geophysicist must, at first, be content with simplifications and over-idealizations of his problems. Thus, for the electrical method, one attempts to calculate the distribution of currents when a homogeneous earth is overlaid by one or more plane layers of different conductivities; when there are one or more parallel vertical planes across which the conductivity suddenly changes; when there is a submerged vertical dyke, or a submerged ledge; or when an otherwise homogeneous earth contains a buried sphere or ellipsoid.

Figure 8 shows, schematically, cases of current flow which have been satisfactorily solved both for point and long lines electrodes. The dotted region indicates the normal earth, while the cross hatched regions indicate ore bodies or structures of differing conductivities. The layered structures have been treated by several investigators although, for commercial reasons, a complete analysis has not been published.[6] The layered structure containing a buried sphere has been treated by the writer.[7] The point electrode in the presence of a buried sphere has recently been solved exactly.[8] Similar solutions for a buried ellipsoid would be of great usefulness since an ellipsoid can take on, as special cases, long needle like or flat lens like shapes.

The mathematical analysis involved in the layered structure was indicated by Maxwell;[9] and, for the simpler case of infinite conductivity ratio, by Riemann.[10] This analysis involves infinite series of the type

$$\sum_{-\infty}^{+\infty}{}_n(-1)^n\left\{[\rho^2+(z+2nh-a)^2]^{-1/2}-[\rho^2+(z+2nh+a)^2]^{-1/2}\right\}$$

which are rather slowly convergent and very tedious to compute. In this expression z measures distance perpendicular to one of the horizontal planes bounding the layer, $\rho^2=x^2+y^2$, where x and y are Cartesian coordinates in a horizontal plane, h is the thickness of the layer, and a is the spacing between electrodes. These series have been satisfactorily handled for large ρ, but their study for small ρ is a pressing problem.

The current flow problems, in the case of long line electrodes and suitable non-homogeneities, are two-dimensional problems involving the solution of the Laplace equation $\nabla^2\Phi=0$. Here the method of conjugate functions applies. Much that has been written concerning the theory of conformal mapping has come from the pens of mathematicians who had little interest or information about the application of this theory to electrical problems, or from the pens of physicists who had only a mild respect or appreciation for the mathematical

[6] See, however, J. N. Hummel, *Der Scheinbare Spezifische Widerstand bei Vier Planparallelen Schichten*, Zeitschrift für Geophysik, vol. 5 (1929), Heft 5/6, 5–228.

[7] Not published in full. See, however, *Certain Applications of the Surface Potential Method*, American Institute of Mining and Metallurgical Engineers, Tech. Pub. 121 (1928).

[8] Unpublished thesis by Dr. J. H. Webb, now of Williams College.

[9] James Clerk Maxwell, *Electricity and Magnetism*, 1873, Vol. I, § 317, page 367.

[10] Riemann, *Gesammelte Werke*, page 59.

aspects. We need an accurate and illuminating exposition of the Schwartz-Liouville theory of the mapping of polygons and of the application of this theory to electrical problems, along with an encyclopaedic collection of examples.

There are many simple shapes of ore mass for which the solution is needed. Fig. 9 indicates six of these unsolved problems, the cross hatched regions again

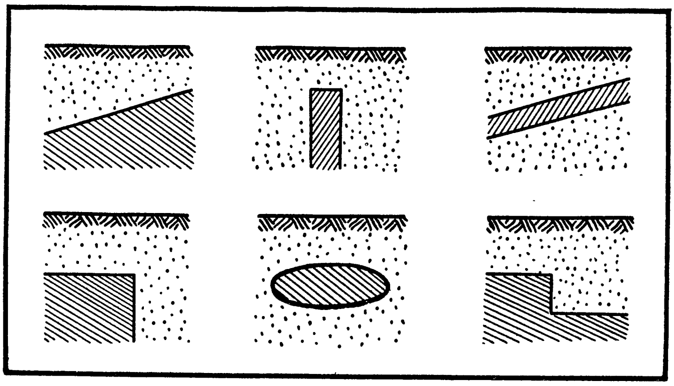


FIG. 9

being the ore masses of higher conductivity, and the dotted regions being the normal earth. One desires the current flow, in all these cases, whether the current enters the earth at a point electrode, or along a line electrode. It is interesting to note that although the horizontal layer has been treated rather fully, the slanting layer has so far withstood all attempts at solution. The ordinary method of images is not applicable to the slanting layer.[11]

It would also be of great interest to treat some simple cases of non-uniform but continuously changing conductivity. If the conductivity is variable, the governing differential equation is no longer linear. Such a problem was formerly not of much practical interest, but is now exceedingly interesting. The most promising mode of attack seems to be through an integral equation set-up, treating the continuously changing density as a limiting case of multi-layered structures.

The electromagnetic method wishes to know, as a first simple approximation, the induced currents and the total earth and air fields, when one uses, as a sending device, an oscillating electric or magnet dipole, or a long overhead wire carrying alternating current. This problem is fairly solved for the case of a homogeneous plane earth. And we have fairly satisfactory approximations to cover the case of a sphere buried in an otherwise homogeneous plane earth. The long line over a layer of finite thickness and conductivity has just been solved.[12] A dipole over such a layer has not yet been solved, at least in usable form.[13] In

[11] See, however, the last footnote of the paper.

[12] Doctor's dissertation, as yet unpublished, by Dr. H. P. Evans, University of Wisconsin.

[13] Since this paper was written, this problem has been solved. See *Asymptotic dipole radiation formulas*, by W. Howard Wise, in the Bell System Technical Journal, VIII, 4, page 662 (October 1929).

fact, a doublet over an infinitely thin layer was not treated until 1922 when Max Abraham, in the last paper he wrote,[14] calculated the induced currents for this case, but made an error in the calculation of the fields. A doublet located over a homogeneous earth in which is buried an *ellipsoidal* mass presents a problem of great interest, and one which has not been usefully solved. These problems of the induced fields in the presence of oscillating dipoles are analogous to the classical problems of the diffraction of light by an obstacle. They are more difficult, however, since the various media have *finite*, but differing, conductivities; and also since they are diffraction problems of the Fresnel, rather than the more usual Fraunhofer type—that is, the source is at a finite distance rather than at infinity. One realizes the probable difficulty of these geophysical problems when he remembers that no one of the classical diffraction problems was solved exactly until Sommerfeld obtained the solution for the straight edge.[15] Any mathematician would do a real service who would study and extend the Sommerfeld multiple-space reflection method, carrying it through for finite conductivities, using sources at finite distances, and, most important of all, discovering how these complicated formal solutions can be made available for computation.

D. Conclusions

These scattered and hasty remarks can hardly be expected to give more than a shadowy suggestion of the sort of problems that confront the theoretical geophysicist. The ancient searcher after ore sought to find the hidden treasures of the earth by using a forked hazel stick, whose magic properties would direct him to the buried veins. The geophysicist of today, with his dip-needle, his volt-meter, his radio apparatus, his slide-rule, and his text-book on partial differential equations, is but the modern counterpart of the old practicer of the black arts who went about with his forked hazel wand. Many years ago when his international fame rested solely on his ability as a mining engineer, President Herbert Hoover, together with Mrs. Hoover, translated from the Latin a very famous old treatise on mining, written by Georgius Agricola during the twenty years preceding its publication in 1566. This book contains the first description of the application of divining rods to the location of ore. The last figure (Fig. 10) shows a wood-cut occurring in this interesting (and now rare) book. You will observe the various experts going about with their hazel wands, and the shallow trenches which are being dug to test the indications. With a freedom from superstition that seems amazing for his age, and with a scientific clarity that would grace any age, Agricola considers the evidence for and against the forked twig, and concludes "Therefore a miner, since we think he ought to be a good and serious man, should not make use of an enchanted twig, because if he is prudent and skilled in the natural signs, he understands that a forked stick is

[14] M. Abraham, Zeitschrift für Angewandte Mathematik und Mechanik, 1922, page 109.

[15] Riemann-Weber, *Die Differential-und Integralgleichungen der Mechanik und Physik*, 1927, Vol. 2, page 433.

of no use to him." We must not hastily judge, however, that Agricola would condemn the modern forked stick. His truly scientific spirit, and the attitude

FIG. 10

proper for us, are alike indicated by this ancient author's remark: "Since this matter remains in dispute and causes much dissension amongst miners, I consider it ought to be examined on its own merits."

AVERAGE CURVES AND THEIR HARMONIC ANALYSIS

By LOUIS BRAND, University of Cincinnati

This paper gives an analytic solution of an important problem arising in the design of alternating current generators. My colleague, Professor W. C. Osterbrock of the department of electrical engineering, states the problem as follows:

"In the alternating current generator magnetic flux is set up by a revolving field structure excited from a separate source by a continuous current. The electromotive force is generated in coils distributed in slots on a stationary armature. The value of the electromotive force in any conductor at any time is $e=Blv$ electrostatic units, where B is the magnetic flux density at the conductor (gauss), l the length of the conductor (cm.), and v the peripheral velocity of the field structure (cm./sec.). This will be a function of period 2π of the angular displacement along the armature such that

$$f(x+\pi)=f(x-\pi)=-f(x). \tag{I}$$

"The various coils forming one group are distributed in slots covering one pole pitch (π radians) and are connected in series so that their electromotive forces add. The problem then arises to find the sum of n such functions as given above, each displaced from the preceding by an interval π/n."

This problem is solved graphically by drawing the original *base curve* $y=f(x)$ and its successive displaced positions and adding the ordinates at any point to obtain the sum curve. However the problem may be approached analytically by finding the limit of the average ordinate at any point as n become infinite. The ordinates of this *limit curve*, $y=F(x)$, multiplied by n, will then give a satisfactory approximation to the sum curve for a finite n if n is not too small.

When the base curve $y=f(x)$ is shifted a distance π to the right, all the ordinates between $x-\pi$ and x pass over the point x. The limit curve is clearly the average of these ordinates; hence

(1) $$F(x) = \frac{1}{\pi}\int_{x-\pi}^{x} f(u)du.$$

Consider first the case when $f(x)$ satisfies the condition (I). Then

$$\pi F(x) = \int_{x-\pi}^{0} f(u)du + \int_{0}^{x} f(u)du,$$

in which the first integral equals

$$\int_{x}^{\pi} f(v-\pi)dv = \int_{\pi}^{x} f(u)du = \int_{0}^{x} f(u)du - \int_{0}^{\pi} f(u)du.$$

Hence, on writing

$$h = \frac{1}{\pi}\int_{0}^{\pi} f(u)du,$$

(2) $$F(x) = \frac{2}{\pi}\int_{0}^{x} f(u)du - h.$$

From (2) we see that

(3) $$F(0) = -h, \qquad F(\pi) = F(-\pi) = h;$$

$$F(x+\pi) = \frac{2}{\pi}\int_{0}^{\pi} f(u)du + \frac{2}{\pi}\int_{\pi}^{x+\pi} f(u)du - h = h - \frac{2}{\pi}\int_{0}^{x} f(u)du;$$

(4) $F(x+\pi) = -F(x).$

Hence $F(x)$ also has the property (I). Moreover

(5) $$F'(x) = 2f(x)/\pi.$$

From the standpoint of generator design it is important to have an harmonic

analysis of $F(x)$. In view of (I) the Fourier expansion of $f(x)$ in the interval $(-\pi, \pi)$ contains only the sines and cosines of odd multiples of x:

$$f(x) \sim \sum_{k=0}^{\infty}[a_{2k+1}\cos(2k+1)x + b_{2k+1}\sin(2k+1)x],$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx, \qquad b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx.$$

From (4) we see that the expansion of $F(x)$ has precisely the same form. Moreover on integrating by parts and making use of (3) and (5) we find

$$A_n = \frac{2}{\pi}\int_0^\pi F(x)\cos nx dx = -\frac{4}{\pi^2 n}\int_0^\pi f(x)\sin nx dx = -\frac{2}{\pi n}b_n,$$

$$B_n = \frac{2}{\pi}\int_0^\pi F(x)\sin nx dx = \frac{4}{\pi^2 n}\int_0^\pi f(x)\cos nx dx = \frac{2}{\pi n}a_n.$$

Therefore

$$F(x) \sim \frac{2}{\pi}\sum_{k=0}^{\infty}\left(\frac{a_{2k+1}}{2k+1}\sin(2k+1)x - \frac{b_{2k+1}}{2k+1}\cos(2k+1)x\right).$$

Owing to the factor $1/(2k+1)$, the upper harmonics are far less important in $F(x)$ than in $f(x)$. This, of course, is very desirable when the objective is a pure sinusoidal electromotive force.

We consider next the case when $f(x)$ is an odd function of period 2π:

$$\text{(II)} \qquad f(-x) = -f(x).$$

From (1) and (II) we now have

$$\pi F(x) = \int_{x-\pi}^0 f(u)du + \int_0^x f(u)du = \int_0^x f(u)du - \int_{\pi-x}^0 f(-v)dv,$$

$$\text{(6)} \qquad F(x) = -\frac{1}{\pi}\int_x^{\pi-x} f(u)du.$$

Hence

$$\text{(7)} \qquad F(0) = -h, \qquad F(\pi) = F(-\pi) = h;$$

$$F(-x) = -\frac{1}{\pi}\int_{-x}^{\pi+x} f(u)du = \frac{1}{\pi}\int_x^{-\pi-x} f(-v)dv = -\frac{1}{\pi}\int_x^{-\pi-x} f(u)du;$$

and, on adding

$$-\frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(u)du = 0$$

to the last integral, we find that

$$F(-x) = F(x). \tag{8}$$

Moreover,

$$F'(x) = \frac{f(x) + f(\pi - x)}{\pi}. \tag{9}$$

Since $f(x)$ is an odd function and $F(x)$ an even function, their Fourier expansions are sine and cosine series respectively:

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx dx,$$

$$F(x) \sim \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos nx, \quad A_n = \frac{2}{\pi}\int_0^{\pi} F(x) \cos nx dx.$$

On integrating by parts and making use of (7) and (9), we find

$$A_n = -\frac{2}{\pi n}\int_0^{\pi} \frac{f(x) + f(\pi - x)}{\pi} \sin nx dx$$

$$= -\frac{1}{\pi n}\left[b_n - (-1)^n b_n\right],$$

$$A_n = \begin{cases} 0 & \text{when } n \text{ is even}, \\ -\dfrac{2}{\pi n} b_n & \text{when } n \text{ is odd}; \end{cases}$$

$$\tfrac{1}{2}A_0 = \frac{1}{\pi}\int_0^{\pi} F(x)dx = \frac{1}{\pi}\left[\pi F(\pi) - \frac{1}{\pi}\int_0^{\pi} xf(x)dx - \frac{1}{\pi}\int_0^{\pi} xf(\pi - x)dx\right],$$

or upon putting $u = \pi - x$ in the last integral, $A_0 = 0$. This, of course, is to be expected since, from (6), $F(\frac{1}{2}\pi) = 0$. Thus we have

$$F(x) \sim -\frac{2}{\pi}\sum_{k=0}^{\infty} \frac{b_{2k+1}}{2k+1} \cos (2k + 1)x.$$

In this case the even harmonics are absent and the odd harmonics are again reduced by the factor $1/(2k+1)$.

Lastly in the important case in which $f(x)$ satisfies both conditions (I) and (II), we see that

$$f(x) \sim \sum_{k=0}^{\infty} b_{2k+1} \sin (2k + 1)x$$

and that $F(x)$ has again the expansion just given.

As an example let

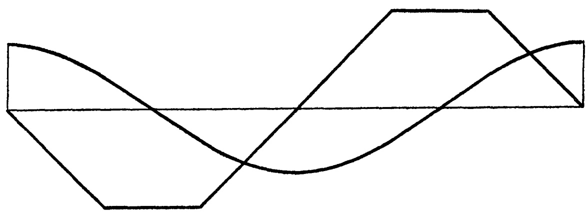
$$f(x) = \begin{cases} x & \text{from } x = 0 \quad \text{to } x = \frac{1}{3}\pi, \\ \frac{1}{3}\pi & \text{“} \quad x = \frac{1}{3}\pi \text{ “} \quad x = \frac{2}{3}\pi, \\ \pi - x & \text{“} \quad x = \frac{2}{3}\pi \text{ “} \quad x = \pi. \end{cases}$$

Then

$$f(x) = 2\pi^{-1}\sqrt{3}\left(\sin x - \frac{\sin 5x}{5^2} + \frac{\sin 7x}{7^2} - \frac{\sin 11x}{11^2} + \cdots\right),$$

$$F(x) = -4\pi^{-2}\sqrt{3}\left(\cos x - \frac{\cos 5x}{5^3} + \frac{\cos 7x}{7^3} - \frac{\cos 11x}{11^3} + \cdots\right).$$

The excellent "cosine wave" shown in the figure is made up of parabolas and



straight lines. The graphical method noted above led to the belief that it was probably an actual cosine wave.

ALGEBRAIC FIRST INTEGRALS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

By Solomon Hurwitz, New York, N. Y.

It was shown by Abel that if y is an algebraic function of x and if $u = \int y\,dx$ is also an algebraic function of x, then u is a rational function of x and y. Fuchs[1] stated an analogous theorem for differential equations of the first order: If $F(x, y, y') = 0$, where F is a polynomial, has for a general integral $V(x, y) = c$, V being an algebraic function of x and y, then there exists a general solution $R(x, y, u) = c$, where $u(x,y) = y'$ is the algebraic function defined by our differential equation, and R is a rational function of x, y and u. In this note we prove a theorem for differential equations of order n which yields Fuchs' theorem as an immediate corollary. Our method of proof is different from that of Fuchs.

Theorem: *If $F(y^{(n)}, y^{(n-1)}, \cdots, y', y, x) = 0$, where F is a polynomial, has an algebraic first integral $V(y^{(n-1)}, \cdots, y', y, x) = c$, V being an algebraic function in its arguments, then there exists a first integral $R(y^{(n-1)}, \cdots, y', y, x, u) = C$,*

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[1] Sitzungsberichte der Berliner Akademie, vol. 2 (1884), pp. 1171–1177.

where $u(y^{(n-1)}, \cdots, y', y, x) = y^{(n)}$ *is the algebraic function defined by our differential equation and* R *is a rational function in its arguments.*

I: The necessary and sufficient condition that $V=c$ be a first integral is that

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}y' + \frac{\partial V}{\partial y'}y'' + \cdots + \frac{\partial V}{\partial y^{(n-1)}}u = 0. \tag{1}$$

For, we have

$$\frac{\partial V}{\partial x} + \frac{\partial V}{\partial y}y' + \frac{\partial V}{\partial y'}y'' + \cdots + \frac{\partial V}{\partial y^{(n-1)}}y^{(n)} = 0.$$

Hence if $y^{(n)} = u$, it follows that (1) must be satisfied. The condition is necessary. If (1) is satisfied it follows that $y^{(n)} = u$. The condition is therefore sufficient.

II. By hypothesis there exist equations of the form:

$$V^m + \beta_1(y^{(n-1)}, \cdots, y', y, x, u)V^{m-1} + \beta_2 V^{m-2} + \cdots + \beta_m = 0, \tag{2}$$

where the β's are rational in $y^{(n-1)}, \cdots y', y, x$ and u. Let

$$V^p + \alpha_1(y^{(n-1)}, \cdots, y', y, x, u)V^{p-1} + \alpha_2 V^{p-2} + \cdots + \alpha_p = 0$$

be an equation of the class (3) which is of minimum degree in V. By differentiating we get:

$$V^{p-1}\left[p\frac{\partial V}{\partial x} + \frac{\partial \alpha_1}{\partial x}\right] + V^{p-2}\left[(p-1)\alpha_1\frac{\partial V}{\partial x} + \frac{\partial \alpha_2}{\partial x}\right] + \cdots + \left[\alpha_{p-1}\frac{\partial V}{\partial x} + \frac{\partial \alpha_p}{\partial x}\right] = 0,$$

$$V^{p-1}\left[p\frac{\partial V}{\partial y} + \frac{\partial \alpha_1}{\partial y}\right] + V^{p-2}\left[(p-1)\alpha_1\frac{\partial V}{\partial y} + \frac{\partial \alpha_2}{\partial y}\right] + \cdots + \left[\alpha_{p-1}\frac{\partial V}{\partial y} + \frac{\partial \alpha_p}{\partial y}\right] = 0,$$

. . . .

. . . .

$$V^{p-1}\left[p\frac{\partial V}{\partial y^{(n-1)}} + \frac{\partial \alpha_1}{\partial y^{(n-1)}}\right] + V^{p-2}\left[(p-1)\alpha_1\frac{\partial V}{\partial y^{(n-1)}} + \frac{\partial \alpha_2}{\partial y^{(n-1)}}\right]$$
$$+ \cdots + \left[\alpha_{p-1}\frac{\partial V}{\partial y^{(n-1)}} + \frac{\partial \alpha_p}{\partial y^{(n-1)}}\right] = 0.$$

Multiplying these equations by $1, y', \cdots, y^{(n-1)}, u$, respectively and adding, we get

$$V^{p-1}\left[\frac{\partial \alpha_1}{\partial x} + \frac{\partial \alpha_1}{\partial y}y' + \cdots + \frac{\partial \alpha_1}{\partial y^{(n-1)}}u\right] + V^{p-2}(A_2) + \cdots + (A_p) = 0. \tag{3}$$

The coefficients of the V's in this equation are rational in $y^{(n-1)}, \cdots y', y, x$, and u; for, if $\alpha_i = R_i(y^{(n-1)}, \cdots y', y, x, u)$,

$$\frac{\partial \alpha_i}{\partial y^{(k)}} = \frac{\partial R_i}{\partial y^{(k)}} + \frac{\partial R_i}{\partial u}\,\frac{\partial u}{\partial y^{(k)}}\,.$$

The first two partial derivatives on the right are obviously rational in $y^{(n-1)}, \cdots, y', y, x$, and u, and by means of the given differential equation we can express $\partial u/\partial y^{(k)}$ rationally in the same quantities. Hence (3) belongs to the class (2) and is of degree $p-1$ in V. Its coefficients must therefore vanish identically. That is

$$\frac{\partial \alpha_1}{\partial x} + \frac{\partial \alpha_1}{\partial y} y' + \cdots + \frac{\partial \alpha_1}{\partial y^{(n-1)}} u = 0.$$

By (I), $\alpha_1 = R(y^{(n-1)}, \cdots, y', y, x, u) = C$ is a first integral of our differential equation.

QUESTIONS AND DISCUSSIONS

EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A PROPOSED NOTATION FOR EXPONENTIALS

By WILLIAM R. RANSOM, Tufts College

The notation "exp x" to represent "the natural base, $e = 2.718+$, with the exponent x," has found some favor because of the ease with which it can be printed or typewritten. Writers who need a greater variety of exponentials may welcome a more complete notation in which the base may be indicated. Suppose for example that

$$(B)(\exp = x)$$

were to find favor for representing "the base, B, with the exponent, x." This permits any degree of complication in either base or exponent, makes it unnecessary to distinguish in size or level between base and exponent, and creates no confusion with existing notations

The symbol ")(" is easily managed on any typewriter and is easy for a printer[1] to procure. It may be thought of as locking a parenthesized base to a parenthesized exponent, while the ordinary parentheses, which embrace the whole symbol, have their usual function of indicating that a single quantity is denoted by the array. The symbol ")(" indicates and separates base and exponent, just as the solidus, "/", indicates and separates numerator and denominator.

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[1] The Banta Publishing Co., of Menasha, Wis., which prints this Monthly, has the symbol in stock for use on a monotype machine.

When the base is $e=2.718+$, the notation (e)(exp $=x$) may condense to the established form "exp x," just as the symbol "$\log_n x$" condenses to the familiar "$ln\ x$" when $n=e$.

Editorial Comment on Professor Ransom's Paper.

Professor Ransom suggests a new symbol for exponentials with a general base which, it will be granted, is distinctive. It cannot be so readily granted that it has the merit of brevity. Moreover, unlike its prototype "exp x," it cannot be immediately set up on a linotype.

QUESTION No. 58: Does the exponential with general base occur frequently enough in mathematics to warrant the general adoption of a special new notation for it, and if so, is Professor Ransom's suggested symbol likely to meet with sufficient approval to be generally adopted?

R. E. G.

THE ARC OF THE ELLIPSE

By ROGER A. JOHNSON, Hunter College

An error in the new Encyclopaedia Britannica leads to the consideration of some expressions for the approximate length of arc of the ellipse which are indeed to be found in the literature,[1] but which are perhaps not so well known to all readers of the MONTHLY as they deserve.

In the article, "Ellipse," in the Encyclopaedia, occurs the following sentence: "Kepler's approximation $\pi(a+b)$ [to the length of an ellipse whose semi-axes are a and b] is about 1/200 too large, and $\pi\sqrt{(a^2+b^2)}$ is about as much too small; their mean errs by about 1/3000."

There is obviously an error in the second expression given; we see below that a factor 2 should appear under the radical sign. Moreover, as to each approximation, it does not seem plausible that the error should be the same, independent of the eccentricity of the ellipse. Further consideration shows that apart from the typographical error just indicated, the Encyclopaedia statement is erroneous in several respects.

If the semi-axes of an ellipse are a and b, its eccentricity is $e=c/a$, where $c^2=a^2-b^2$. Then it is well known that the length of a quadrant of an ellipse is

$$s = a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\phi}\,d\phi = \frac{1}{2}\pi a\left[1-\frac{1}{4}e^2-\frac{3}{64}e^4-\frac{5}{256}e^6-\frac{175}{16384}e^8\cdots\right];$$

and values of this function have been tabulated (as for instance in Peirce's *Short Table of Integrals*, page 121). Now let R represent the radius of a circle whose circumference equals that of the ellipse, so that R/a is given by the expression in brackets. Now it is to be shown that good approximations for R are (1) the arithmetic mean of a and b; (2) the square root of their mean square;

[1] See, for instance, Appell, *Eléments d'analyse Mathématique*, 1905, page 186, where the approximations herein designated as R_1 and R_3 are to be found.

(3) their geometric mean. A truly remarkable approximation is the mean of (1) and (2).

Expressing each of these in terms of a and e, and expanding each by the binomial theorem, we have:

$$R_1 = \frac{a+b}{2} = a\left[1 - \frac{1}{4}e^2 - \frac{4}{64}e^4 - \frac{8}{256}e^6 - \frac{320}{16384}e^8 \cdots\right],$$

$$R_2 = \left(\frac{a^2+b^2}{2}\right)^{1/2} = a\left[1 - \frac{1}{4}e^2 - \frac{2}{64}e^4 - \frac{2}{256}e^6 - \frac{40}{16384}e^8 \cdots\right],$$

$$R_3 = (ab)^{1/2} = a\left[1 - \frac{1}{4}e^2 - \frac{6}{64}e^4 - \frac{14}{256}e^6 - \frac{616}{16384}e^8 \cdots\right]$$

$$\frac{1}{2}(R_1 + R_2) = a\left[1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \frac{180}{16384}e^8 \cdots\right].$$

It follows then that Kepler's approximation R_1 is too small, and that R_2 is too large, the error in each case being of the order of $e^4/64$. The mean of R_1 and R_2 is too small by approximately $5e^8/16384$.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College of the City of New York.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

The Rhind Mathematical Papyrus, British Museum 10057 and 10058. Vol. I. Free Translation and Commentary, by A. B. Chace, Chancellor, Brown University, with the assistance of H. P. Manning, Associate Professor of Mathematics, Brown University, Retired. Bibliography of Egyptian Mathematics, by R. C. Archibald, Professor of Mathematics, Brown University. Mathematical Association of America, Oberlin, Ohio, U. S. A., 1927. Pages 1–210. Volume II, Photographs, Transcription, Transliteration, Literal Translation, by A. B. Chace · · · L. Bull, Associate Curator in the Egyptian Department, Metropolitan Museum, New York, H. P. Manning, · · · Bibliography of Egyptian and Babylonian Mathematics (Supplement), by R. C. Archibald · · · . The Mathematical Leather Roll in the British Museum, by S. R. K. Glanville, Department of Egyptian and Assyrian Antiquities, British Museum, · · · 1929.

The publication of Chace's edition of the Rhind papyrus is an event of importance to all mathematicians interested in the history of their science. It marks the culmination of the efforts of many students of Egyptian mathematics, endeavoring to present to the modern reader the complete and exact form of the ancient papyrus, to supply an exact translation of it and to give a philosophic

interpretation of Egyptian processes in mathematics. The Chace edition represents all that painstaking study and liberality in financial expenditure could accomplish at the present time.

Before the discovery of the Rhind papyrus, little was known of Egyptian mathematics, except the fragmentary statements of Greek writers. Thales and Pythagoras had travelled in Egypt and acquired Egyptian lore. But the Egyptian mathematics which percolated into Greek books which have come down to the present time seem unworthy of a people which had erected the Egyptian pyramids. The Rhind papyrus was found about 1858, in a building near the great monument Ramesseum, at Thebes. It came into the possession of A. Henry Rhind from whom its name is derived. In 1864 it was purchased by the trustees of the British Museum. The papyrus roll was broken in some places and parts were missing. The trustees of the British Museum, in 1869, authorized the preparation of a facsimile edition and a descriptive text of the papyrus. Samuel Birch, Keeper of the Egyptian antiquities of this Museum, prepared plates, but the progress of the work was delayed. In the Spring of 1872, the archeologist August Eisenlohr of Heidelberg visited England and secured from Birch proofs taken from these plates. After five years of study, in which he was assisted by Moritz Cantor, the historian of mathematics, Eisenlohr brought out his edition of the Rhind papyrus in 1877. He did this, however, without securing the consent of the trustees of the British Museum to use Birch's plates. Besides the reproduction of the papyrus itself, Eisenlohr gave a commentary, as well as a translation of the papyrus into German and a transcription of the hieratic script in which the papyrus is written into the more picturesque hieroglyphic script. During the half century following, Eisenlohr's very creditable publication was the only source from which mathematicians could obtain a knowledge of the contents of the Rhind papyrus. Meanwhile, in 1898, the British Museum issued a beautiful lithographic "facsimile" of the papyrus. By "facsimile" we are not to understand an exact reproduction by the aid of photography; it was simply an imperfect hand copy, containing many little deviations from the original. As Eisenlohr used the British Museum plates, his representation of the papyrus was no better; besides he made some slight alterations of his own.

During the fifty years following 1877, great strides were made in the mastery of the languages of ancient Egypt. It became evident that a new translation and commentary of the Rhind papyrus was desirable. In 1923 this was accomplished in a masterly way by T. Eric Peet, professor of Egyptology in the University of Liverpool. By a strange good fortune some of the missing parts of the papyrus were found in 1922, in the possession of the New York Historical Society. Copies of these fragments were prepared, and Peet succeeded in finding the proper places in the papyrus for 25 of the 40 fragments. Using the papyrus itself, not the "facsimile" of it, Peet gave a hieroglyphic transcription and a translation into English. Critics found this a very able piece of work, notwithstanding occasional slips of perhaps minor importance.

For some years preceding the appearance of Peet's translation and commentary, Arnold B. Chace, Chancellor of Brown University, had made an intensive study of the Rhind papyrus. In some details his conclusions were not in entire agreement with those of Peet. Chace and his co-workers felt that there was still room for a new edition of the papyrus, especially as no real facsimile of the papyrus, and of the fragments possessed by the New York Historical Society, were available to the general reader. As the outcome of Chace's enthusiasm and liberality, we have the present magnificent edition, of which the second volume contains the photographic reproduction of the papyrus as it appears at the present time in the British Museum, and of the New York fragments. There is a satisfaction in examining an exact copy. When told that a pair of legs walking forward represents "addition," and when walking backward represents "subtraction," we like to see how much of this statement is due to the imagination of the modern investigator, and how much to the graphic technique of the ancient scribe. Our inspection leads to the conclusion that the modern imaginative faculty operated in full force.

The different parts of the papyrus are shown by 31 photographs. In making the hieroglyphic transcription, the transliteration, and literal translation, 109 plates are used. This is not the place to enumerate the many minute new interpretations. Suffice it to say that a dozen small fragments not located by Peet, are in the Chace edition assigned to what seem to be their proper places in the papyrus.

As indicated on the title page Chancellor Chace had as collaborators, H. P. Manning and R. C. Archibald, both of Brown University, and Ludlow Bull of the Egyptian department of the Metropolitan Museum in New York. Mr. S. R. K. Glanville of the British Museum supplied a brief account of a mathematical leather roll in that Museum. It is an Egyptian document which appears to be of greater significance than at first thought.

A valuable feature of the Chace edition is the bibliography of Egyptian mathematics prepared by Archibald. The many informative and critical remarks are particularly valuable to readers who have not the time or opportunity to consult the original sources. The very last entry is a pre-publication notice of V. V. Struve's account of the Golenishchev mathematical papyrus of the Moscow Museum of Fine Arts, which will appear in *Quellen und Studien zur Geschichte der Mathematik, Abteilung A*: *Quellen*, Berlin, 1930. Parts of this Egyptian papyrus have been deciphered and published before. It contains startling revelations on early Egyptian mathematics, and may rival in importance the Rhind papyrus.

FLORIAN CAJORI

Über die Verhältniszahl des Goldenen Schnitts, die Reihe der mit ihr zusammenhängenden ganzen Zahlen und eine aus dieser abgeleitete Reihe. By Ludwig Kaiser. Leipzig, B. G. Teubner. Paper, viii+124 pages. Price: RM 7.50.

If the line-segment AB be divided internally at C in what Euclid called

extreme and mean ratio, *i.e.* so that $(AB/AC)=(AC/CB)$, it is easy to show that the value of each of these ratios is the positive root of $q^2=q+1$. The author gives nearly forty pages to a discussion of the arithmetic of the number q so defined. About its outcroppings in geometry, phyllotaxis, aesthetics, and elsewhere he has nothing to say.[1] The treatment is elementary and not hurried; theorems are found by exploration, illustrated in numbers, and proved usually by complete induction. It is shown that any integral power of q can be expressed in the form $aq+b$ where a and b are consecutive terms of the doubly endless Fibonacci series

$$g_1 = g_2 = 1, \qquad g_n = g_{n-1} + g_{n-2}, \qquad (n = \cdots, -2, -1, 0, 1, 2, 3, \cdots).$$

The derived series mentioned in the title is

$$g_1' = 1,\ g_2' = 3, \quad g_n' = g_{n-1}' + g_{n-2}' = g_{n-1} + g_{n+1} = (g_{2n}/g_n).$$

The author uses the word "Kepler" where we have the habit of saying "Fibonacci" or "Pisano," uses g to remind us of golden and, if he is aware that these are the Lucas u and v series concerning which Dickson gives more than a hundred references in chapter XVII of volume I of his *History of the Theory of Numbers*, he does not anywhere show it.

Because the positive integral powers of q can be expressed as periodic continued fractions of the form $(c; d, d, d, \cdots)$, the properties of continued fractions are worked out in detail and, since c and d are found always to belong to one or the other of the above series, the illustration of these properties leads to many interesting theorems.

The remainder of the book is devoted to a study of the Fibonacci numbers but no chance is missed to establish analogous relations for the terms of the associated series wherever possible. Among the topics are: general term, sum, periodicity quâ any integral modulus, types of period, appearance and repetition of primes, and the so-called Pell equation $x^2-5y^2=1$. The author does not make the claim that any of his results are new. Appended tables contain the factored forms of g_n and g_n' $(n=1, 2, \cdots, 40)$ and a summary of the characteristics of the remainder-periods for all prime moduli up to 41 and for about twenty other primes.

The book can be read with profit by any student who knows the elements of the theory of numbers. The style is pleasant; the author, like a good teacher, feels free to pause for the examination of any sort of interesting by-product but when a proof is begun it is carried directly forward and the conclusion is nailed down w. z. b. w. The headings of each of the twenty-five chapters are as ample as the title of the book itself and make an index unnecessary. The

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[1] See this MONTHLY, vol. 25 (1918), pp. 232–8, where Professor R. C. Archibald gives an extensive bibliography of such applications. Incidentally he shows that the glamorous words "golden section" are less than a hundred years old. See also Tropfke, *Geschichte der Elementar-Mathematik*, vol. IV, p. 187.

appearance of the page and the printing and proof-reading are of that degree of excellence which we have learned to expect from the house of Teubner.

NORMAN ANNING

Premières Leçons de Géométrie Analytique et de Géométrie Vectorielle à l'usage des élèves de la classe de mathématiques et des candidats aux grandes écoles. By E. Lainé. Librairie Vuibert, Paris, 1929. 47 pages.

Recently there has been a tendency in the classical preparation leading up to the university courses in France to introduce vector notation in analytic geometry, kinematics, rational mechanics and differential geometry along with the usual analytic notation, such as one finds, for instance, in Appell's *Traité de Mécanique Rationnelle.* In these introductory courses the theory of systems of vectors is treated quite fully. In the present book just enough of vector analysis is given so that the average student can get a glimpse of some of its possibilities. It gives the usual discussion of vectors: vector addition, scalar multiplication, scalar product, vector product, and derivative of a vector. It applies the elementary theory to the study of straight lines, circles, planes, circular motion and systems of vectors. The theory is followed by a set of 23 problems, several of which are taken from *baccalauréat* examination papers, designed for students who have a knowledge of elementary calculus.

There are no important typographical errors. On page 28, line 12, x^0 should read x_0. On page 33, figure 19, the letters G and G_1 are omitted. The derivation of the expressions for the components of the vector product of two vectors (page 17) might have been simplified.

The geometric and vector system applications might well find their place in American elementary analytic geometry and mechanics texts, since vector analysis, dealing directly with geometric objects instead of indirectly by means of their components, gives a greater simplicity and elegance of presentation in these subjects. A more advanced text along the same line, as clearly written as this one, would be extremely useful to both students and instructors.

W. E. BYRNE

A History of Mathematical Notations. Volume II. Notations Mainly in Higher Mathematics. By Florian Cajori, Ph.D. Chicago, The Open Court Publishing Company, 1929. xvii+367 pages. $6.00.

This work is the second volume of Professor Cajori's *History of Mathematical Notations.* As the author indicates in his introduction, "The task of making a complete collection of signs occurring in mathematical writings from antiquity down to the present time transcends the endurance of a single investigator." Indeed, the invention of new symbols is as confusing as the change in the styles of women's clothes. After the mathematical world had been educated to the meaning and value of symbols, many an author exercised his ingenuity on the making of them. Surely the study of such a work as this reveals the ill effects

of this multiplicity of signs and should "afford a more intense conviction of some form of organized effort to secure uniformity."

As has been stated in a review of the first volume of this work, this *History of Mathematical Notations* provides an inexhaustible mine of facts and bibliography for the historian of mathematics and for the teacher of its history. The painstaking, careful and exhaustive labor extending over many years and resulting in a work of such magnitude places the mathematical-historical world in a position of deep indebtedness to the author. If all the uses to which these are put could be gathered together and presented in one continuous scroll, appreciation due his results might be adequately shown. Perhaps a greater reward would lie in his own expression "that this *History* will not have been written in vain" if advance in the spread of new results in mathematics can be brought about through international action on adoption of symbols.

The symbols associated with almost any topic in mathematics will be found in one or the other volumes of this work. Among these, none are more fully and convincingly presented than the symbols of the Differential and Integral Calculus. "The history of the growth of the calculus notations is not only interesting, but it may serve as a guide in the invention of fresh notations in the future" and here are included symbols for fluxions, differentials and derivatives, partial differentials and partial derivatives with the names of many prominent mathematicians beginning with Newton and Leibniz.

The author presents an intensive study entitled: "Survey of Mathematical Symbols used by Leibniz" because of the "leading rôle played by Leibniz in the development of notations" and because "no mathematician has seen more clearly than Leibniz the importance of good notations in mathematics." (pp. 180–1). This article makes fascinating reading and is a relief from the succession of symbols whose multiplicity makes one's head fairly reel, although this amount of detail is of incalculable value to the student of a symbol. Following this special study are nine pages of tables of symbols used by Leibniz in his manuscripts and in the papers he published.

Many interesting phases of the invention of symbols are found. There is, for instance, the casual appearance of symbols now well-known as in the case of π. There is the adoption of a symbol already used by a writer through the influence of a great mathematician as in the case of i. There is the credit accorded to a man as the sole originator of a symbol when other men had anticipated him as in the case of the notation for partial derivatives now in use. We find the appearance of a symbol for a concept that had been long present in the writings of mathematicians, such as the familiar symbol "to signify infinite number." Our attention is drawn to the fact that: "In the nineteenth century the need of more compact notations asserted itself" and there resulted such representations as Salmon's $(a_1, b_2, c_3 \cdots)$. Exception is taken to the extremes to which certain leaders in mathematical thinking are going in the use of symbols in the phrase "notations for matters that can more conveniently be expressed by ordinary words or in less specialized symbolism." (p. 77).

The reviewer must find some matters from which to dissent. Attention has

been called in an article appearing in School Science and Mathematics, vol. 24 (1924), p. 509, to an instance of reading into the works of early mathematicians modern concepts. This is here illustrated by the statements that "Euclid in his *Elements* represented general number by segments of lines" and that "In Leonardo of Pisa's *Liber Abaci* (1202), the general representation of given numbers by small letters is not uncommon." (p. 1.) Is not this unjustifiable enlargement of the field of symbolic representation! Jordanus Nemorarius did use letters to represent numbers quite as they are used at present, and he belongs in a different category from the earlier mentioned authors.

Some fifteen pages are devoted to "The Evolution of the Dollar Mark." It may be hypercritical to question the presence of this symbol in the work although the title leaves a loophole for its entrance. To have the author's views and evidence so completely presented in permanent form is an asset to a library, however. Such a widely divergent theory, backed by a mass of manuscript evidence, is suggested by an equally eminent authority in this field that a complete presentation of the latter is highly desirable. The two positions should be weighed over against each other.

The connected account is interrupted at times by matter (p. 239) which makes the work seem more encyclopedic in character, for the sporadic use of certain symbols by an erratic writer has no bearing on the general problem.

Considerable space toward the end is devoted to lessons to be learned and to the solution of the whole problem. Quotations from many writers are given. A fine section entitled: "Empirical Generalizations on the Growth of Mathematical Notations" shows the author's masterly and scholarly grasp of the problem that he presents. The conclusion of the whole matter as its beginning is: "Agreements to be reached by International Committees the only Hope for Uniformity in Notations." It cannot be overemphasized that: "In the endeavor to secure the universal adoption of mathematical symbolism international cooperation is a *sine qua non.*" Moreover, "no attempt should be made to set up at any one time a full system of notation for any department of mathematics." (p. 349). The last words in the book are too significant to omit: " · · · when in their publications, mathematicians, by a gentlemen's agreement, shall abide by the decisions of such committees." (p. 350).

The contents are: Topical Survey of Symbols in Arithmetic and Algebra (Advanced Part); Symbols in Modern Analysis; Symbols in Geometry (Advanced Part); The Teachings of History; with many subheadings. Twenty illustrations enrich the subject matter.

The strange symbols which pepper the pages of this work indicate the immense amount of labor and expense which went into the printing of it. The unique contribution to mathematical learning which The Open Court Publishing Company makes through its publications is again strikingly illustrated in this one. These are needed and only through such a high-minded agency can the need be met.

LAO G. SIMONS

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3419. *Proposed by the late F. P. Matz.*

Determine the law of force in order that the orbit be a Cassinian oval.

3420. *Proposed by the late G. B. M. Zerr.*

Let $x=f(t)$ be the equation giving the horizontal distance of a projectile in a resisting medium in terms of the time t. Prove that the vertical distance is given by

$$y = -gf(t)\int\frac{dt}{f'(t)} + g\int\frac{f(t)}{f'(t)}dt + Af(t) + B,$$

where A and B are constants.

3421. *Proposed by Otto Dunkel, Washington University.*

A convex polygon of n sides may be divided into triangles by its diagonals which intersect only at their extremities. Derive an expression for the number of ways in which this may be done.

3422. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Prove that the polar reciprocal of a sphere with respect to a given sphere is a quadric of revolution. Discuss the nature of this quadric.

3423. *Proposed by Morgan Ward, California Institute of Technology.*

Let A and B be two permutable elements of an abstract group, of orders a and b respectively, and let c be the order of $C=AB=BA$. Show that m' divides c, and c divides m, where m is the L.C.M. of a and b, and m' is the product of all of those prime factors of m which appear in a and b raised to different powers, every such factor being raised to the power to which it appears in m.

3424. *Proposed by W. J. Greenstreet, Editor of the Mathematical Gazette.*

Two straight rods OA, OB in the vertical plane make equal angles, α, with the vertical drawn up from the point O. A third rod, PQ, of fixed length l with rings at its ends slides on the two rods. Show that the third rod may assume a position of equilibrium if its centre of mass lies within a portion of the rod PQ of length $l \sin 2u \operatorname{cosec} 2\alpha$, where u is the angl^ of friction for rings and rods.

UNSOLVED PROBLEMS

2854 [1920, 377]. *Proposed by C. N. Mills.*

Solve the simultaneous equations for x and y

$$x^n + y^n = a_n, \qquad x^{n-1} + y^{n-1} = a_{n-1}.$$

2856 [1920, 377]. *Proposed by O. S. Adams, U. S. Coast and Geodetic Survey.*

Show that for the real domain defined by $+1>x>-1$, and s a positive integer,

$$\frac{1}{(1-x^s)^{1/s}}\int_0^x \frac{dx}{(1-x^s)^{(s-1)/s}} = x + \sum_{n=1}^{\infty}\frac{2(s+2)(2s+2)\cdots(ns-s+2)}{(s+1)(2s+1)\cdots(ns+1)}x^{ns+1},$$

and

$$\frac{1}{(1-x^s)^{(s-1)/s}}\int_0^x \frac{dx}{(1-x^s)^{1/s}} = x + \sum_{n=1}^{\infty}\frac{n!s^n}{(s+1)(2s+1)\cdots(ns+1)}x^{ns+1}.$$

SOLUTIONS

3309 [1928, 93]. *Proposed by Otto Dunkel, Washington University.*

If (1, 2, 3, 4, 5, 6) denotes the determinant of the 6th order whose ith row is

$$x_i^2,\ x_iy_i,\ y_i^2,\ x_i,\ y_i,\ 1, \text{ and if } (i,j,k) = \begin{vmatrix} x_i, & y_i, & 1 \\ x_j, & y_j, & 1 \\ x_k, & y_k, & 1 \end{vmatrix},$$

show that

$$(1,2,3,4,5,6) = (6,1,2)(2,3,4)(4,5,6)(1,3,5) - (1,2,3)(3,4,5)(5,6,1)(2,4,6)$$

is an identity in the 12 independent variables x_i, y_i.

values, the values of the constants. Hence we must have $\phi \equiv c_1\psi$, where c_1 is independent of x_1, y_1, and ϕ and ψ are regarded as polynomials in x_1, y_1.

Now regard x_2, y_2 as variables in the two polynomials while the remaining variables assume fixed values from the region R. Then as before we conclude that $\phi \equiv c_2\psi$, where c_2 is independent of x_2y_2. If in the two equations $\phi \equiv c_1\psi$, $\phi \equiv c_2\psi$ we suppose that the twelve variables have the same values which are taken from R, then $(c_2 - c_1)\psi = 0$. But since ϕ does not vanish for these values neither does ψ, and hence $c_2 = c_1$. This shows that neither c_1 nor c_2 depends upon (x_1, y_1) or (x_2, y_2), at least for the region R. Reasoning in the same way we show that $c_1 = c_2 = c_3$ and each is independent of the variables with the subscripts 1, 2, 3; and so on until finally we have $\phi \equiv c\psi$, where c is an absolute constant, at least for R. But since ϕ and ψ have each the same term with the same coefficient unity, we have $c = 1$, and then $\phi \equiv \psi$. Hence the corresponding coefficients in ϕ and ψ are the same for the region R; and we have therefore $\phi \equiv \psi$ without any restriction upon the range of the twelve variables.

As an application of this result suppose that one of the second order determinants is zero, say (123), then the determinant ϕ breaks up into the product of four determinants each of the second order.

Also solved by Paul Wernicke.

3360 [1928, 564]. *Proposed by Bernardo Ig. Baidaff, Buenos Aires.*

Resolve the system $x_i = a^i + bx_{i-1} + b^2x_{i-2} (i = 1, 2, \cdots, n)$, with $x_{-1} = x_0 = 0$. Generalize.

Solution by Paul S. Dwyer, Antioch College.

It is easily found that

$$
\begin{aligned}
x_1 &= a, \\
x_2 &= a^2 + ab, \\
x_3 &= a^3 + a^2b + 2ab^2, \\
x_4 &= a^4 + a^3b + 2a^2b^2 + 3ab^3, \\
&\cdots\cdots\cdots\cdots
\end{aligned}
\tag{1}
$$

The above results when continued suggest

$$x_i = \sum_{j=0}^{i-1} k_j a^{i-j} b^j, \qquad i = 1, 2, 3, \cdots, \tag{2}$$

where $k_0 = k_1 = 1$ and $k_j = k_{j-1} + k_{j-2}$, $j > 1$. The above results in (1) are given by (2), and we shall now show that (2) gives the solution of the system of equations in the problem. Inserting the values of x_i, x_{i-1}, x_{i-2} in the given equation, we shall show that (2) is identically equal to

$$a^i + b\sum_{j=0}^{i-2} k_j a^{i-j-1} b^j + b^2 \sum_{j=0}^{i-3} k_j a^{i-j-2} b^j. \tag{3}$$

The first two terms of (2) are $k_0a^i + k_1a^{i-1}b$, while in (3) the corresponding terms

are $a^i+k_0a^{i-1}b$, and these are equal since $k_0=k_1=1$. For any other value of j in (2) we consider $j-1$ and $j-2$ in the first and second sums of (3), and we obtain $(k_{j-1}+k_{j-2})a^{i-j}b^j=k_ja^{i-j}b^j$, $1<j\leqq i-1$. Hence (2) satisfies the given equation.

Note by Otto Dunkel: The numbers k_j are the famous rabbit numbers which form the well known Fibonacci sequence mentioned several times in this Monthly. For example see [1921, 329] the solution of 2809 [1920, 80], where a formula is worked out for such numbers and references to the literature on the subject are given. The formula referred to when simplified reduces to

$$k_j = \frac{1}{2^j}\sum_{p=0}^{m} {}_{j+1}C_{2p+1}5^p,$$

where m is the greatest integer in $\frac{1}{2}j$ and where ${}_{j+1}C_{2p+1}$ is the binomial coefficient.

A generalization is obtained in the equation

$$y_i = c^i + y_{i-1} + y_{i-2} + \cdots + y_{i-m}, \quad m\geqq 1, \quad i = 1,2,3,\cdots,$$

with the initial conditions $y_0=y_{-1}=\cdots=y_{1-m}=0$. This equation goes over into the form of the problem by setting $a=bc$, $x_i=b^iy_i$. The solution of this equation is given by

$$y_i = \sum_{j=0}^{i-1} k_jc^{i-j},$$

where $k_j=k_{j-1}+k_{j-2}+\cdots+k_{j-m}$ with the initial conditions $k_{-1}=k_{-2}=\cdots=k_{1-m}=0$, $k_0=1$. The equation in the k's may be solved by the methods given in the article, *Solutions of a probability difference equation*, in this Monthly, vol. 32 (1925), pp. 354–370. This particular equation is considered on page 365 of that article.

A simpler form of solution may be obtained by using the general methods employed for handling a difference equation with constant coefficients. The y equation may be written

$$f(U)y_i = c^{m+i},$$

where $Uy_i=y_{i+1}$, $U^2y_i=y_{i+2}$, $\cdots$, and where $f(t)=t^m-t^{m-1}-t^{m-2}-\cdots-t-1$. If c is not a root of this equation in t, a particular solution of the difference equation above is given by $f(c)z_i=c^{m+i}$. The homogeneous equation $f(U)y_i=0$ admits the solution

$$-f(c)y_i = s_1k_{i-1} + s_2k_{i-2} + \cdots + s_mk_{i-m},$$

where $s_1, s_2, \cdots, s_m$ are arbitrary constants and k_i is defined as above. Hence the general solution is

$$f(c)y_i = c^{m+i} - [s_1k_{i-1} + s_2k_{i-2} + \cdots + s_mk_{i-m}].$$

The constants s_j are to be determined so that $y_0=y_{-1}=\cdots=y_{1-m}=0$ and it is easily found that $s_j=c^j+c^{j+1}+\cdots+c^m$.

If $f(c)=0, f'(c)\neq 0$ as shown in the article mentioned above [1925, 357]. The particular solution in this case is $f'(c)z_i=ic^{m+i-1}$. The solution of the equation with the given initial conditions is in this case

$$f'(c)y_i = ic^{m+i-1} + s_1' k_{i-1} + s_2' k_{i-2} + \cdots + s_{m-1}' k_{i-m+1},$$

where $s_j'=(m-j)c^{j-1}+(m-j-1)c^j+\cdots+c^{m-2}$.

For the special case of the problem where $m=2$ these solutions assume the simple form

$$(a^2 - ab - b^2)x_i = a^{i+2} - ab^{i+1}k_{i-1} - a^2b^ik_i,$$

if the coefficient of x_i is not zero. If it is zero, then the solution may be written in the form

$$(2a - b)x_i = ia^{i+1} + b^{i+1}k_{i-1}.$$

The second case, in which $f(c)=0$, may be obtained from the first by differentiation with respect to c and then performing certain reductions. It is simpler, however, to proceed as above.

3383 [1929, 338]. *Proposed by S. A. Corey, Des Moines, Iowa.*

A bag contains $(m+n)$ balls, each of which may be either black or white with equal probability, m being the number of balls of one color and n being the number of balls of the other color. A white ball is dropped into the bag, and then a ball is drawn out at random and found to be white. What is now the chance that of the original balls n were white and m black?

Solution by R. E. Moritz, University of Washington.

According to the conditions of the problem the original balls in the bag are either m black and n white, or n black and m white, and the probability of either alternative is exactly one-half. Of a sufficiently large number $(2N)$ of such bags approximately N will contain n white balls and the other N will contain m white balls. Suppose a white ball is dropped into each of the $2N$ bags and a random drawing is made from one of the bags, if the drawing occurs from a bag containing now $n+1$ white balls, the probability of the white ball being drawn is $(n+1)/(m+n+1)$, while if the drawing occurs from a bag containing $m+1$ white balls the probability of the ball being white is $(m+1)/(m+n+1)$.

Take N a multiple of $m+n+1$, say $N=k(m+n+1)$, and suppose that one drawing is made from each of the bags, then the drawings from the bags containing $n+1$ white balls will yield $N(n+1)/(m+n+1)=k(n+1)$ white balls and the drawings from bags containing $m+1$ white balls will yield $N(m+1)/(m+n+1)=k(m+1)$ white balls, in other words, on the long run, out of every $m+n+2$ white balls that are drawn, $n+1$ come from bags which originally contained n white balls, the remaining $m+1$ come from bags which originally contained m white balls.

Hence the probability that a given bag originally contained n white balls is $(n+1)/(m+n+2)$.

Also solved by Andrew G. Clark, Michael Goldberg and Paul Wernicke.

3384 [1929, 338]. *Proposed by Nathan Altshiller-Court. University of Oklahoma.*

A circle (C) touches an equilateral hyperbola (H) in A and passes through the diametric opposite B of A on (H). (1) Prove that the third point common to the two curves is the diametric opposite of A on (C). (2) The lines joining A to the ends of the diameter of (C) perpendicular to AB are parallel to the asymptotes of (H). (3) If a line through A meets the two curves again in P and Q, show that BP, BQ are equally inclined to BA.

I. Solution by A. Pelletier, Montreal, Canada.

Let O and O' be the centers of (H) and (C), respectively; let OM and OM' be the intercepts on the asymptotes of the tangent at A. Draw the diameter AA' of (C), and let its intercepts on the asymptotes be OD and OD'. Since the curve is an hyperbola $MA=AM'$, and since it is equilateral each of these lengths is equal to OA. Then $OO'=DO'=O'D'$; for $\angle OD'O'=\angle AM'O=\angle AOM'=\angle D'OO'$. Hence $AD=AO'-DO'=O'A'-O'D'=D'A'$. This shows that A', the diametric opposite of A on (C), lies also on (H).

Draw EE', the diameter of (C) passing through O. Since $O'E=O'A$ and $O'O=O'D$, the chord EA is parallel to the asymptote OD, and, consequently, AE' is parallel to the other asymptote OD'.

Draw any line through A meeting (C) in P and (H) in Q. Let the intercepts of PQ and BQ on the asymptotes be OK', OK, and OL', OL, respectively. From the properties of the hyperbola it follows that $K'A=QK$, $BL'=LQ$, and the triangle LKQ is isosceles, since KL passes through O. Also $\angle ABP=\angle M'AP=\angle MAQ$. The two triangles MAK and OBL have $\angle MKA=\angle OLB$, and $\angle AMK=\angle BOL$. Hence $\angle ABQ=\angle OBL=\angle MAQ=\angle ABP$. This completes the proof.

II. Solution by Otto Dunkel, Washington University.

Let the tangents to the circle (C) at A and B meet in T. Since AT is also tangent to (H) at A, and since M, the mid-point of AB, is the center of the hyperbola (H), the tangent to (H) at B is parallel to AT. The line through M parallel to AT is the diameter of (H) conjugate to AB, and hence these two diameters are harmonically separated by the asymptotes. But, since the asymptotes are perpendicular to each other, they bisect the angles between the two diameters. Draw the line TM cutting (C) in E and E', then it follows that AE and AE' are parallel to the asymptotes.

The hyperbola (H) is generated by two projective pencils BQ and AQ with centers A and B on (H). Consider the difference of angles $\angle ABQ-\angle BAQ$. When the ray BQ is along the tangent at B this difference is equal to $\angle BAT$; when Q is at A this difference is again equal to $\angle BAT$. When the ray AQ passes through E, $\angle BAQ=90°-\frac{1}{2}\angle BAT$, and $\angle ABQ=90°+\frac{1}{2}\angle BAT$, since AE has the direction of an asymptote. Hence the difference is again $\angle BAT$. This suffices to show that the two pencils at A and B have opposite sense and

are equal. Hence $\angle ABQ - \angle BAQ = \angle BAT$. But since P is on (C) $\angle BPA = \angle BAT$, and then $\angle ABQ = \angle BAQ + \angle BPA = 180° - \angle ABP$. This proves (3), and (1) easily follows. For, when AQ passes through the intersection C of (H) and (C), BP and BQ coincide in BC, and BC is therefore perpendicular to AB. Hence AC is a diameter of (C).

Also solved by J. H. Neelley, Margaret M. Young, and Paul Wernicke.

3385 [1929, 397]. *Proposed by J. Rosenbaum, Milford, Conn.*

In any pentagon, $A_1A_2 \cdots A_5$, $P_1, P_2 \cdots P_5$ are the middle points of the sides $A_1A_2, A_2A_3, \cdots, A_5A_1$. The points $M_1, M_2, \cdots, M_5$ are the midpoints of $P_1P_3, P_2P_4, \cdots, P_5P_2$. Prove that the lines $A_5M_1, A_1M_2, \cdots, A_4M_5$ are concurrent at a point O, such that the vectors $OA_1, OA_2, \cdots, OA_5$ form a closed polygon.

Solution by Otto J. Ramler, The Catholic University of America.

Choose as the origin of vectors the centroid O of the vertices of the pentagon, and let a_i, p_i, m_i represent the vectors from O to A_i, P_i, M_i, respectively. Then $\sum a_i = 0$, and the vectors OA_i form a closed polygon. From the definition of the points we have $2p_i = a_i + a_{i+1}$, $2p_{i+2} = a_{i+2} + a_{i+3}$, $4m_i = (2p_i + p_{i+2})$. Hence $4m_i = a_i + a_{i+1} + a_{i+2} + a_{i+3} = -a_{i-1}$, and this means that $A_{i-1}M_i$ passes through O.

Also solved by J. W. Clawson. Michael Goldberg, W. M. Miller, and A. Pelletier.

3386 [1929, 397]. *Proposed by H. E. Trefethen, Colby College.*

If the incircle passes through the centroid of a triangle, find positive integral values for the sides a, b, c.

Solution by Norman Anning, University of Michigan.

By hammer and tongs or by reference to Hobson's *Plane Trigonometry*, 3rd edition, p. 200, the value of GI, the distance from centroid to incenter, can be found. When it is set equal to the inradius there results

$$(a + b + c)(5a^2 + 5b^2 + 5c^2 - 6bc - 6ca - 6ab) = 0.$$

There are no positive integers which satisfy $a+b+c=0$. If we put $a=cx$, $b=cy$, our problem reduces to that of finding rational points on the conic

$$5x^2 - 6xy + 5y^2 - 6x - 6y + 5 = 0.$$

Every rational point on this conic will yield a solution of the problem and $a=2b=2c$ is a sample of the only kind of solution which must be discarded as trivial.

The straight line $x=2+lt$, $y=1+mt$, where l and m are rational, will intersect the conic in the point $(2, 1)$ and in another rational point for which the parameter t has the value

$$(8m - 8l)/(5l^2 - 6lm + 5m^2).$$

When we use this value for t, we find the general solution

$$a = 2l^2 - 4lm + 10m^2,$$
$$b = 5l^2 - 14lm + 13m^2,$$
$$c = 5l^2 - 6lm + 5m^2,$$

where l and m are any integers. By inspection, the set (2, 5, 5) is a solution. It is probably the simplest.

Also solved by P. J. Federico, Michael Goldberg, William Hoover, O. J. Ramler, Paul Wernicke, and the Proposer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Under the auspices of the Edinburgh Mathematical Society, a Mathematical Colloquium will be held in St. Andrews, Scotland, from July 19th to 30th, 1930. The following courses of lectures have been arranged: "Rational curves and surfaces," by H. F. Baker; "Arithmetical properties of curves and surfaces," by H. W. Richmond; "The wave mechanics," by C. G. Darwin; "Elementary mathematics from the higher standpoint," by H. W. Turnbull; "Recent developments in symmetric functions, determinants, and algebraic equations," by A. C. Aitken; "Theory of functions," probably by Professor G. H. Hardy; Informal talks by Professor E. T. Whittaker and others—discussions on the lecture courses.

The fee for the Colloquium (including all the lecture courses) is £1, of which 5s. is payable as the registration fee when application is made. Application should be made to E. T. Copson, Esq., 144 North Street, St. Andrews, as early as possible, and in no case later than June 30th. Early application is particularly advisable in the case of those who propose to stay at the Hall (see below), as this Hostel accommodation is limited.

Members of the Colloquium may stay at the University Hall, which has been reserved entirely for this purpose. The Hostel is divided into three separate wings for ladies, for gentlemen, and for members accompanied by their wives. The cost of board and lodging for the period of the Colloquium (dinner on July 19th to breakfast on July 30th) will be £5 10s. per head; some reduction will be made for shorter periods.

Arrangements will be made for golf, tennis, excursions and other recreations.

Professor Harry W. Tyler, of the Massachusetts Institute of Technology, will retire at the end of the present academic year. Fifty years have lapsed since he entered that institute as a freshman in the course in chemistry at the age of seventeen. After graduation he was an assistant on the staff for two

years, studied a year at Göttingen, took his doctorate in philosophy at Erlangen in 1889, meanwhile having held the grade of instructor on the Institute staff in absentia. In 1899, at the age of twenty-seven, he was made an assistant professor; at twenty-nine, an associate professor; at thirty, a professor: at thirty-eight, head of the department of mathematics; and at forty-seven, Walker Professor of Mathematics. Since its foundation in 1901, he has been active in the work of the College Entrance Examination Board, of which he is now vice-chairman.

Following his retirement from the Institute, Professor Tyler expects to devote his time to furthering the interests of the American Association of University Professors, of which he is general secretary, and as a consultant to the Library of Congress on scientific literature. In so doing he will carry out his wish to follow the line of "maximum usefulness rather than of least resistance," for his retirement is not due to impaired health.

The following courses in mathematics are announced for the summer of 1930:

University of California at Los Angeles, July 1–August 10. By Professor Paul H. Daus: Foundations of arithmetic. By Professor Sophia H. Levy: The teaching of mathematics.

University of Chicago. In addition to the regular courses in differential and integral calculus the following advanced courses are announced for the Summer Quarter, 1930, first term June 16–July 23, second term July 24–August 29. By Professor H. E. Slaught: Differential equations, Theory of definite integrals. By Professor L. E. Dickson: Theory of numbers. By Professor E. P. Lane: Higher plane curves. By Professor E. Bompiani: Analytic projective geometry, Projective differential geometry of hyperspace. By Professor F. D. Murnaghan: Vector analysis, Electrodynamics. By Professor L. M. Graves: Theory of functions of a complex variable, Calculus of variations I. By Professor W. Bartky: Theoretical mechanics II, Modern theories of analytic differential equations II. By Professor Sanger: Solid analytic geometry. By Professor H. S. Everett: Introduction to higher algebra, Algebraic invariants.

Columbia University, July 7–August 15. In addition to courses in trigonometry, solid geometry, analytic geometry, calculus, and methods of teaching secondary mathematics, the following advanced courses are offered: By Professor W. B. Fite: Differential equations. By Professor J. F. Ritt: Functions of a complex variable. By Professor B. O. Koopman: Differential geometry. By Professor P. A. Smith: Fundamental concepts of mathematics. By Dr. M. S. Demos: Introduction to higher algebra.

University of Colorado, first term, June 23–July 26; second term, July 28–August 29. In addition to the usual elementary work in algebra, trigonometry, analytic geometry, and calculus, the following courses will be offered. First term—By Professor Light: Teachers' course in mathematics; History of mathematics; Differential geometry. By Professor Cohen: Differential equations;

Modern algebra; and a graduate course to be arranged in accordance with the desires of the applicants. Second term—By Professor Light: Statistics; Theory of finance; Differential geometry (continued). By Professor Cohen: Theory of equations; Differential equations (continued); Modern algebra (continued).

Cornell University, July 5 to August 15. In addition to the usual elementary work, the following advanced courses will be offered. By Professor J. I. Hutchinson: Higher algebra. By Professor Virgil Snyder: Teachers' course; Projective geometry II. By Professor F. R. Sharpe: Projective geometry I. By Professor W. A. Hurwitz: Theory of functions of a complex variable. By Professor D. C. Gillespie: Advanced calculus. Reading and research work will be directed by Professors J. I. Hutchinson, Virgil Snyder, F. R. Sharpe, W. A. Hurwitz, W. B. Carver, D. C. Gillespie, C. F. Craig, C. F. Roos, and F. R. Bamforth.

University of Illinois, June 16 to August 9. In addition to the usual courses in college algebra, trigonometry, analytic geometry, and calculus, the following advanced courses are offered. By Professor J. B. Shaw: Linear algebra; Modern topics in mathematics. By Professor Arnold Emch: Projective geometry. By Associate Professor A. R. Crathorne: Functions of a complex variable; Statistics. By Assistant Professor H. R. Brahana: Analysis situs. By Dr. L. L. Steimley: Differential equations. By Dr. H. W. Bailey: Fundamental concepts.

University of Iowa, first term, June 9 to July 17. In addition to courses in college algebra, trigonometry, analytic geometry, and calculus, the following subjects are offered. By Miss Ruth Lane: Subject matter and teaching of mathematics. By Professor Weida: Statistics. By Professor Dines: Advanced calculus; Linear inequalities. By Professor Ward: Differential geometry; Differential equations. By Professor Wylie: Celestial mechanics; Descriptive astronomy; Mathematics of finance. By Professor Woods: Modern Geometry; Seminar in geometry. By Professor Reilly: Algebra for high school teachers; Integration and summation; Seminar in interpolation. By the Staff: Reading and research. Second term, July 21 to August 21. By Dr. Conkwright: Differential equations. By Professor Weida: Advanced algebra; Statistics. By Professor Woods: Constructive geometry; Projective geometry. By Professor Chittenden: Advanced calculus; The theory of dimension; Seminar in abstract sets. By the staff: Reading and research.

Johns Hopkins University, June 30 to August 8. By Professor Alonzo Church, of Princeton University: Introduction to Point set theory; College Algebra; Modern geometry (Text, Johnson's *Modern Geometry*).

University of Maine, July 7 to August 15. In addition to the usual elementary work, the following advanced courses are offered. By Associate Professor Bryan: Teachers' course; Higher algebra; Seminar in the teaching of analytic geometry and calculus. By Associate Professor Jordan: Celestial mechanics. By Professor Willard: Advanced calculus; Differential equations; Theory of functions of a complex variable.

Massachusetts Institute of Technology: First period, June 17 to July 29. Courses in calculus and differential equations covering the prescribed work of the first two years. Second period, July 30 to September 10. Courses given in first period repeated. August 11 to September 13. Courses in algebra, solid geometry and trigonometry, in preparation for fall entrance examinations in those subjects. July 7 to August 1. Courses in methods of teaching mathematics in the Junior High School and the Senior High School. June 17 to July 8. Courses in advanced calculus and theoretical aeronautics. July 9 to July 29. Course in theoretical aeronautics continued. July 7 to August 4. Differential equations; intended primarily for army officers.

University of Michigan, June 30 to August 22. In addition to courses in algebra, trigonometry, analytic geometry, elementary calculus, statistics, and finance, the following advanced courses will be offered. By Professor J. W. Bradshaw: Figures of solid geometry; Projective geometry. By Professor P. Field: Vector analysis; Engineering problems. By Professor W. B. Ford: Advanced calculus; Infinite series. By Professor T. H. Hildebrandt: Calculus of variations; Complex Variables. By Professor L. C. Karpinski: Teaching algebra; History of geometry. By Professor T. R. Running: Graphical methods; Empirical formulas. By Professor H. C. Carver: Advanced mathematical theory of statistics. By Professor L. A. Hopkins: Differential equations; Celestial mechanics. By Professor R. L. Wilder: Theory of equations; Foundations of mathematics. By Professor Norman H. Anning: Differential equations. By Professor C. J. Coe: Integral equations. By Professor J. A. Nyswander: Probability; Finite differences.

University of Minnesota, first term, June 17 to July 26. In addition to the usual elementary work, the following courses will be offered. By Professor Dunham Jackson: Advanced algebraic theory. By Professor W. E. Milne, University of Oregon: Differential Equations. By Assistant Professor Gladys Gibbens: Synthetic Metric Geometry. By Professors Jackson, Milne, and Gibbens: Reading in advanced mathematics. Second term, July 28 to August 30. By Professor Raymond W. Brink and Associate Professor Roger A. Johnson: Reading in advanced mathematics.

Northwestern University, June 23 to August 16. In addition to courses in trigonometry, analytic geometry, dfferential and integral calculus, the following advanced courses are offered. By Professor T. F. Holgate: Modern pure geometry. By Professor D. R. Curtiss: Advanced Calculus. By Mr. J. F. Denney: Mathematics of statistics.

Ohio State University, June 17 to August 29. In addition to courses in trigonometry, analytic geometry, calculus, and methods of teaching secondary mathematics the following advanced courses will be offered. By Professor R. E. Langer, of the University of Wisconsin: Partial differential equations; Fourier's Series. By Professor C. C. MacDuffee: Algebraic geometry; linear algebras. By Professor C. T. Bumer: Advanced calculus; vector analysis.

By Professor H. M. Beatty: Advanced Euclidean geometry. Professors Langer and MacDuffee will also direct courses in reading and research.

University of Pennsylvania, July 7 to August 16. In addition to elementary courses, the following advanced courses are offered:— By Professor G. H. Hallett: Theory of finite groups. By Professor F. H. Safford: Partial differential equations. By Professor M. J. Babb; Differential equations. By Professor W. L. G. Williams, of McGill University: Limits and series and theory of numbers.

University of Pittsburgh, June 16 to August 11. In addition to the usual undergraduate courses, the following advanced courses will be offered. By Professor K. D. Swartzel: Functions of a complex variable; Teaching of mathematics. By Professor F. A. Foraker: Modern synthetic geometry; solid analytic geometry. By Associate Professor Taylor: Advanced calculus; functions of a real variable. By Assistant Professor Culver: Differential equations; Theory of equations.

Syracuse University. In addition to the usual courses in mathematics through the calculus the following courses are offered. By Professor F. F. Decker: The teaching of algebra and geometry in secondary schools; Introduction to the theory of invariants or introduction to modern algebra. By Professor A. D. Campbell: Theory of functions of a real variable.

University of Wisconsin, six weeks session, June 30 to August 8. By Professor T. Bennett: Differential equations; College geometry. By Professor H. P. Evans: Vector analysis. By Professor Warren Weaver: Advanced calculus; Theory of probability. Special nine weeks session for graduates, June 30 to August 29. By Professor E. B. Skinner: The Lie theory of differential equations; Theory of finite groups. By Professor H. W. March: Mathematical theory of wave propagation; Partial differential equations of mathematical physics. By Professor L. R. Ingersoll of the Department of Physics: Kinetic theory of gases, with electronic applications; Lectures in selected topics in physical optics; Mathematical theory of heat conduction. (Only one of the two last named courses will be given, the choice depending upon the demand.)

Professor W. O. Beal, of the department of astronomy of the University of Minnesota, died on February 15, 1930, at the age of fifty-six. He was a charter member of the Association.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

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MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.
INDIANA, Earlham College, May 2-3.
IOWA, Ames, Iowa, May 2-3.
KANSAS, February 15.
KENTUCKY, Lexington, Ky., April 5.
LOUISIANA-MISSISSIPPI, Cleveland, Miss., March 7-8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.
MICHIGAN, Ann Arbor, Mich., March 22.
MINNESOTA, Carleton College, May 17.
MISSOURI.
NEBRASKA, Peru, Neb., May 9.
OHIO, Columbus, Ohio, April 3.
PHILADELPHIA, Philadelphia, Pa., November 29.
ROCKY MOUNTAIN, Denver, Colo., April 11-12.
SOUTHEASTERN, Atlanta, Ga., May 2-3.
SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, Calif., March 8.
TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

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DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-sixth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at St. John's College, Annapolis, Md., on Saturday, December 7, 1929. Sessions were held in the morning and in the afternoon; Professor W. F. Shenton, Chairman of the Section, presided at both sessions.

Fifty-one persons attended the meeting, including the following thirty-four members of the Association: O. S. Adams, G. F. Alrich, R. N. Ashmun, W. J. Berry, G. A. Bingley, C. C. Bramble, Paul Capron, C. N. Claire, J. L. Clayton, G. R. Clements, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, J. B. Eppes, P. J. Federico, Michael Goldberg, W. M. Hamilton, F. E. Johnston, L. M. Kells, W. D. Lambert, C. L. Leiper, E. S. Mayer, F. D. Murnaghan, C. H. Rawlins, Jr., H. M. Robert, Jr., R. E. Root, J. B. Scarborough, W. F. Shenton, J. H. Taylor, John Tyler, P. Wernicke, J. Williamson, H. J. Winslow, E. W. Woolard.

During the intermission between the morning and the afternoon sessions, those attending the meeting were entertained at luncheon by the Annapolis members. Following luncheon, there was an interesting and delightful trip through the colonial museum of the Hammand-Harwood house, under the guidance of Professor R. T. H. Halsey of St. Johns College. Preceding the reading of papers at the afternoon session, a vote of thanks was passed in appreciation of the provisions made by St. Johns College for the meeting, and of the exceptionally fine program arranged by the Annapolis members.

The following seven papers were presented:

1. "On the Bertrand paradox," by Professor Tobias Dantzig, University of Maryland.

2. "Solutions of equations by continued fractions," by Professor Paul Capron, U. S. Naval Academy.

3. "Differentiation and substitution," by Professor John Tyler, U. S. Naval Academy.

4. "The rectangular hexagon," by Dr. Paul Wernicke, U. S. Patent Office.

5. "Isometric projection as an aid in solid geometry," by Professor Walter F. Shenton, American University.

6. "Oscillations of a rotating shaft," by Professor R. E. Root, Postgraduate School, U. S. Naval Academy.

7. "Packing of spheres and hyperspheres," by Michael Goldberg, Bureau of Ordnance, Navy Department.

Abstracts of two of these papers follow:

5. The isometric projection of descriptive geometry is the orthogonal projection of a space figure upon a plane with which the three fundamental axes

make equal angles. The projection makes possible the natural method of making all measurements along any of the three fundamental axes, or parallel to them, to the same scale. The paper was illustrated by sets of drawings showing how excellently this device may be used to draw quadric surfaces of various sorts.

7. The problem of the close packing of circles and spheres was generalized to the packing of hyperspheres in space of n dimensions. The distance between layers is the altitude of a regular simplex (which is the n-space analog of the regular triangle in two-space, and the regular tetrahedron in three-space) whose edges are equal to the diameter of the hyperspheres. The number of unit hyperspheres that can be packed into a given volume divided by the number of unit hypercubes in the same volume equals $[2^n/(n+1)]^{1/2}$. The portion of the hyperspace which is taken by a close packing of hyperspheres is given by $2[n\Gamma(n/2)]^{-1}[(n+1)(2/\pi)]^{-1/2}$. As n increases from unity, the pore space between hyperspheres increases and approaches 100% as a limit.

EDGAR W. WOOLARD, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The tenth regular meeting of the Southern California Section was held at the University of Southern California, Los Angeles, California, on Saturday, March 8, 1930. Professor E. E. Allen presided.

The attendance was sixty-two, including the following thirty-eight members of the Association: O. W. Albert, E. E. Allen, L. D. Ames, M. A. Basoco, Harry Bateman, Clifford Bell, E. T. Bell, Jessie R. Campbell, Annie C. Clark, R. H. Daus, Iva B. Ernsberger, Raymond Garver, H. H. Gaver, Harriet E. Glazier, L. E. Gurney, E. R. Hedrick, G. H. Hunt, Glenn James, Jack Levine, G. R. Livingston, Deca Lodwick, W. E. Mason, A. D. Michal, W. B. Orange, T. S. Peterson, Lena E. Reynolds, W. P. Russell, G. E. F. Sherwood, H. M. Showman, Marcus Skarstedt, D. V. Steed, F. C. Touton, H. C. Van Buskirk, Mabel G. Whiting, W. M. Whyburn, H. C. Willett, Clyde Wolfe, Euphemia Worthington.

The meeting began with a luncheon at the Student Union Building, after which it adjourned to Mudd Memorial Hall for a short business meeting and the program. The following officers were elected for the year 1930–31: Chairman, G. E. F. Sherwood, University of California at Los Angeles; Vice-chairman, H. C. Van Buskirk, California Institute of Technology; Program Committee, Marcus Skarstedt, Whittier College and A. D. Michal, California Institute of Technology. The next meeting was scheduled to be held at Occidental College, March 14, 1931.

The following program was presented:

1. "On the transformation of certain curves into straight lines, using only the straight edge," by Professor L. E. Gurney, University of Southern California, by invitation.

2. "The work of the new international commission on the teaching of mathematics," by Professor E. R. Hedrick, University of California at Los Angeles.

3. "An invariant class of n functionals under the group of general linear functional transformations," by Mr. T. S. Peterson, California Institute of Technology.

4. "Success in mathematics as conditioned by the method of work," by Professor F. C. Touton, University of Southern California.

Abstracts of these papers follow:

1. This paper presented a construction using only the straight edge and a given length, which transformed the curve $y=a_1x^n+ \cdots +a_nx+a_{n-1}$ into a curve of degree $n-1$. By repeated applications, the curve was transformed into a straight line. Other curves can be transformed into straight lines by first applying a projective transformation. The construction was used to solve problems of mapping and curve fitting. The illustrations were presented by large well-constructed drawings and an interesting mechanical device.

2. Professor Hedrick reviewed the work of the original international commission on the teaching of mathematics, and explained the problem of the new commission, whose main task is with reference to the teaching of secondary mathematics. Interpreted properly, this involves the requirements for teaching mathematics in junior colleges and most colleges, since a great deal of such work is secondary according to international standards. Professor Hedrick indicated some of the difficulties of the commission and some of the anomalies peculiar to the State of California.

3. In an earlier paper (American Journal of Mathematics, vol. 51 (1929)) the author has indicated a particular invariant class of n functionals ($n=1$, 2, 3, 4), namely, the so-called resolvents of a quadratic functional form. It is the purpose of this note to generalize and demonstrate the truth of the proposition in all its generality.

4. Professor Touton urged that in teaching mathematics, we give more attention to the thought processes and less to formal well-ordered proofs. Our purpose should be to guide thinking in new fields rather than listening to the presentation of the finished product. Verbal problems in algebra afford an illustration. Here we are liable to inject the unknown too quickly into the problem, while what is needed most, is really a short hand expression of all the situations that arise. The solving of originals in geometry presents a second illustration. A genetic proof, showing the thought processes, is more important than the finished product, if success in mathematics is to be measured by the ability of the student to think.

P. H. DAUS, *Secretary*

PRESENT TENDENCIES IN PROJECTIVE GEOMETRY

By ERNEST P. LANE, University of Chicago

1. *Introduction.* The aim of this address is to convey in a short period of time some impression of the present tendencies in projective geometry. It seems that this purpose can be most effectively accomplished by contrasting, first of all, the comparative youth of projective geometry with the greater age and antiquity of the classical metric geometry. Then the development of synthetic and analytic projective geometry in the last century will be briefly surveyed. The rise of projective differential geometry will be described, and the present status of this science considered. Finally, some conclusions will be drawn as to the present tendencies, and the probable course of future development, of projective geometry.

2. *Origin of projective geometry.* Metric geometry, which studies those properties of figures that are invariant under motion of the figures, is one of the oldest of all the sciences. We have had it handed down to us that this geometry originated in Egypt, and that it was introduced into Greece by Thales of Miletus (600 B.C.). However historically accurate this tradition may prove to be, the fundamental ideas involved are sufficiently correct to indicate the antiquity of this kind of geometry. In Greece, and again in Egypt at Alexandria, the ancient geometry flourished more or less for about 1000 years. In this time lived such illustrious men as Thales of Miletus, Pythagoras (500 B.C.), Euclid (300 B.C.), Apollonius of Perga (250 B.C.), Archimedes of Syracuse (250 B.C.), and Pappus of Alexandria (350 A.D.).

On the other hand, projective geometry, which studies those properties of figures which are invariant under projection and section, is of much more recent origin. In fact, the science of projective geometry was conceived in the mind of a French army officer confined in a Russian military prison in 1813–14 A.D. The name of this man was Poncelet (1788–1867). He was a lieutenant of engineers in Napoleon's army and started with Napoleon on the disastrous invasion of Russia in the summer of 1812. He was wounded in battle and left on the field for dead at Krasnoi. But he lived and was taken as a prisoner to Saratoff where, without books or papers or any kind of aid, he began to develop in his own mind a theory of projective geometry. He returned to France in September 1814, and in 1822 published his treatise on the projective properties of figures, which is the first on the list of great books embodying the science of projective geometry.

It would be untrue to say that there was no geometry of a projective nature before the time of Poncelet. Since the metric group of transformations is a subgroup of the projective group, it follows that all projective invariants are also metric invariants. Therefore it is scientifically correct to include projective theorems among metric theorems. Not only is it scientifically correct to do so, but it is historically true that even in classical times there were certain features of metric geometry which we now recognize to be of a projective nature, but

the ancient geometers were not conscious as we are of the projective aspects of these portions of their science. There was no organized science or theory of projective geometry in existence before the work of Poncelet.

By way of instances of projective outcroppings in metric geometry before the time of Poncelet we may mention the following problem and theorems. The problem to which we refer is to divide a given line segment internally and externally in the same ratio. This is essentially the same problem as the projective problem, to construct the fourth harmonic of a given point with respect to two given points on a straight line. Moreover, one of the fundamental theorems of projective geometry, namely, that the cross ratio of four collinear points is invariant under projection and section, is due to Pappus of Alexandria. About 1000 years after him lived Desargues of Lyons (1593–1662), to whom we owe two well known theorems of projective geometry. Of these the first is perhaps the more familiar. It can be stated as follows: If two triangles are so situated that lines joining pairs of vertices are concurrent, then pairs of sides intersect in collinear points. The second theorem of Desargues states that, if a complete quadrangle is inscribed in a conic, then a transversal meets the conic and pairs of opposite sides of the quadrangle in four pairs of points in involution. Finally, there is the famous theorem of Pascal, which was discovered by him in 1640 at the age of sixteen, and which states that, if a simple hexagon is inscribed in a conic, then pairs of opposite sides intersect in collinear points.

3. *Development of synthetic and analytic projective geometry.* Synthetic geometry is pure geometry. It uses intuition as a guide and logic as the instrument by which its results are obtained. To be contrasted therewith is analytic geometry, which introduces a coordinate system and employs the methods and machinery of algebra and analysis to demonstrate geometrical theorems. Naturally, we have *synthetic projective geometry* and also *analytic projective geometry.*

Immediately after the publication of Poncelet's book in 1822, synthetic projective geometry had a phenomenal growth, flourishing most luxuriantly in the first half of the last century. Some of the most distinguished synthetic projective geometers were Gergonne, Möbius, Steiner, Chasles, Von Staudt, and Cremona, of whom Gergonne was born first in 1771 and Cremona died last in 1903.

Analytic projective geometry seems to have grown most rapidly a little later, toward the middle and latter half of the last century. Some of the most distinguished analytic projective geometers were Plücker, Hesse, Clebsch, Grassmann, and Salmon, of whom Plücker was born first in 1801 and Salmon died last in 1904.

Space and time do not permit to give an adequate account and appreciation here on this occasion of the mighty works accomplished by these intellectual giants. It must suffice to say that they are an inspiration to those who live after them.

4. *Projective differential geometry.* Differential geometry studies the proper-

ties of a configuration in the neighborhood of a general one of its elements. So, for example, the differential geometry of a curve studies the curve in the neighborhood of a general one of its points. The well known definition of the tangent line at a point of a curve as the limit of the secant line through this point and a neighboring point, as the second point approaches the first along the curve, illustrates the processes of differential geometry. Clearly, the definition requires a knowledge of the curve only in the neighborhood of the point of tangency. Moreover, the definition involves a limiting process, and such limiting processes are characteristic of differential geometry. In view of this fact it is not surprising that differential geometry uses the differential calculus extensively, and has for the most part been developed since the invention of the calculus.

The terms *metric differential geometry* and *projective differential geometry* are now self-explanatory. The former grew to maturity before the latter was born. Isolated instances of projective differential properties of figures have been known for a long time, to be sure. The aforementioned definition of a tangent, for instance, is quite old, and is of a projective differential nature. But it was not until the last quarter of the last century that projective differential geometry was consciously studied, and an attempt was made to create a science of this subject. Cockle, Laguerre, Brioschi, and Halphen, whose lives were spent within the last century, discovered projective differential theorems. In the cases of the first three of these men, the results referred to were more or less incidental. Halphen was the first ever to undertake a systematic projective differential investigation. He studied rather thoroughly the projective differential geometry of plane and space curves about 1878 and 1880.

Wilczynski, from about 1900 to about 1923, studied the projective differential geometry of curves, ruled surfaces, general surfaces, and rectilinear congruences, confining his investigations usually to ordinary space. He founded what may be called the American school of projective differential geometers, and perfected a method of his own in the new science, which was adopted and used by the geometers of his school. According to his method the projective differential geometry of a configuration is studied by means of the invariants and covariants of a completely integrable system of linear homogeneous differential equations under a suitably chosen group of transformations. For the purpose of setting up complete systems of invariants and covariants he used infinitesimal transformations according to Lie's theory of continuous groups.

Since about 1916 there has arisen in Italy a new school of projective differential geometers, of whom Fubini at Torino is the leader, and among whom are included Bompiani at Rome, Terracini at Torino, and others. Čech at Brno in Czechoslovakia should be mentioned in relation with this school of geometers. According to their method, the projective differential geometry of a configuration is studied by means of a system of invariant differential forms. They employ, as is natural for the study of a system of differential forms, the absolute calculus of Ricci.

5. *Present tendencies.* We are now prepared to understand what seem to be the present tendencies in projective geometry. The emphasis at present is on analytic projective geometry, and, more specifically, on analytic projective differential geometry. In this field there are three recognizable tendencies, each of which will be discussed briefly.

First of all, there is a tendency toward unification of the methods and results of the Italian and American schools, which is being accelerated by exchange of publications and by the oral interchange of ideas made possible when professors of each country visit the universities of the other. Moreover, there is a close scientific connection between the methods after all. For, one of the problems that arises in the Italian work is to determine the configuration defined by a given set of differential forms. This determination is accomplished by setting up a system of differential equations whose solution gives the configuration. These are precisely the differential equations with which the American school would start as fundamental. As an instance, the Italians show that the theory of a surface referred to its asymptotic net in ordinary space may be based on a consideration of the following system of differential forms:

$$2\beta\gamma\, du\, dv, \quad 2\beta\gamma(\beta\, du^3 + \gamma dv^3), \quad p\, du^2 + q\, dv^2.$$

Then they show that a surface defined by this system of forms can be determined by integrating the following system of partial differential equations:

$$\begin{aligned} x_{uu} &= px + \theta_u x_u + \beta x_v, \\ x_{vv} &= qx + \gamma x_u + \theta_v x_v \qquad\qquad (\theta = \log \beta\gamma). \end{aligned}$$

The Americans would begin with these equations as fundamental and would obtain the forms later.

In the second place, particularly in the work of E. Bortolotti, there seems to be a movement toward unifying the theories of projective differential geometry and the modern non-riemannian geometry. This is quite recent and at present little can be said concerning it.

Finally, there is a tendency to attempt to extend the methods already found effective in ordinary space to hyperspace. The method of the Italian school is available for many configurations in ordinary space, as has already been indicated, and for curves and hypersurfaces in n-space. But in spite of the fact that considerable work has been done on the projective differential geometry of varieties in n-space by Segre, Bompiani, and others, no comprehensive systematic theory exists. For $n>4$ the Italian method breaks down for a variety of k dimensions with $1<k<n-1$, either because of the lack of an invariant quadratic differential form, or else because of the lack of an absolute calculus for an n-ary p-adic differential form. On the other hand, the American method is theoretically applicable, but the amount of labor involved when n is large seems at present to be, in most situations, prohibitive. Perhaps the method can be refined and improved so as to carry us much farther into the projective

differential geometry of general varieties in hyperspace than we have yet advanced. Recent progress makes us optimistic and hopeful of more successes yet to be obtained by means of this method.

ON THE HISTORY OF DETERMINANTS

By G. A. MILLER, University of Illinois

The history of determinants is an unusually interesting part of the history of elementary mathematics in view of the fact that it illustrates very clearly some of the difficulties in this history which result from the use of technical terms therein without exhibiting the definite meaning which is to be given to these terms. Many modern writers have based their definitions of a determinant on the existence of a square matrix. This was done, for instance, in the widely used Weber-Wellstein *Encyklopädie der Elementarmathematik*, volume 1, 4th edition, 1922, page 304. From this point of view a determinant does not exist without its square matrix, and, judging from many of the textbooks on elementary mathematics, it is likely that many students consider the square matrix as an essential part of a determinant, so that the term determinant conveys to them a dual concept composed of a square matrix and a certain polynomial associated therewith. When they speak of the rows and columns of a determinant they naturally are thinking of its matrix and when they speak of the value thereof they are naturally thinking of the polynomial implied by the term determinant.

When a student who is familiar with no definition of the term determinant except the dual one noted in the preceding paragraph meets with the common statement that the discovery of determinants is usually ascribed to G. W. Leibniz, he naturally concludes that a square matrix and a polynomial were associated by G. W. Leibniz in about the same way as they are associated at the present time. This is, however, not the case. In fact G. W. Leibniz associated a polynomial with two square matrices and he derived this polynomial therefrom in a way which differs widely from the one now followed in expanding a determinant. Hence the question arises whether it is desirable to associate the name of G. W. Leibniz with the discovery of determinants. To throw some light on this question it may be desirable to consider here the motives which led to some of the early developments which are now commonly associated with the beginnings of the theory of determinants.

The three subjects which are commonly associated with the early history of determinants are: The solution of a system of n linear equations in n unknowns, the elimination of the unknowns from a system of $n+1$ linear equations in n unknowns, and linear transformations. The first of these subjects is naturally one of the oldest in the history of mathematics and when general methods relating thereto are considered they are apt to have something in common with

the polynomials which are now used to exhibit the general solution of such a system of equations. Hence some mathematical historians have been inclined to trace the history of determinants to methods used by the ancient Chinese and the ancient Japanese in regard to the solution of a system of linear equations. In view of the great disparity between their methods and those now commonly employed it is difficult to exhibit any definite contact between their methods and our modern ideas of determinants.

Much closer approaches to such contacts are exhibited by the work of G. W. Leibniz since he employed a notation which is almost equivalent to our double subscript notation and thereby was able to write the general formulas for the eliminant of a system of linear equations as well as for the solution of such a system. There is, however, a wide difference between the motives involved in finding such general formulas and those relating to the study of the advantages gained by associating a square matrix and a polynomial as is now being commonly done in the use of determinants. Hence it would appear to be entirely justifiable to say that the work of G. W. Leibniz had no direct connection with determinants, and thus to distinguish sharply between work motivated by the desire to find general rules for obtaining the polynomials involved in solving a system of general linear equations or in eliminating the unknowns from such a system, and the work relating directly to studying the advantages resulting from the association of a square matrix and a well defined polynomial related thereto.

We do not mean to imply that the dual definition of the term determinant, according to which this term implies both a square matrix and a certain polynomial associated therewith, is the only one for which there is good authority. In fact, some of the best authorities define the term determinant without any reference to the existence of a square matrix. Our object is only to exhibit here some of the advantages resulting from this definition as regards the history of elementary mathematics, and especially as regards the simplification of the history of the subject of determinants itself. In particular, it may be noted here that if this definition had been adopted in the very useful work by Thomas Muir which appeared in four volumes under the title *The theory of determinants in the historical order of development*, a large number of the titles contained therein could have been omitted. These omissions would include all the titles of works which antedate the beginning of the nineteenth century and hence they would affect very fundamentally the commonly accepted view as regards the origin of determinants.

In particular, our common method of solving a system of n general linear equations in n unknowns by means of determinants is commonly spoken of as Cramer's Rule. In view of the fact that the dual view of determinants noted above was unknown at the time of G. Cramer (1704–1752) it is clear that this rule could not have been known in its present form at the time when G. Cramer lived and hence could not be due to him in this form. We have here, therefore, a striking instance of the danger of misleading the reader by being too generous

to certain individuals in giving credit. If the square matrix is an essential element of a determinant then it must be said that determinants were first used about half a century after the death of G. Cramer. On the other hand, the polynomials which are such an important part of determinants were used by G. Cramer and his use thereof had a great influence on the development of determinants. Many of the methods and theorems which have been named after certain mathematicians have naturally become much richer in the course of the development of our subject than could have been foreseen at the time when these mathematicians lived, and credit should usually not be withheld on account of this additional richness, but in the present case this additional credit affects fundamentally the definition of the term involved and hence it seems questionable whether the noted rule should now be called Cramer's Rule.

While the modern student of mathematics usually becomes familiar with the theory of determinants for the first time in connection with the solution of a system of n linear equations in n unknowns, it is probable that the subject of linear transformations influenced most profoundly the early development of determinants if we adopt the dual meaning of the term determinant noted above. In fact, it appears that the first association of a square matrix and the corresponding polynomial in the modern sense is due to C. F. Gauss and was employed in connection with linear transformations. Moreover, the fundamental subject of multiplication of determinants was developed in connection with linear transformations. The date at which determinants were first multiplied is also greatly affected by our definition of the term determinant since the polynomials which result from successive linear transformations were studied before the multiplication of determinants, as the term is now commonly understood, was considered. In the special case when the determinants are of the third order the theorem for multiplication can easily be inferred from the work of C. F. Gauss as given in the fifth chapter of his noted *Disquisitiones Arithmeticae.*

The main points which we aimed to exhibit in the present article are that unless the contrary is explicitly stated the term determinant should be used in the histories of elementary mathematics with the dual meaning implying both a square matrix and a certain polynomial associated therewith, and that the history of determinants would thereby be greatly simplified. While the general formulas involved in the solution of n linear equations in n unknowns are closely related to the determinants it is undesirable to regard the efforts to develop such formulas as developments in the theory of determinants. The separation of these developments and the theory of determinants enables us to distinguish sharply between the development of a general algebraic notation which made it possible to use the modern mathematical formulas and the special notation relating only to the subject of determinants. Just as the use of our ordinary complex numbers was first greatly stimulated by their appearance in work relating to the solution of the cubic equation and not by their appearance in the solution of the quadratic with which we now generally associate them most

closely, so the development of the theory of determinants was first greatly stimulated by linear transformations and not by the usefulness of determinants in the solution of a system of linear equations.

The use of double subscripts may serve the same purpose as that of rows and columns in the square matrix of a determinant and it is of some interest to note that G. W. Leibniz used number pairs as coefficients which served the same purpose as double subscripts. The credit commonly given him for the discovery of determinants is partly based on this fact. In view of the fact that Webster's *New International Dictionary* is widely used by teachers of mathematics it may be desirable to refer here to a slight error which appears therein under the term "determinants". It is correctly stated thereunder that Jacobi employed determinants "powerfully", but the date 1823 assigned for this employment is not quite correct. As Jacobi was born on December 10, 1804, he became 19 years old late in 1823 and hence if the given date were correct it would imply that he was a youthful prodigy and hence his biography would be of especial interest to teachers of our subject. As a matter of fact, however, his important contributions to determinants were published much later after he had made very valuable contributions toward the advancement of mathematics along other lines. The date 1823 should therefore be replaced by 1841 in the later editions of Webster's New International Dictionary.

A MATHEMATICAL THEORY OF THE TRANSMISSION OF SUCCESSIVE IMPULSES THROUGH A MUSCLE

By H. E. BUCHANAN, Tulane University

1. *Introduction.* We assume that a muscle of uniform physiological properties will transmit, with uniform velocity, an impulse resulting from a stimulus applied at one end. This assumption has been justified for impulses in nerves by Gasser and Erlanger[1] who have given an interesting formula to represent the transmission. It is generally accepted by physiologists[2] that such impulses are transmitted uniformly through a muscle if the muscle is thoroughly recovered from any previous impulse.

2. *Definitions.* If a second stimulus be applied immediately after the first the muscle refuses to respond. The *absolutely refractory period* is a time interval during which the muscle is in contraction and refuses to respond to a second stimulus. The *relatively refractory* period is a time interval, beginning with the end of the absolutely refractory period, during which the muscle is less than

[1] American Journal of Physiology, vol. 73 (1925), pp. 613–634.

[2] The problem of impulse transmission in muscle was suggested to me by Dr. Richard Ashman of the Tulane University College of Medicine and the numerical data were supplied by him.

normally responsive to stimuli and during which the impulses are transmitted with less than normal velocity.

Given a muscle, OP, of length b. Any effective stimulus applied at O will

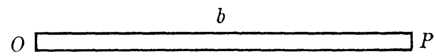

FIG. 1.

create an impulse which will be transmitted to P, the space transversed being given by the equation

$$s_1 = vt. \tag{1}$$

The velocity of transmission is $ds_1/dt = v$, and the total time required to travel from O to P is $T_1 = b/v$.

A second stimulus applied before the end of the relatively refractory period creates an impulse which will be transmitted to P more slowly and the corresponding T_2 will be larger. The velocity of propagation of this second impulse will depend in some manner on the length of time the muscle has rested.

If a third impulse is started following the second it is found, experimentally, that there is a new absolutely refractory period and a different total time of transmission T_3. The curves, A and B, in figure 2 give the graphical representation of T_2 and T_3 for various starting conditions. They were determined experimentally.

The purpose of this paper is to set up a mathematical theory for the transmission of these three successive impulses and in particular to determine whether or not the less precipitous descent of curve B in figure 2 can be explained in terms of the character of the propagation of the s_2 impulses.

3. *The second impulse.* The velocity of the second impulse at any point in the muscle certainly depends in some way on the time which has elapsed since the first impulse passed that point diminished by the absolutely refractory period. The simplest of all the possible laws of dependence is that of direct proportion. Assuming that this is the correct law we have the following as the differential equation which describes the motion of the second impulse:

$$\frac{ds_2}{dt} = K\left(t - A - \frac{s_2}{v}\right) = -\frac{Ks_2}{v} + Kt - KA, \tag{2}$$

where A is the absolutely refractory period and K is the proportionality factor. We measure the time from the instant of departure of the first impulse. The solution of equation (2), subject to the condition that $s_2 = 0$ when $t = A + \alpha$, by well known methods is

$$s_2 = \frac{v^2 - Kv\alpha}{K} e^{-K(t-A-\alpha)/v} + vt - \frac{v^2 + KvA}{K}. \tag{3}$$

The velocity at any instant of time is found by differentiating equation (3):

$$\frac{ds_2}{dt} = v - (v - K\alpha)e^{-K(t-A-\alpha)/v}. \tag{4}$$

The constant K must be determined so that $s_2 = b$ when $t - A - \alpha = T_2$. This will obviously give a different K for every value of α. It can be seen by inspection of equation (4) that $K\alpha < v$, for otherwise ds_2/dt could become larger than v, which is impossible.

From curve A, figure 2, we obtain the following data: $s_2 = 1$, $v = 2.1$, $T_2 = .6$, $\alpha = .02$. When these values are substituted in equation (3) and the equation rearranged we find

$$4.41 - .042K - (4.41 - .302K)e^{.2875} = 0.$$

The value of K which satisfies this is, approximately, 14.7. A similar computation for $s_2 = 1$, $v = 2.1$, $T_2 = .5$, $\alpha = .24$ gives $K = 7.8$. From the curve A it appears that for values of $\alpha \geqq 1$, K must equal v/α since, for all such values of α, ds_2/dt must equal v.

4. *The third impulse.* Again we assume that the velocity at any point of the muscle varies directly as the time which has elapsed since the second impulse passed that point diminished by the new absolutely refractory period, B. We might set up the differential equation which describes the motion of the third impulse by solving equation (3), after replacing s_2 by s_3, for $T - A - \alpha$ as a power series in s_3 and subtracting the result from $t - A - B - \alpha$, but this method leads to serious complications. We can secure an approximation which is probably more accurate than any of the experimental observations by dividing s_3 by the average velocity of s_2 and subtracting from $t - A - B - \alpha$. This is what was done to get equation (2), but in that case the average velocity of s_1 was the constant v and no approximation was introduced. Since s_2 starts out with a velocity $K\alpha$ and ends with a velocity very nearly equal to v, the average is $\frac{1}{2}(K\alpha + v)$. Hence we write for the differential equation of the third impulse:

$$\frac{ds_3}{dx} = \frac{-2Ls_3}{K\alpha + v} + Lx - LB, \qquad x = t - A - \alpha, \tag{5}$$

and B is the new absolutely refractory period. The solution of this equation subject to the condition that $s_3 = 0$ when $x = B + \beta$ is

$$\begin{aligned} s_3 = {} & \frac{K\alpha + v}{4L}(K\alpha + v - 2L\beta)e^{-2L(x-B-\beta)/(K\alpha+v)} \\ & + \frac{K\alpha + v}{2}x - \frac{(K\alpha + v)(K\alpha + v + 2LB)}{4L}. \end{aligned} \tag{6}$$

The velocity at any instant is

$$\frac{ds_3}{dx} = \frac{K\alpha + v}{2} - \frac{(K\alpha + v - 2L\beta)}{2}e^{-2L(x-B-\beta)/(K\alpha+v)}. \tag{7}$$

5. *The relation of curves A and B.* An examination of curve B in figure 2 shows that for equal values of α and β the time of transmission of the second impulse from O to P is shorter than it is for the third. One would naturally expect this to be true since the s_3 impulse would be held back by the s_2 impulse more than s_2 is held back by s_1 because s_2 moves more slowly than s_1. We inquire as to whether or not our mathematical theory accounts for this fact.

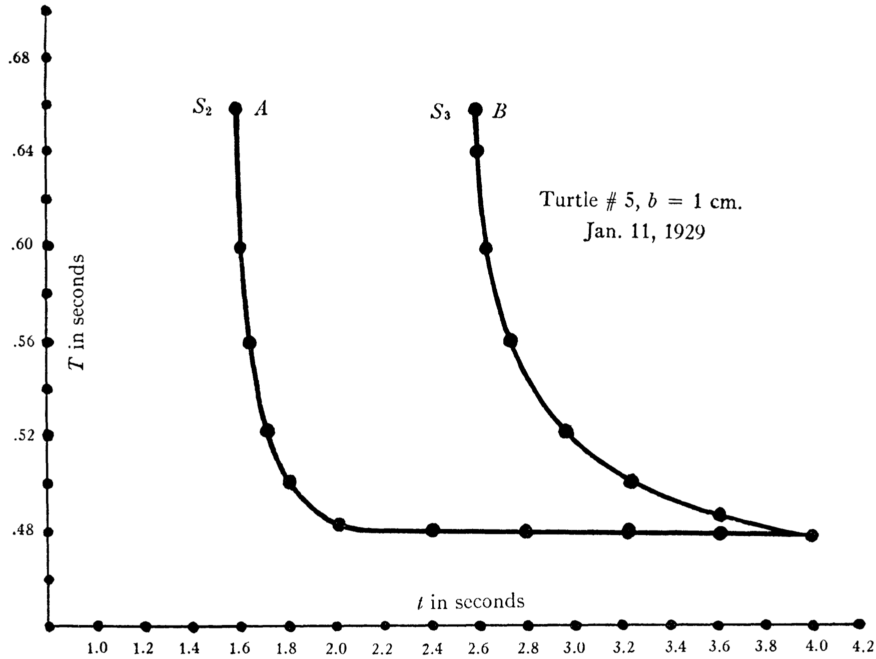

FIG. 2.

In equation (6) let $\alpha=\beta=.02$, $K=L$, $B=1$, $s_3=1$, $v=2.1$. Solving for $x-B-\beta$ we find $T_3=.88$. The corresponding value for T_2 was .6. Similarly for $\alpha=\beta=.24$ we find $T_3=.51$. The corresponding value of T_2 was .5. One could make a general discussion for every value of $\alpha=\beta$ and $K=L$, but these two numerical examples are sufficient to show that the less precipitous descent of curve B results from the character of curve A because it depends on K which was determined by curve A.

6. *The S shaped curves.* The graph of $s_1=vt$ is a straight line. For $v=2.1$ the graph is shown by the line labeled s_1 in figure 3. The graph of s_2 for $A=1.6$ and $\alpha=.02$ is also shown. Dr. Ashman[3] suspected the existence of S shaped graphs for some of the s_3 functions. Equation (7) may be used to show that there are such curves for certain values of α and β.

[3] Loc. cit.

Let α be small, say .02, and let β be relatively large, say .95. Then β and B are approximately the same size. At $x=B+\beta$, $ds_3/dx=L\beta$, which must be very nearly equal to v. Hence $L=v/\beta$ approximately. The third impulse then starts off with a large velocity, say $v_3=2$. By equation (7) this velocity has decreased to 1.76 when .3 seconds have elapsed. This is obviously due to the fact that the rapidly moving s_3 is catching up with the more slowly moving s_2 and consequently is getting into more and more refractory muscle. At about .4 seconds s_3 must begin to regain its velocity since s_2 has completed its course 1.75 seconds before. Hence there are s shaped curves for small values of α and

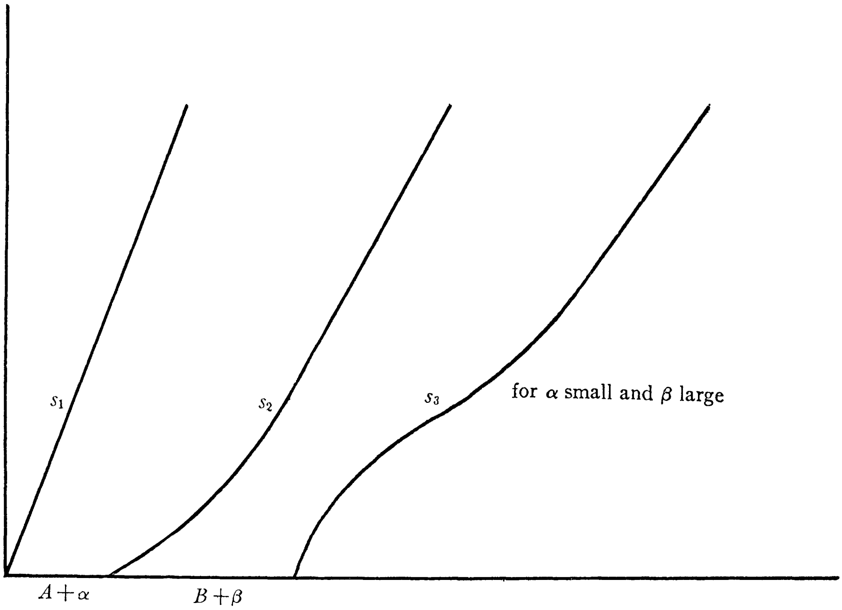

FIG. 3.

large values of β. However not all the s_3 curves are of that type. Many of them resemble the s_2 curves very closely.

7. *The formula of Gasser and Erlanger.* Drs. Gasser and Erlanger[4] of Washington University, Saint Louis, gave an empirical formula connecting space and time for the second impulse in a nerve. From the experiments of these gentlemen it seems that the mathematical theory of this paper holds as well for impulses in nerves as in muscles, with, of course, suitable adjustments of the constants. Their formula is

$$t = \frac{y}{A} + \frac{1}{m}\log_{10}\frac{y}{C}\cdot$$

In our notation this formula becomes

Loc. cit.

$$t = \frac{s_2}{v} + \log_e s_2 - \log_e C. \tag{8}$$

Differentiating with respect to t gives

$$\frac{ds_2}{dt} = v - \frac{v^2}{s_2 + v}.$$

Thus the impulse can reach the velocity v only when s_2 is very large with respect to v^2. Such a situation could never exist in any of the data that the author has seen in muscles and it seems improbable that it could ever be realized in either muscles or nerves. Further, in equation (8) when $s_2 = 0$, t becomes negatively infinite. Thus Gasser and Erlanger's formula does not give accurate results for small values of s_2. They were aware of this defect.

Contrasted to the formula of Gasser and Erlanger, formulas (3) and (4) give accurate results at both ends of the muscle. They exhibit the various constants concerned with the motion and can be adjusted to fit any data. Formulas (6) and (7) are not so accurate but still serve a very useful purpose in discussing the character of the s_3 curves.

8. *Conclusion.* In conclusion the author ventures to hope that his excursion into the field of physiology may prove interesting to the mathematician and not distasteful to the physiologist. Possibly such efforts may lay a foundation for a more mathematical treatment of certain phases of the biological sciences and at the same time serve to humanize and broaden the field of mathematics itself.

INFINITE SERIES FORMED BY ASSOCIATING THE TERMS OF A SERIES OF FUNCTIONS WITH THE POINTS OF A SEQUENCE

By NORMAN MILLER, Queens University

We consider a series $\sum u_n(x)$ of functions of a real variable x and a sequence $(x_1, x_2, x_3, \cdots)$ of points on the x-axis. By taking the values of the successive terms of the series at the successive points of the sequence we obtain the series of constants $\sum u_n(x_n)$ whose convergence or divergence is the subject of discussion. The most interesting cases are those in which the points of the sequence from a certain point on lie entirely within an interval of convergence or an interval of divergence of the series $\sum u_n(x)$. In what follows some simple general cases are first discussed, with certain sufficient conditions for the convergence of the resulting series $\sum u_n(x_n)$, which on account of their origin are in the present discussion termed pointwise series. The method of forming such series is then illustrated by obtaining a number of examples of convergent and of divergent series which form useful material for the application of the classical convergence tests.

THEOREM 1: *Let it be assumed* (1) *that the series* $\sum u_n(x)$ *converges absolutely in the interval* $a \leqq x \leqq b$, (2) *that there exist in this closed interval a finite number of points*, $y_1, y_2, \cdots, y_k$ *such that for values of* x *in* ab $|u_n(x)|$ *has its upper limit at one of these points, and that for every* n. *Then if* (x_n) *denotes any sequence of points in* ab *the series* $\sum u_n(x_n)$ *converges absolutely.*

PROOF: The series $\sum u_n(x)$ may be divided into k series of which the ith includes those terms whose absolute values have their upper limits at y_i $(i=1, 2, \cdots k)$. Since the points of the sequence (x_n) are in one-to-one correspondence with the terms of the series, a corresponding division of these points into k partial sequences (x_n'), (x_n''), $\cdots (x_n^{(k)})$ may be made. From the constant term series $\sum u_n(x_n)$ we get then k partial series $\sum u_n^{(i)}(x_n^{(i)})$, $(i=1, 2, \cdots k)$, of which at least one is infinite. Since the series $\sum |u_n^{(i)}(x_n^{(i)})|$ is dominated by the series $\sum |u_n^{(i)}(y_i)|$, whose convergence follows from the first hypothesis, it follows that each of the partial series and hence the series $\sum u_n(x_n)$ converges absolutely.

Remarks on Theorem 1: The second hypothesis of the theorem is satisfied by power functions x^n, by exponential functions n^x, n^{-x}, and by all functions which are monotone in the interval ab.

For a power series $\sum a_n x^n$ the hypotheses are satisfied if a and b are within the interval of convergence or if one or both of these points are end points of the interval of convergence provided the series converges absolutely at these points.

An ordinary Dirichlet series $\sum(a_n/n^x)$ conforms to the hypotheses of the theorem provided the series converges absolutely at the point $a (a<b)$.

It is easy to build up arbitrary convergent series whose terms do not conform to the second hypothesis of the theorem. Thus if $u_n(x)=0$ when $x \neq 1/n$ and $u_n(1/n)=1$, the series $\sum u_n(x)$ converges for all values of x while the pointwise series formed by taking the successive terms at the points of the set $(1/n)$ diverges. Again if $u_n(x)=1/n^2$ when $x \neq 1/n$ and $u_n(1/n)=1/n$, the series $\sum u_n(x)$ converges uniformly along the whole axis of x. But again the pointwise series $\sum u_n(1/n)$ diverges. Other examples in which the terms of the series are continuous occur among Fourier series. Thus the series $\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots$ converges uniformly in the interval $-\frac{1}{2}\pi \leqq x \leqq \frac{1}{2}\pi$, but the series formed by taking the values of the terms at the successive points of the sequence $((-1)^{n-1}\pi/2n)$ diverges. A theorem which applies to a trigonometric series is the following:

THEOREM 2: *In the series* $\sum u_n(x)$ *suppose that* $u_n(x)=a_n v_n(x)$, *where* a_n *is constant, and* (1) *that* $\sum a_n$ *converges absolutely*, (2) *that* $|v_n(x)| \leqq M$, *a constant for* $a \leqq x \leqq b$, $n=1, 2, 3 \cdots$. *Then if* (x_n) *is any sequence of points in* ab $\sum u_n(x_n)$ *converges absolutely.*

For we have $|u_n(x_n)| = |a_n| \cdot |v_n(x_n)| \leqq M|a_n|$, and $\sum |a_n|$ converges.

It follows that any pointwise series formed from a trigonometric series in any interval converges provided the series of coefficients converges absolutely. If

however the series of coefficients does not converge absolutely a sequence of points can always be found (if the interval is not restricted) over which the pointwise series will diverge.

Examples: If the series $\sum u_n(x)$ converges within the interval ab and diverges at b and if (x_n) is a sequence of points in ab having the limit point b, then the pointwise series $\sum u_n(x_n)$ may converge or diverge depending on the rapidity with which the points of the sequence approach their limit. A similar statement holds in case ab is an interval of divergence and b a point of convergence of the series $\sum u_n(x)$. The method is a useful one for obtaining series for practice in testing convergence. The following table exhibits a number of series formed from certain power series and Dirichlet series. The first column gives the series of functions $\sum u_n(x)$, the second gives the end point of a convergence or divergence interval, the third gives the character of the series (convergent or divergent) at this point, the fourth gives the n^{th} point of a sequence having this point as limit point, the fifth records the character of the resulting pointwise series which is itself given in the sixth column. It will be noticed that the sixth column includes, for the sake of completeness, a few series of an elementary character.

I	II	III	IV	V	VI
$\sum x^n$	1	D	$1-\frac{1}{n}$	D	$\sum\left(\frac{n-1}{n}\right)^n$
(Similar results			$1-\frac{1}{n^\alpha}, 0<\alpha<1$	C	$\sum \frac{(n^\alpha-1)^n}{n^{\alpha n}}, 0<\alpha<1$
hold for $\sum\frac{x^n}{n}$)			$1-\frac{1}{\log n}$	C	$\sum\left(\frac{\log n-1}{\log n}\right)^n$
$\sum\frac{x^n}{n^p}$	1	C	$1+\frac{1}{n}$	C	$\sum\frac{(n+1)^n}{n^{p+n}}, p>1$
$p>1$			$1+\frac{1}{n^\alpha}, 0<\alpha<1$	D	$\sum\frac{(n^\alpha+1)^n}{n^{p+\alpha n}}, p>1, 0<\alpha<1$
			$1+\frac{1}{\log n}$	D	$\sum\frac{(\log n+1)^n}{n^p(\log n)^n}, p>1$
$\sum n^p x^n$	1	D	$1-\frac{1}{n}$	D	$\sum\frac{(n-1)^n}{n^{n-p}}, p>0$
$p>0$			$1-\frac{1}{n^\alpha}, 0<\alpha<1$	C	$\sum\frac{(n^\alpha-1)^n}{n^{\alpha n-p}}, p>0, 0<\alpha<1$
			$1-\frac{1}{\log n}$	C	$\sum n^p\left(\frac{\log n-1}{\log n}\right)^n, p>0$
$\sum\frac{x^n}{n!}$	∞	D	n	D	$\sum\frac{n^n}{n!}$
			$n^\alpha, 0<\alpha<1$	C	$\sum\frac{n^{\alpha n}}{n!}, \alpha<1$
			$\log n$	C	$\sum\frac{(\log n)^n}{n!}$

I	II	III	IV	V	VI
$\sum \frac{x^n}{n^n}$	∞	D	n	D	$\sum 1$
			$\alpha n,\ 0 < \alpha < 1$	C	$\sum \alpha^n,\ 0 < \alpha < 1$
			$(\log n)^p,\ p > 0$	C	$\sum \frac{(\log n)^{np}}{n^n},\ p > 0$
$\sum \frac{1}{n^x}$	1	D	$1 + \frac{1}{n}$	D	$\sum \frac{1}{n\sqrt[n]{n}}$
(Similar results			$1 + \frac{1}{n^\alpha},\ 0 < \alpha < 1$	D	$\sum \frac{1}{n^{1+1/n^\alpha}},\ 0 < \alpha < 1$
hold for $\sum \frac{1}{n^x \sqrt[n]{n}}$			$1 + \frac{1}{\log n}$	D	$\sum \frac{1}{ne}$
and			$1 + \frac{1}{(\log n)^\alpha},\ 0<\alpha<1$	C	$\sum \frac{1}{n^{1+1/(\log n)^\alpha}},\ 0 < \alpha < 1$
$\sum \frac{1}{n^x \log n}$)			$1 + \frac{1}{\log \log n}$	C	$\sum \frac{1}{n^{1+1/\log\log n}}$
$\sum \frac{a^n}{n^x}$	∞	C	n	C	$\sum \frac{a^n}{n^n},\ a > 1$
$a > 1$			$n^\alpha,\ 0 < \alpha < 1$	D	$\sum \frac{a^n}{n^{n^\alpha}},\ a > 1,\ 0 < \alpha < 1$
			$\log n$	D	$\sum \frac{a^n}{n^{\log n}},\ a > 1$
$\sum \frac{n!}{n^x}$	∞	C	n	C	$\sum \frac{n!}{n^n}$
			$n^\alpha,\ 0 < \alpha < 1$	D	$\sum \frac{n!}{n^{n^\alpha}},\ 0 < \alpha < 1$
			$\log n$	D	$\sum \frac{n!}{n^{\log n}}$
$\sum \frac{(n!)^p}{n^x}$	∞	C	n	D	$\sum \frac{(n!)^p}{n^n},\ p > 1$
$p > 1$			$kn,\ k \geqq p$	C	$\sum \frac{(n!)^p}{n^{kn}},\ k \geqq p > 1$
$\sum \frac{1}{n^x (\log n)^p}$	1	C	$1 - \frac{1}{n}$	C	$\sum \frac{\sqrt[n]{n}}{n(\log n)^p},\ p > 1$
$p > 1$			$1 - \frac{1}{\log n}$	C	$\sum \frac{e}{n(\log n)^p},\ p > 1$
			$1 - \frac{1}{(\log n)^\alpha},\ 0 < \alpha < 1$	D	$\sum \frac{n^{1/(\log n)^\alpha}}{n(\log n)^p},\ p > 1,\ 0 < \alpha < 1$
			$1 - \frac{1}{\log \log n}$	D	$\sum \frac{n^{1/\log\log n}}{n(\log n)^p},\ p > 1$

A NOTE ON MATRIX POWER SERIES.

By I. M. SHEFFER[1], Pennsylvania State College

Let $A : \|\alpha_{ij}\|$ be a square matrix of order k, and let

$$f(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1}$$

be a power series with non-zero radius r. We consider the condition that the *matrix power series*

$$f(A) = \sum_{n=0}^{\infty} f_n A^n \tag{2}$$

be convergent. This problem has been studied by Hensel[2] and others, but a simpler treatment than theirs seems possible.

The characteristic equation for A is

$$\Delta(\lambda) \equiv \begin{vmatrix} \alpha_{11} - \lambda & \alpha_{12} \cdots \alpha_{1k} \\ \cdots & \cdots \\ \cdots & \cdots \\ \alpha_{k1} & \alpha_{k2} \cdots \alpha_{kk} - \lambda \end{vmatrix} = 0, \tag{3}$$

or, on expanding,

$$\Delta(\lambda) \equiv \delta_0 + \delta_1\lambda + \cdots + \delta_k\lambda^k = 0, \tag{4}$$

where the δ_i are readily determined. It is well-known that a matrix satisfies its characteristic equation

$$\Delta(A) \equiv \delta_0 I + \delta_1 A + \cdots + \delta_k A^k = 0, \tag{5}$$

where I is the identity matrix. On multiplying through by A^n we obtain the relations

$$\delta_0 A^n + \delta_1 A^{n+1} + \cdots + \delta_k A^{n+k} = 0, \quad n = 0, 1, \cdots. \tag{6}$$

Now set

$$A^n = \|\alpha_{ij}^{(n)}\|, \quad n = 0, 1, \cdots, \tag{7}$$

so that

$$\alpha_{ij}^{(0)} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}; \quad \alpha_{ij}^{(1)} = \alpha_{ij}. \tag{8}$$

Then (6) can be re-written in terms of the $\alpha_{ij}^{(n)}$:

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[1] National Research Fellow.

[2] K. Hensel, *Über Potenzreihen von Matrizen*, Journal für die Reine und Angewandte Mathematik, vol. 155 (1926), pp. 107–110.

$$(6') \qquad \delta_0\alpha_{ij}^{(n)} + \delta_1\alpha_{ij}^{(n+1)} + \cdots + \delta_k\alpha_{ij}^{(n+k)} = 0, \; n = 0,1, \cdots .$$

That is, for each (i, j), $\alpha_{ij}^{(n)}$ considered as a function of n satisfies the difference equation of order k with constant coefficients:

$$(9) \qquad \delta_k u_{n+k} + \delta_{k-1}u_{n+k-1} + \cdots + \delta_1 u_{n+1} + \delta_0 u_n = 0, \; n = 0,1, \cdots .$$

Let the characteristic equation (4) have the zeros $\lambda_1, \lambda_2, \cdots, \lambda_q$, of order $s_1, s_2, \cdots, s_q$, so that $s_1 + \cdots + s_q = k$. Then, as is well-known in the theory of such difference equations, the general solution of (9) is given by

$$(10) \qquad u_n = \sum_{p=1}^{q}(k_{p0}\lambda_p^n + k_{p1}n\lambda_p^{n-1} + \cdots + k_{p,s_p-1}n(n-1)\cdots(n-s_p+2)\lambda_p^{n-s_p+1}),$$

where the k_{pr}'s are arbitrary constants.

Let us now return to the series (2). It converges[3] if for each (i, j) the series $\sum_0^\infty f_n\alpha_{ij}^{(n)}$ converges, and then (definition)

$$(11) \qquad f(A) = \left\| \sum_0^\infty f_n\alpha_{ij}^{n} \right\| \qquad (i,j = 1, \cdots, k).$$

On appealing to (10) and to the fact that $\alpha_{ij}^{(n)}$ satisfies (9), we see:

THEOREM 1. *Series* (2) *converges, and to the sum-matrix* (11), *if the following series converge:*[4]

$$\sum f_n\lambda_p^n, \sum f_n n\lambda_p^{n-1}, \cdots, \sum f_n n(n-1)\cdots(n-s_p+2)\lambda_p^{n-s_p+1}, \; p = 1,2, \cdots, q.$$

COROLLARY. *Series* (2) *converges if*

(i) $|\lambda_p| \leqq r, p = 1, \cdots, q$;

(ii) *for every* λ_p *such that* $|\lambda_p| = r$, *the power series*[5] *for* $f(z), f'(z), \cdots, f^{(s_p-1)}(z)$ *converge at* $z = \lambda_p$.

The converse of this theorem is not strictly true, as may be seen by the following example: Choose $A = I$ (the identity), and let $f(z) = \sum_0^\infty f_n z^n$ be any analytic function satisfying the conditions: (i) the radius of convergence is unity; (ii) $\sum_0^\infty f_n$ converges; (iii) $\sum_0^\infty nf_n$ diverges. Now $f(A) = f(I) = I(\sum_0^\infty f_n)$, so that $\sum_0^\infty f_n I^n$ converges. But I has $\lambda = 1$ as a k-fold root of its characteristic equation, so that were the converse to hold universally, the series for $f'(z), \cdots, f^{(k-1)}(z)$ should all converge for $z = 1$. But this is not the case.

Let M be a matrix of order k. It generates a linear transformation in vector-

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[3] We take this as the *definition* of the convergence of (2).

[4] It may be noted that the convergence of the last of these series implies the convergence of all that precede it, as Hensel (loc. cit.) points out, so that it suffices merely to demand that $\sum f_n n^{s_p-1}\lambda_p^n$ converges, $p = 1, 2, \cdots, q$.

[5] See footnote 4.

space of k dimensions: $\sum_{j=1}^{k} m_{ij}x_j = y_i$ or $M(x) = y$ carries vector x into vector y. A direction is *invariant* under M if there is a scalar λ such that for every vector $\mathbf{x}$ in this direction, $M(\mathbf{x}) = \lambda\mathbf{x}$.

LEMMA. *If matrix A of order k has k linearly independent invariant directions, there exists a matrix Φ with non-vanishing determinant such that*

$$A = \Phi^{-1}A^*\Phi, \tag{12}$$

where

$$A^* = \left\| \begin{matrix} \lambda_1 \cdot & & 0 \\ & \cdot & \\ 0 & & \cdot\ \lambda_k \end{matrix} \right\|, \tag{13}$$

$\lambda_1, \cdots, \lambda_k$ *being the zeros*[6] *of the characteristic equation* (3).

Proof: Setting $\Phi = \| \phi_{ij} \|$ and expanding the equation $\Phi A = A^*\Phi$, we obtain a system of k equations in the k unknowns $\phi_{i1}, \phi_{i2}, \cdots, \phi_{ik}$ (i fixed $= 1, 2, \cdots, k$), the determinant being $\Delta(\lambda_i) = 0$. The hypothesis on the invariant directions assures that there are k linearly independent sets $(\phi_{i1}, \cdots, \phi_{ik})$, $i = 1, 2, \cdots, k$, so that Φ has a non-vanishing determinant.

DEFINITION. A is *regular* if it has k linearly independent invariant directions.

COROLLARY 1. *If A is regular then a_{ij} is a linear function of the k quantities $\lambda_1, \lambda_2, \cdots, \lambda_k$.*

COROLLARY 2. *If A is regular, then*

$$P(A) = \Phi^{-1}\left\| \begin{matrix} P(\lambda_1) \cdot & & 0 \\ & \cdot & \\ 0 & & \cdot\ P(\lambda_k) \end{matrix} \right\|\Phi \tag{14}$$

for every polynomial $P(z)$.

From the definition of convergence it follows that $f(A) = \sum_0^\infty f_n A^n = \lim_{s=\infty} \sum_{n=0}^{s} f_n A^n$; whence from corollaries 1 and 2,

$$(15)\, f(A) = \lim_{s=\infty} \Phi^{-1}\left\| \begin{matrix} \sum_{n=0}^{s} f_n\lambda_1^n & & 0 \\ & \cdot & \\ 0 & & \sum_{n=0}^{s} f_n\lambda_k^n \end{matrix} \right\|\Phi$$

$$= \Phi^{-1}\lim_{s=\infty}\left\| \begin{matrix} \sum_0^s f_n\lambda_1^n & & 0 \\ & \cdot & \\ 0 & & \sum_0^s f_n\lambda_k^n \end{matrix} \right\|\Phi = \Phi^{-1}\left\| \begin{matrix} f(\lambda_1) \cdot & & 0 \\ & \cdot & \\ 0 & & \cdot\ f(\lambda_k) \end{matrix} \right\|\Phi.$$

Hence we have the sharper result:

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[6] They need not be distinct.

THEOREM 2: *If A is regular then a necessary and sufficient condition that $\sum_0^\infty f_n A^n$ converge is that the series for $f(z)$ converge at $z=\lambda_1, \cdots, \lambda_k$.*

GRAPHICAL INTEGRATION OF DIFFERENTIAL EQUATIONS IN A POLAR COÖRDINATE SYSTEM

By E. A. KHOLODOVSKY, New York, N. Y.

The methods of graphical integration of differential equations in the Cartesian coördinate system, in other words when the Cartesian coordinate system is chosen for the geometrical representation of functions, are well known and have been fully expounded by J. Massau, R. Mehmke, C. Runge, H. Sanden, Fr. Willers, and others.[1] In some cases the differential equations are of such a form that the geometrical constructions for graphical solution of the equations are easier and simpler, if we interpret geometrically the variables of the equations as polar coördinates. In particular that occurs when the equation contains trigonometric functions of the independent variable.

Graphical integration of differential equations in a polar coordinate system is based on the same principles as in a Cartesian system; and in our exposition we shall limit ourselves to the peculiarities of graphical integration of differential equations in a polar coördinate system without repeating the explanations and proofs which are common to both systems.

1. *Ordinary differential equations of first order*

A. *Method of characteristics*

Let a differential equation $f(x, y, dy/dx)=0$ and the initial values of the variables x_0 and y_0 be given. Suppose the function whose analytical expression is $y=\phi(x)$ is the solution of this equation. For convenience we substitute the letters θ and ρ for the letters x and y; then $dy/dx=d\rho/d\theta=\rho'$ and the equation $f(\theta, \rho, \rho')=0$, the initial values, θ_0, ρ_0, and the integral of the equation, $\rho=\phi(\theta)$. We interpret θ and ρ, geometrically, as polar coördinates of the points of the curve whose equation is $\rho=\phi(\theta)$. The problem is to construct the curve which passes through the point $M_0(\theta_0, \rho_0)$ and whose equation is $\rho=\phi(\theta)$.

The construction of the curve $\rho=\phi(\theta)$ is analogous to the solution of this problem in a Cartesian coördinate system and is based on the principle of continuity and on the geometrical meaning of the derivative ρ' or of other analytical expressions containing this derivative. The method of characteristics consists in a Cartesian coördinate system of drawing a system of isoclinal lines (charac-

[1] Encyklopädie der Mathematischen Wissenschaften, B. II$_3$, N2; J. Massau, *Mémoire sur l'intégration graphique* (1885); B. Mehmke, *Leitfaden zum graphischen Rechnen* (1924); C. Runge, *Graphical Methods* (1912), *Graphische Methoden* (1914); H. Sanden, *Praktische Analysis* (1914); Fr. Willers, *Graphische Integration* (1920).

teristics). At each point of an isoclinal line the tangent to the required integral curve has the same slope. Practicality of the graphical method depends very much upon the simplicity of drawing these characteristics.

If we transform the given differential equation $f(\theta, \rho, \rho')=0$ into the form $F[\theta, \rho, p(\rho')]=0$, where $p(\rho')$ is a function of ρ', the geometrical meaning of which is known, and if the drawing of the curve $F(\theta, \rho, K)=0$, where K is a constant, is easy, it is advantageous to apply the method of characteristics in the polar coördinate system.

Depending upon the geometrical meaning of the function $p(\rho')$ we apply different constructions to get the simplest one. Let us consider several particular cases.

Case (*a*). Let the given equation $F(\theta, \rho, \rho')=0$ be of such a form that the construction of the curve $F(\theta, \rho, K)=0$, where K is a constant, is easy. In this case $p(\rho')=\rho'$. Then, taking into consideration the fact that the geometrical meaning of ρ' is the value of the subnormal of the required curve, $\rho=\phi(\theta)$, we substitute in the given differential equation, $F(\theta, \rho, \rho')=0$, arbitrary constant values, K_0, K_1, K_2, $\cdots$ for ρ', taking $K_0=\rho_0'$; the initial value of ρ_0' we compute, or construct, from the given equation. In this way we obtain a system of equations:

$$(A)\quad \begin{array}{ll} (a) & F(\theta,\rho,K_0) = 0, \\ (b) & F(\theta,\rho,K_1) = 0, \\ (c) & F(\theta,\rho,K_2) = 0. \\ & \cdots\cdots\cdots \\ & \cdots\cdots \end{array}$$

We construct (Fig. 1) the curves (*a*), (*b*), (*c*), $\cdots$, which we call characteristics, represented by these equations, the first curve passing through the point $M_0(\theta_0, \rho_0)$. On the perpendicular through O to the vector OM_0 we lay the segment $ON_0=K_0$. (If K_0 is positive, we lay it in the positive direction, that is so that the angle between the vectors OM_0 (first) and ON_0 (second) is positive; if K_0 is negative we lay it in the opposite direction.) Through the point M_0 we draw[2] M_0m_1 perpendicular to M_0N_0. It is evident that this line M_0m_1 is the tangent to the required curve, $\rho=\phi(\theta)$ at the point M_0, as the coördinates of the point M_0 and the value of the subnormal, K_0, satisfy the equation (*a*) of the system (*A*).

We may continue this tangent to the intersection with the curve (*b*); considering this point of intersection as the point of the required curve we may construct in the same manner the tangent through this point, taking into consideration the value K_1 of the subnormal, and so on. We shall obtain a more correct graph if we continue the tangent through the point M_0 to some point m_1, approximately in the middle between the curves (*a*) and (*b*), and draw a

[2] It is not necessary to draw the line M_0N_0.

tangent through this point, taking the length $On_1 = K_1$ as the value of the subnormal. At the point M_1 of intersection of this tangent with the curve (b) we draw a tangent in the same way as for the point M_0, taking the length $ON_1 = K_1$

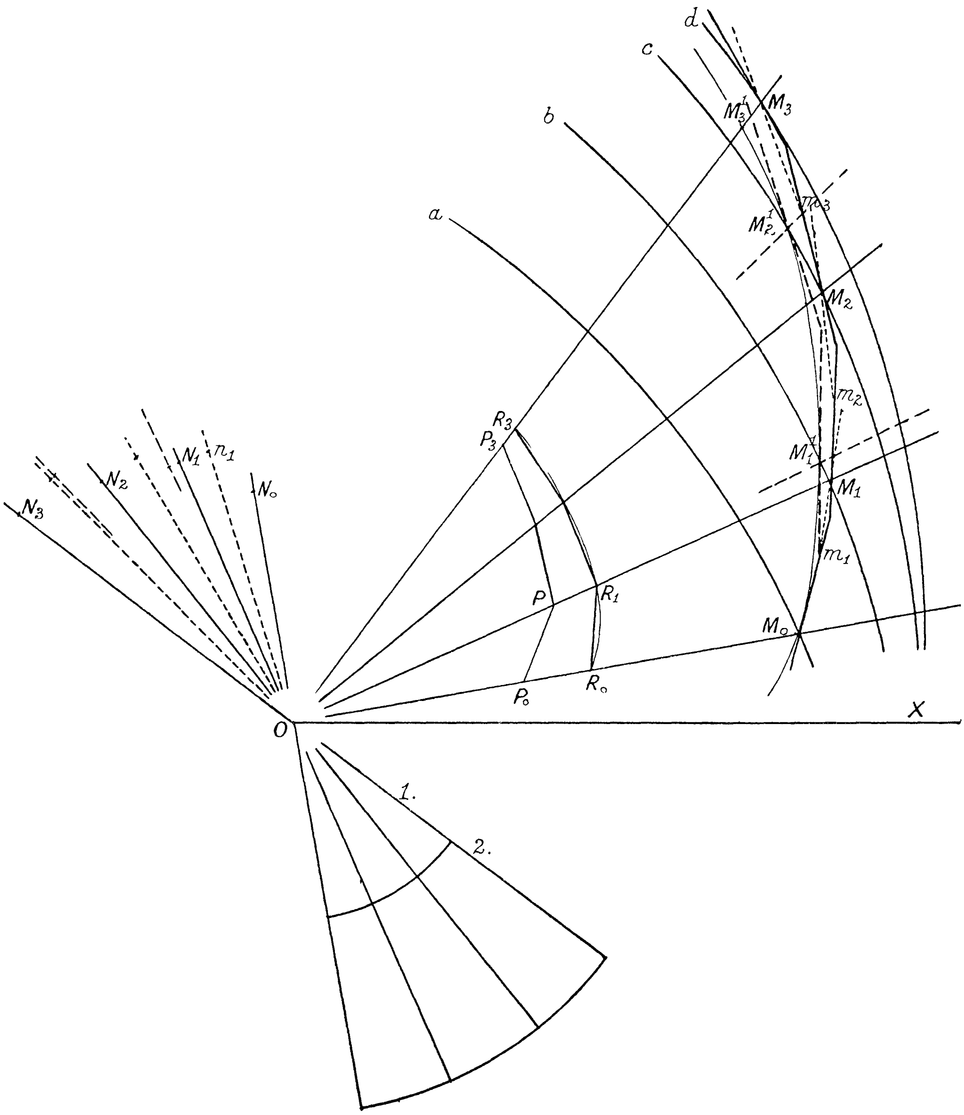

FIG. 1

as the subnormal for this point. Proceeding in the same manner we obtain a broken line of tangents through the points M_0, M_1, M_2, $\cdots$. A curve tangent to this broken line at the points M_0, M_1, M_2, $\cdots$ is the first approximation

to the required integral curve which we shall denote by the equation $\rho=\varphi_1(\theta)$. (On the diagram this curve is not drawn, for better clearness of the sketch.)

To get a better approximation we proceed as in a Cartesian coördinate system: On the rays OM_0, OM_1, OM_2, $\cdots$ we put the lengths $OP_0=K_0$, $OP_1=K_1$, $\cdots$. The curve $P_0P_1 \cdots P_3$ (on the diagram the curve is not drawn) whose equation can be written $\rho=\varphi_1'(\theta)$ represents the change of the derivative $\varphi_1'(\theta)$. Integrating this curve by methods of graphical integration in a polar coördinate system,[3] we obtain the second approximation. On the diagram (Fig. 1) the curve R_0R_3 represents the curve $\rho=\sqrt{[2\varphi_1'(\theta)]}$ and the curve M_0M_3' the second approximation, $\rho=\varphi_2(\theta)$. If the second curve coincides with the first, we consider it as the required integral curve. If the correctness is not sufficient, we get in the same way the third approximation, taking into consideration the points M_1', M_2', $\cdots$ of the intersections of the integral curve with the characteristics (b), (c), $\cdots$ and so on. A reasoning analogous to the reasoning in a Cartesian coördinate system proves that in general we can obtain an approximation of desirable exactness.[4] It is advisable to construct with sufficient accuracy a short part of the required curve and then proceed with further construction.

Case (b). $p(\rho')=\rho/\rho'$. The given equation $f(\theta, \rho, \rho')=0$ is transformed into $F(\theta, \rho, \rho/\rho')=0$. It is known that $\rho/\rho'=\tan\mu$ where μ is the angle between the positive direction of the radius vector and the positive direction of the tangent. With initial values θ_0, ρ_0 we construct or compute from the given differential equation the initial value $\rho_0/\rho_0'=\tan\mu_0=K_0$ and we construct the curves

$$(a) \qquad F(\theta,\rho,K_0) = 0,$$

$$(b) \qquad F(\theta,\rho,K_1) = 0,$$

$$(c) \qquad F(\theta,\rho,K_2) = 0,$$

$$\cdots\cdots\cdots$$

$$\cdots\cdots\cdots$$

where K_1, K_2, $\cdots$ are arbitrary values; $K_1=\tan\mu_1$, $K_2=\tan\mu_2$, $\cdots$. The point $M_0(\theta_0, \rho_0)$ is on the curve (a). At this point we construct the angle μ_0 (Fig. 2), at the point[5] m_1 an angle equal to μ_1, and so on.

The broken line $M_0M_1M_2 \cdots$ represents the tangents to the required integral curve. To get the second approximation we construct the curve $\rho=\varphi_1'(\theta)$, using the formula $\tan\mu=\rho/\rho'$, $\rho'=\rho\cot\mu$. To perform this construction we may, on the perpendicular to the radius vector OM_0, put the length $OM_0^1=OM_0$

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[3] E. A. Kholodovsky, *Graphical integration and differentiation of functions in a polar coordinate system*, in this Monthly, vol. 36 (1929), pp. 3–21.

[4] C. Runge, *Graphical Methods*, p. 124.

[5] The meaning of letters and signs in this and other cases is the same, as in case (a).

and through the point M_0' draw a line parallel to M_0M_1; the intersection of this line with the vector OM_0, the point P_0, is the point of the curve $\rho=\varphi'(\theta)$ on the ray OM_0, and so on. Then we proceed as in the case (*a*).

The values of ρ', certainly, can be found in all cases by means of constructing the subnormals.

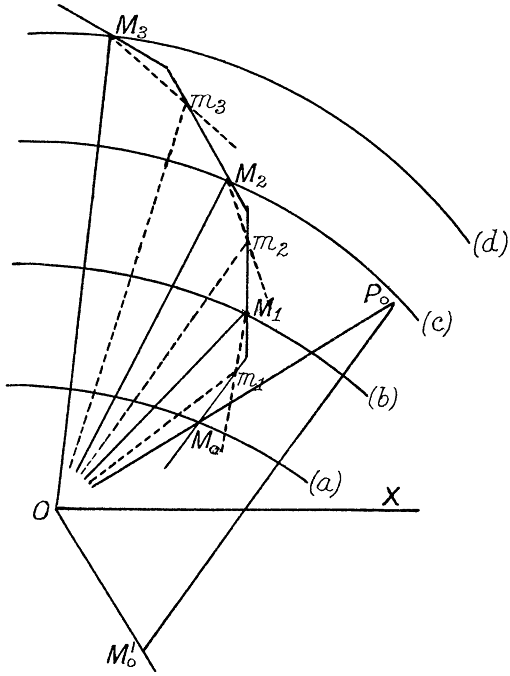

Fig. 2

Case (*c*): $p(\rho')=(\rho^2+\rho'^2)^{1/2}$. The given differential equation is transformed into $F[\theta,\ \rho,\ (\rho^2+\rho'^2)^{1/2}]=0$. The geometrical meaning of the expression $(\rho^2+\rho'^2)^{1/2}$ (equal to K) is the length of the normal, and the construction is analogous to case (*a*); the characteristics having been drawn, we find the points $N_0,\ N_1,\ N_2,\ \cdots$ (Fig. 1) as $M_0N_0=K_0,\ M_1N_1=K_1,\ M_2N_2=K_2,\ \cdots$.

For the second approximation we construct the curve $\rho=\phi_1',\ (\theta)$ by means of the lengths $ON_0=\rho_0',\ ON_1=\rho_1',\ ON_2=\rho_2'\ \cdots$ as in case (*a*).

Case (*d*): $p(\rho')=\rho^2/\rho'$. We put $\rho^2/\rho'=K$ (the subtangent) and we proceed as in case (*a*). On the diagram (Fig. 3) the subtangent, $OT_0=K_0,\ OT_1=K_1$, and so on.

For the second approximation we have to construct the subnormals, $ON_0=\rho_0'$, and so on.

Case (*e*): $p(\rho')=(\rho/\rho'(\rho^2+\rho'^2)^{1/2}$. The construction is analogous to that in case (*d*) if the given equation can be reduced to the equation $F(\theta,\ \rho,\ (\rho/\rho')(\rho^2+\rho'^2)^{1/2}=0$, as the geometrical meaning of the expression $(\rho/\rho')(\rho^2+\rho'^2)^{1/2}$

(equal to K) is the length of the tangent. The points T_0, T_1, T_2, $\cdots$ (Fig. 3) are obtained as $M_0T_0=K_0$, $M_1T_1=K_1$, $M_2T_2=K_2$, $\cdots$.

For the second approximation we construct the subnormals as in case (d).

B. *Method of successive approximation*

When the given differential equation $F(\theta, \rho, \rho')$ can not be represented in a form which allows us easily to construct the curves of the system (A), we can

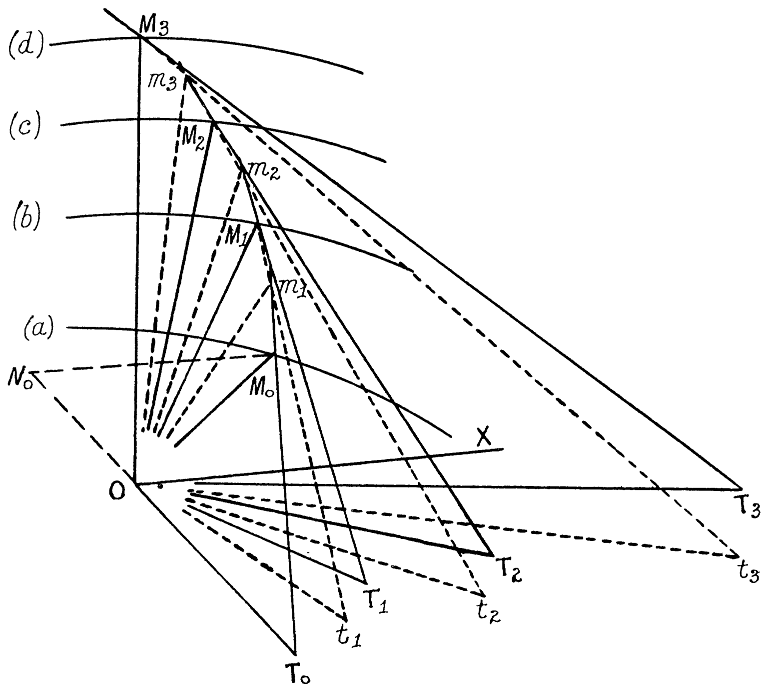

FIG. 3

apply another method of construction of tangents (successive approximation). We construct the tangent M_0M_1 (Fig. 4) through the initial point M_0 as in case (a), after the value of the subnormal has been found from the given equation by measuring or computing. We take point M_1 on this tangent at a sufficiently small arbitrary distance from M_0, measure the coordinates θ_1 and ρ_1 of this point and we find the value of ρ_1' from the given equation by measuring or computing. The value ρ_1' is the subnormal of the required curve at the point M_1. In the same way as before we draw the tangent M_1M_2 through the point M_1, and so forth.

Also as in other cases we can obtain the second approximation, the third, and so on, constructing the curve of derivatives ρ' and integrating it.

2. *A system of two ordinary differential equations of the first order with two unknown functions*[6]

Given a system

$$\rho' = f(\theta,\rho,\nu), \qquad \nu' = g(\theta,\rho,\nu).$$

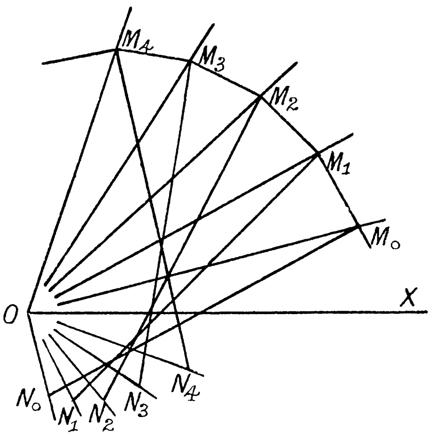

FIG. 4

and the initial values θ_0, ρ_0, ν_0. The problem is to construct the curves $\rho=\varphi(\theta)$ and $\nu=\psi(\theta)$ satisfying this system. We shall use the method of constructing consecutive tangents to the required curves. From the equations of the given system we find (by computing or constructing) ρ_0' and ν_0' for $\theta=\theta_0$. We construct the points $M_0(\theta_0, \rho_0)$ and $m_0(\theta_0, \nu_0)$ (Fig. 5). For convenience we have

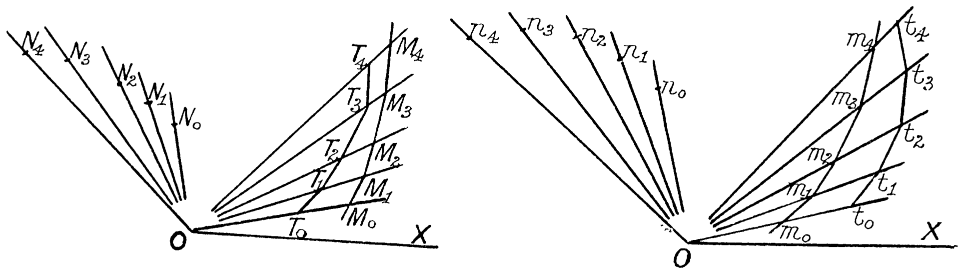

FIG. 5

taken different drawings for the curves $\rho=\varphi(\theta)$ and $\nu=\psi(\theta)$. We construct $ON_0=\rho_0'$ and $On_0=\nu_0'$ and we draw the required tangents through the points M_0 and m_0. We draw the vectors OM_1 and Om_1, making the same angle θ_1 with the polar axis, sufficiently close to the vectors OM_0 and Om_0. We measure the values of θ_1, ρ_1, ν_1 at the points M_1 and m_1 and from the given equations we find corresponding values of ρ_1' and ν_1', and we proceed as before.

To get a better approximation of the integral curves we construct on the same

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[6] Method of successive approximation.

vectors, or on another drawing with vectors making the same angles, the values of $\rho_0' = OT_0$, $\rho_1' = OT_1$, $\cdots$ and $\nu_0' = Ot_0$, $\nu_1' = Ot_1$, $\cdots$. Thus we obtain the curves $\rho = \varphi_1'(\theta)$ and $\nu = \psi_1'(\theta)$, the curves $\rho = \varphi_1(\theta)$ and $\nu = \psi_1(\theta)$ being the first approximations. We integrate them graphically taking for the initial points M_0 and m_0. We obtain new values of the vectors $\rho_1, \rho_2, \cdots, \nu_1, \nu_2, \cdots$ for the same values of the angles $\theta_1, \theta_2, \cdots$. With these values we find new values for $\rho_1', \rho_2', \cdots, \nu_1', \nu_2', \cdots$ and construct a second approximation of the required curves and so on until two consecutive approximations coincide.

The same method may be applied to any system of n differential equations of the first order with n unknown functions.

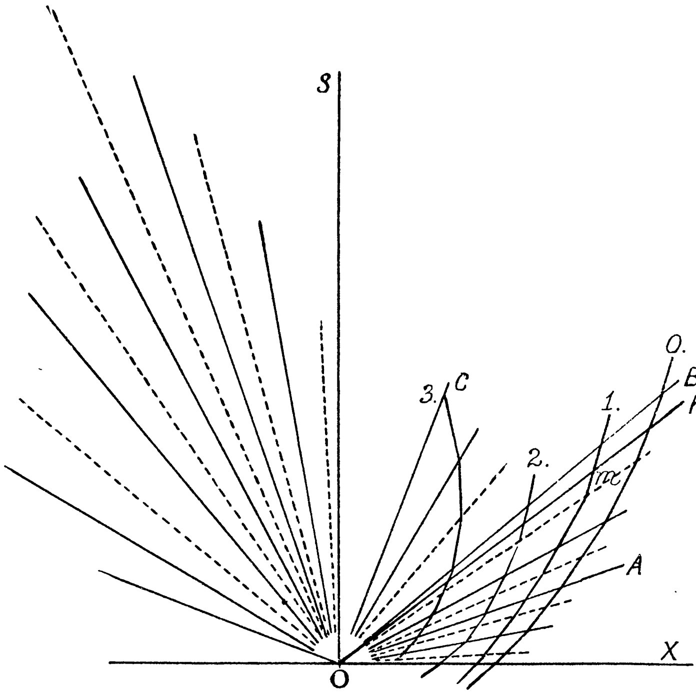

FIG. 6a

3. *Ordinary differential equations of the second order*

Given the differential equation

$$\rho'' = f(\theta, \rho, \rho')$$

and the initial values $\theta_0, \rho_0, \rho_0'$. With the substitution $\rho' = \nu$ we get the system

$$\rho' = \nu, \quad \nu' = f(\theta, \rho, \nu),$$

which can be integrated as shown in §2. The curve $\rho = \varphi(\theta)$ is the solution of the given differential equation.

In the same way an ordinary differential equation of any order can be graphically integrated. If, for instance, a differential equation of third order is given, $\rho''' = f(\theta, \rho, \rho', \rho'')$ we put $\rho' = \nu$, $\nu' = R$ and replace the given equation by the system

$$R' = f(\theta, \rho, \nu, R); \quad \rho' = \nu; \quad \nu' = R.$$

4. *Linear partial differential equations of the first order*

Given partial differential equation

$$X(\partial z/\partial x) + Y(\partial z/\partial y) = Z,$$

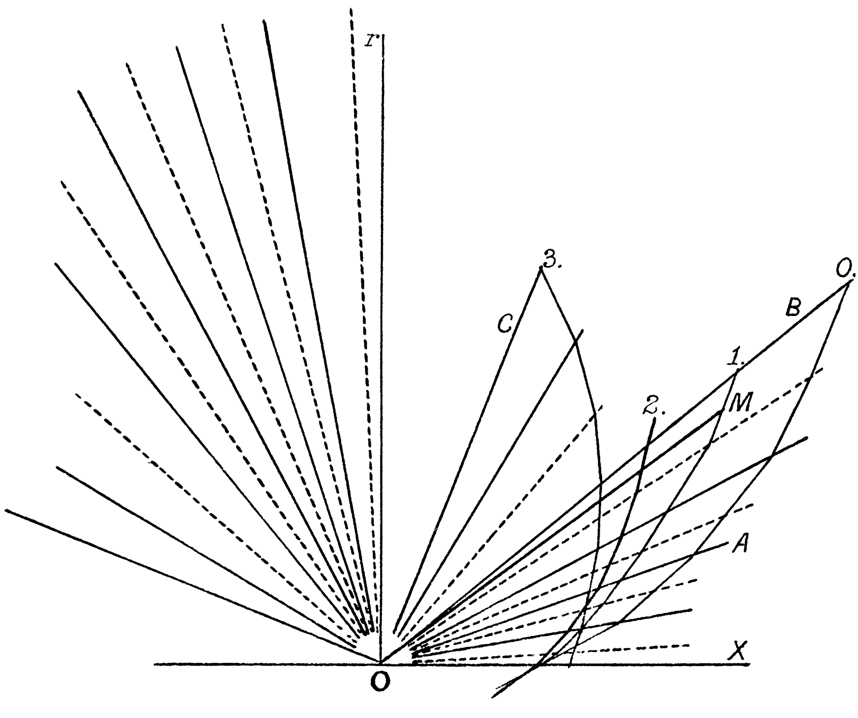

FIG. 6b

where X, Y, and Z are functions of x, y, and z, and initial equations $y = \alpha(x)$, $z = \beta(x)$. From the equations of the characteristic $dx/X = dy/Y = dz/Z$ we get $dz/dx = Z/X$, $dy/dx = Y/X$, taking x as independent variable and y and z as functions.

Substituting θ, ρ, ν for x, y, z and putting $Z/X = f(\theta, \rho, \nu)$, $Y/X = g(\theta, \rho, \nu)$ we obtain the system

$$\nu' = f(\theta, \rho, \nu), \quad \rho' = g(\theta, \rho, \nu),$$

From the initial equations $\rho = \alpha(\theta)$, $\nu = \beta(\theta)$ we obtain (by computing or constructing) a series of values

$$\theta_0, \rho_0, \nu_0; \theta_1, \rho_1, \nu_1; \cdots; \theta_n, \rho_n, \nu_n$$

where θ_0, θ_1, $\cdots \theta_n$ are arbitrary values.

On the diagram (Fig. 6), $\theta_0 = O$, $\theta_1 = \angle XOA$, $\theta_2 = \angle XOB$, $\theta_3 = \angle XOC$. For each group of these values we construct curves $\rho = \varphi_i(\theta)$, $\nu = \psi_i(\theta)$ $(i = 0, 1, 2, \cdots, n)$ as in §3. On the diagram these are the curves 0, 1, 2, 3. To find the value ν for given $\theta = \Theta$, $\rho = \mathrm{P}$, we draw, on the diagram for $\rho = \varphi(\theta)$, the vector OK making the angle Θ with the polar axis and we put on it the length Om equal to P. If the point m lies on a curve $\rho = \varphi_k(\theta)$, the vector OM, drawn on the diagram for $\nu = \psi(\theta)$ and making the same angle θ with the polar axis, intersects the curve $\nu = \psi_k(\theta)$, corresponding to the curve $\rho = \varphi_k(\theta)$, at the point M and is the measure of the required value of ν. If the point m is between two curves, we find the measure of ν by interpolation.

AN UNUSUAL USE OF THE NODAL CUBIC IN THE PLANE

By BESSIE I. MILLER, University of Illinois

Until recently manufacturers of lamp reflectors usually employed a reflector approximating the mathematician's paraboloid of revolution. Such a surface throws light in parallel rays provided there is a point-source of light at the focus. Practically the manufacturer found it satisfactory to use a V-shaped filament lying in a horizontal plane on the axis of the reflector. The axis of the V coincides with the axis of the reflector; the vertex of the V is directed away from the vertex of the reflecting surface. This provides for some latitude in "fore and aft" focussing. Now the manufacturers wish to use two filaments one above the other, so arranged that one can be used for distance lighting during fast travel, the other for conditions occurring in city driving or in passing vehicles moving in the opposite direction. The parabolic reflector of the past fails in this case, since any displacement above or below the axis causes the light to break into divergent beams.

Wm. H. Wood, M.D., of Cleveland, Ohio, who is a specialist in the treatment of disorders caused by the improper functioning of the glands and who is a well-known inventor of commercial and scientific instruments concerned with the reflection of light, has recently added to his long list of patents. He has constructed a new generating curve for an automobile lamp reflector to replace the parabola of the older reflectors. His method was wholly experimental, and the surface eventually obtained was the result of an extraordinarily delicate process used in the polishing of the surface point by point, until the desired corrections on the parabolic reflector were obtained. The engineers to whom was submitted a plane section of the surface obtained by measurement were unable to analyze the curve, but were able to determine a construction for it. It was however not difficult mathematically to find the equation of the curve which, unexpectedly to the inventor, I found to be a nodal cubic, a thing of which he had never heard.

The relation between the old parabola and the new nodal cubic can easily be seen if their equations,

$$y^2 = 4px \tag{1}$$

and

$$y^3 - cy^2 - 4pxy + 4cx^2 = 0, \tag{2}$$

are graphed. The constant c in (2) is the distance above the axis on the latus rectum of (1) at which the second filament is to be placed. The node of (2) is at the vertex of (1). The loop lies almost wholly but not quite in the first quadrant. The upper branch of the loop and a portion of the curve in the fourth quadrant is used in the reflector. The result is that upward divergence of rays from the axis is entirely removed and the beam itself is directed slightly downwards from the horizontal. Hence an on-coming driver is not annoyed.

The manufacturer cannot conveniently use a cubic curve for a generator of a surface, so after determining the cubic it was necessary for me to determine a parabola which approximated the cubic within the region in which it is used. This was done. The formulas both for the particular cubic and parabola have now been patented and the reflectors based on the cubic are on the market.

ON CERTAIN FINITE SUMS OF BINOMIAL COEFFICIENTS AND GAMMA FUNCTIONS

By MIGUEL A. BASOCO, California Institute of Technology

The purpose of this note is to exhibit certain finite sums involving binomial coefficients and gamma functions which are the immediate consequence of the orthogonality property of certain systems of polynomials. Indeed, the results here given are equivalent to the orthogonality property in the sense that one implies and is implied by the other. The explicit results are given for the more common orthogonal polynomials such as those bearing the names of Laguerre, Jacobi, Hermite, etc., but obviously, analogous results may be derived from any such system. Thus the polynomials recently given by Romanovski (Comptes Rendus, t. 188, No. 16, April 1929) will yield the value of certain finite sums, which however, are fairly complicated. It may be pointed out that all of these results would, perhaps, be difficult to establish without reference to their source.

The details of the work; being simple, will be omitted; the definite integrals which have to be evaluated are well known and may be expressed in terms of the beta and gamma functions. In what follows reference is made to Pólya-Szegö, *Aufgaben und Lehrsätze*, II, chapter VI.

1. *The Generalized Laguerre Polynomials.*

These polynomials have the form,

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{x^k}{k!}, \tag{1}$$

where $\alpha > -1$ but otherwise is arbitrary and $\binom{n+\alpha}{r}$ is the coefficient of x^r in the expansion of $(1+x)^{n+\alpha}$. They have the property that

$$\int_0^\infty e^{-x}x^\alpha L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)dx = \begin{cases} 0 \text{ if } m \neq n, \\ \binom{n+\alpha}{n}\Gamma(\alpha+1) \text{ if } m = n. \end{cases} \tag{2}$$

If (1) is substituted in (2) and the order of integration and summation be interchanged, it will follow, after a slight reduction, that the following sum has the value indicated:

$$\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n+\alpha}{n-k}\binom{m+\alpha}{m-l}\frac{(-1)^{k+l}}{k!\,l!}\Gamma(k+l+\alpha+1) = \begin{cases} 0 \text{ if } m \neq n. \\ \binom{n+\alpha}{n}\Gamma(\alpha+1) \text{ if } m = n. \end{cases}$$

For the ordinary Laguerre polynomials, which correspond to the case when $\alpha = 0$, the particularly simple result is obtained:

$$\sum_{k=0}^{n}\sum_{l=0}^{m}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\binom{k+l}{l} = \begin{cases} 0 \text{ if } m \neq n, \\ 1 \text{ if } m = n. \end{cases}$$

2. *The Jacobi (Hypergeometric) Polynomials.*

The Jacobi polynomials may be written in the form:

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x-1}{2}\right)^{n-k}\left(\frac{x+1}{2}\right)^k, \quad \alpha,\beta > -1. \tag{3}$$

They satisfy the following relation:

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)dx = \begin{cases} 0 \text{ if } m \neq n. \\ \dfrac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\,\dfrac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)} \\ \qquad \text{if } m = n. \end{cases} \tag{4}$$

If (3) is substituted in (4), it will be seen, after some reduction, that the following holds:

$$\sum_{k=0}^{n}\sum_{l=0}^{m}(-1)^{k+l}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}\binom{m+\alpha}{l}\binom{m+\beta}{m-l}B(n+m+\alpha-k-l+1,\ \beta+k+l+1) \tag{5}$$

$$= \begin{cases} 0 \text{ if } m \neq n \\ \dfrac{1}{2n+\alpha+\beta+1}\cdot\dfrac{B(n+\alpha+1,\ n+\beta+1)}{B(n+1,\ n+\alpha+\beta+1)} \text{ if } n = m. \end{cases}$$

The case $\alpha=0$, $\beta=0$ is of interest, for then (5) reduces to

$$(6)\quad \sum_{k=0}^{n}\sum_{l=0}^{m}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\binom{k+l}{k}\binom{m+n-k-l}{n-k}=\begin{cases}0 \text{ if } m\neq n\\ \dfrac{(2n)!}{n!^2} \text{ if } m=n,\end{cases}$$

which is a consequence of the orthogonality of the Legendre polynomials, when these are expressed in the form corresponding to (3). It will be seen in the next section that another formula may be obtained when the Legendre polynomials are written in a different form.

If $\alpha=-\frac{1}{2}$, $\beta=-\frac{1}{2}$, the Jacobi polynomials reduce to Tschebyscheff polynomials (to within a constant factor) so that (5) with these values of α, β is essentially a consequence of the orthogonality of the Tschebyscheff polynomials.

3. *The Legendre Polynomials.*

These polynomials may be written in the form:

$$P_n(x)=\sum_{k=0}^{k\leq n/2}(-1)^k\frac{(2n-2k)!}{(n-2k)!(n-k)!k!}x^{n-2k}.$$

If this expression is substituted in the integral,

$$(7)\qquad \int_{-1}^{1}P_m(x)P_n(x)dx=\begin{cases}0 \text{ if } m\neq n,\\ \dfrac{2}{2n+1} \text{ if } m=n,\end{cases}$$

it will be seen that

$$(8)\quad \sum_{k=0}^{k\leq n/2}\sum_{l=0}^{l\leq m/2}(-1)^{k+l}\binom{n}{k}\binom{m}{l}\binom{2n-2k}{n}\binom{2m-2l}{m}\frac{1}{m+n-2k-2l+1}=\begin{cases}0 \text{ if } m\neq n \text{ and } m+n\equiv 0 \bmod 2,\\ \dfrac{2^{2n}}{2n+1} \text{ if } m=n.\end{cases}$$

It is clear that formulas (6) and (8), having their source in the same orthogonality property (7), are equivalent.

4. *The Tschebyscheff Polynomials.*

These polynomials are defined by the expression

$$T_n(x)=\frac{1}{2^{n-1}}\cos n\theta,\quad x=\cos\theta,$$

which may be written in the form

$$(9)\quad T_n(x)=\frac{1}{2^{n-1}}\sum_{k=0}^{k<n/2}\sum_{r=0}^{k}(-1)^{k+r}\binom{n}{2k}\binom{k}{r}x^{n+2r-2k}+\frac{\epsilon(n)(-1)^{n/2}}{2^{n-1}}\sum_{s=0}^{n/2}(-1)^s\binom{n/2}{s}x^{2s},$$

where k ranges from 0 to $\frac{1}{2}(n-2)$ if n is even or to $\frac{1}{2}(n-1)$ if n is odd, and where $\epsilon(n)$ is zero or unity according as n is odd or even.

When (9) is substituted in the integral,

$$\int_{-1}^{1}\frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{\pi}{2^{2n-1}} & \text{if } m = n, \end{cases}$$

it will be seen, after evaluating the necessary integrals and reducing, that the following formula holds:

$$\sum_{k=0}^{k<n/2}\sum_{l=0}^{l<m/2}\sum_{r=0}^{k}\sum_{\rho=0}^{l}(-1)^{k+l+r+\rho}\binom{n}{2k}\binom{m}{2l}\binom{k}{r}\binom{l}{\rho}\binom{m+n+2r+2\rho-2k-2l-1}{\frac{m+n}{2}+r+\rho-k-l}2^{2k+2l-2r-2\rho}$$

$$+\epsilon(n)(-1)^{n/2}2^n\sum_{l=0}^{l<m/2}\sum_{\rho=0}^{l}\sum_{s=0}^{n/2}(-1)^{l+\rho+s}\binom{m}{2l}\binom{l}{\rho}\binom{n/2}{s}\binom{m+2\rho-2l+2s-1}{\frac{m}{2}+\rho-l+s}2^{2l-2\rho-2s}$$

$$+\epsilon(m)(-1)^{m/2}2^m\sum_{k=0}^{k<n/2}\sum_{r=0}^{k}\sum_{\sigma=0}^{m/2}(-1)^{k+r+\sigma}\binom{n}{2k}\binom{k}{r}\binom{m/2}{\sigma}\binom{n+2r-2k+2\sigma-1}{\frac{n}{2}+r-k+\sigma}2^{2k-2r-2\sigma}$$

$$+\epsilon(n)(-1)^{(n+m)/2}2^{n+m}\sum_{s=1}^{n/2}\sum_{\sigma=1}^{m/2}(-1)^{s+\sigma}\binom{n/2}{s}\binom{m/2}{\sigma}\binom{2s+2\sigma-1}{s+\sigma}2^{-(2s+2\sigma)}$$

$$= \begin{cases}\epsilon(n)(-1)^{(n+m+2)/2}2^{n+m-1} \text{ if } m \neq n \text{ and } m+n \equiv 0 \bmod 2 \\ (-1)^{n+1}2^{2n-2} \text{ if } n = m.\end{cases}$$

For either m or n odd this formula obviously reduces to a fairly simple one.

An analogous formula may be derived from the polynomials,

$$U_n(x) = \frac{1}{n+1}T'_{n+1}(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta,$$

and the integral

$$\int_{-1}^{1}\sqrt{1-x^2}U_n(x)U_m(x) = \begin{cases} 0 & \text{if } m \neq n, \\ \dfrac{\pi}{2} & \text{if } m = n. \end{cases}$$

5. *The Hermite Polynomials.*

Employing the notation of the preceding section, the Hermite polynomials may be written in the form:

$$H_n(x) = \sum_{k=0}^{k<n/2}(-1)^k k!\binom{n}{k}\binom{n-k}{k}(2x)^{n-2k} + \epsilon(n)\frac{(-1)^{n/2}n!}{(n/2)!}.$$

The orthogonality relation for these polynomials is

$$\int_{\infty}^{\infty}e^{-x^2}H_n(x)H_m(x)dx = \begin{cases} 0 \text{ if } n \neq m, \\ 2^n n!\sqrt{\pi}, \text{ if } m = n. \end{cases}$$

Proceeding as before the following result is obtained:

$$\sum_{k=0}^{k\leq n/2}\sum_{l=0}^{l\leq m/2}(-1)^{k+l}k!l!\binom{n}{k}\binom{m}{l}\binom{n-k}{k}\binom{m-l}{l}\binom{m+n-2k-2l-1}{\frac{m+n}{2}-k-l}\left(\frac{m+n}{2}-k-l\right)!$$

$$+\frac{\epsilon(n)(-1)^{n/2}n!}{(n/2)!}\sum_{l=0}^{l\leq m/2}(-1)^{l}l!\binom{m}{l}\binom{m-l}{l}\binom{m-2l-1}{\frac{m}{2}-l}\left(\frac{m}{2}-l\right)!$$

$$+\frac{\epsilon(m)(-1)^{m/2}m!}{(m/2)!}\sum_{k=0}^{k\leq n/2}(-1)^{k}k!\binom{n}{k}\binom{n-k}{k}\binom{n-2k-1}{\frac{n}{2}-k}\left(\frac{n}{2}-k\right)!$$

$$=\begin{cases}\epsilon(n)\dfrac{(-1)^{(n+m+2)/2}n!m!}{2(n/2)!(m/2)!} & \text{if } m\neq n \text{ and } m+n\equiv 0 \bmod 2\\[2ex] 2^{n-1}n!+\epsilon(n)\dfrac{(-1)^{n+1}n!^2}{2(n/2)!^2} & \text{if } m=n.\end{cases}$$

In case either m or n are odd this formula obviously simplifies considerably.

ON THE REDUCTION OF THE GENERAL QUARTIC TO BINOMIAL FORM

By RAYMOND GARVER, University of California at Los Angeles

In a recent issue of the Bulletin of the American Mathematical Society[1] I pointed out that the quartic equation, considered from the standpoint of the Tschirnhaus transformation, may be said to occupy a unique position, between the quadratic and cubic on one hand, and the higher degree equations on the other. For the equations of lower degree can be reduced easily to binomial form by simple linear or quadratic transformations, while equations of the fifth degree and higher can be reduced to a form lacking the second, third, and fourth terms by a fourth degree transformation, the determination of whose coefficients requires the solution of no equation of degree higher than the third. A transformation of this type was first set up, for the quintic, by Bring, and later, in general, by Jerrard. Now these two problems, reduction to binomial form and removal of three intermediate terms, become equivalent for, and only for, the quartic equation.

The question as to whether or not this reduction of the quartic was possible still remained, and seemed to be of some interest. First, the Bring-Jerrard process does not apply to the quartic, nor can the direct method of transforming the general cubic to binomial form be extended to the quartic. For it is easy to see that if the attempt is made to remove three intermediate terms with the aid of a simple cubic transformation, a sixth degree equation will be arrived at. And while it is true that Lagrange[2] was able to give an elegant proof that this sixth

[1] Vol. 33 (1927), pp. 677–680.

[2] References to Lagrange and Sylvester are given in my Bulletin paper.

degree equation factors as the product of three quadratics whose coefficients are themselves roots of cubics, it is hardly feasible to carry out this factorization in practice. Secondly, Lagrange's conclusion itself was contradicted later by Sylvester, who said: "Five is the minimum degree of equation from which three terms may be removed without solving an equation above the third degree." I showed, however, that Sylvester's statement was invalidated by the fact that he was actually treating, in his proof, a slightly different problem than the one he proposed. And finally, in my Bulletin paper, I effected the reduction of the general quartic to binomial form by employing two successive transformations, properly chosen. Lagrange's work did not go this far.

It turns out that the matter can be treated somewhat more nicely by proceeding indirectly. We shall start with a binomial equation, in fact, with the form

$$y^4 - 1 = 0, \tag{1}$$

and apply to it the transformation

$$x = Ay^3 + By^2 + Cy. \tag{2}$$

The transformed equation is easily set up; and we shall show that it can be identified with the general quartic (with second term removed),

$$x^4 + a_2x^2 + a_3x + a_4 = 0, \tag{3}$$

by choosing properly the parameters A, B, C. This method of attack was suggested by the fact that the transformation of the Brioschi normal quintic into a general principal quintic is simpler than the direct transformation of the general principal quintic into the Brioschi form.[3] A well-known solution of the cubic, which is due to Euler, employs the same principle. Incidentally, a point of interest in the method of this paper is that it leads to a convenient algebraic solution of the quartic equation.

In setting up the transformed equation which results from applying (2) to (1) we require the values of the sums of certain powers of the roots of (1). If s_k represents, as usual, the sum of the kth powers of the roots it is obvious that $s_k = 0$ unless k is a multiple of 4 and that $s_4 = s_8 = \cdots = 4$. If we sum (2) for the roots of (1) we then have $\sum x_i = 0$, and the transformed equation thus has no term in x^3. The coefficients of x^2 and x may be obtained by multiplying $\sum x_i^2$ and $\sum x_i^3$, respectively, by $-\frac{1}{2}$ and $-\frac{1}{3}$. These summations are evaluated by squaring and cubing (2), and summing over the roots of (1). In forming x^3, it is necessary to retain only the terms involving y^8 and y^4, since the others will disappear upon summing. The numerical work is thus very simple, and we find $-2(B^2 + 2AC)$ as the coefficient of x^2, and $-4B(A^2 + C^2)$ as the coefficient of x.

The constant term in the transformed equation can be obtained with the aid of $\sum x_i^4$; this would be the usual procedure. But we may also work directly from

[3] L. E. Dickson, *Modern Algebraic Theories*, pp. 214–218, 247.

(2). The four roots of the transformed equation are obtained by substituting for y in (2) the values $1, -1, i, -i$, from (1), in succession. The constant term is the product of these four roots, or $(A+B+C)(-A+B-C)(-Ai-B+Ci)(Ai-B-Ci)$. The first two factors give at once $(B^2-A^2-2AC-C^2)$, and the last two, $(B^2+A^2-2AC+C^2)$. The final product can then be put in the form $(B^2-2AC)^2-(A^2+C^2)^2$.

The transformed quartic can be identified with the general quartic (3) provided we can solve the following system of equations for A, B, C, in terms of a_2, a_3, a_4:

$$2(B^2 + 2AC) = - a_2 \tag{4}$$

$$4B(A^2 + C^2) = - a_3 \tag{5}$$

$$(B^2 - 2AC)^2 - (A^2 + C^2)^2 = a_4. \tag{6}$$

If we substitute for A and C from (4) and (5), (6) becomes

$$64B^6 + 32a_2B^4 + 4(a_2^2 - 4a_4)B^2 - a_3^2 = 0. \tag{7}$$

This is exactly the resolvent equation which appears in Descartes' solution of the quartic; it is of course solvable algebraically, and its roots may be denoted by B_1, B_2, B_3 and their negatives. In particular, we shall satisfy (7) by taking $B=B_1$.

We now note that, by (7), $B_1^2+B_2^2+B_3^2=-a_2/2$, $B_1^2B_2^2B_3^2=a_3^2/64$. We may consider B_3 chosen so that $B_1B_2B_3=-a_3/8$. Equations (4) and (5), with $B=B_1$, may now be written:

$$AC = (B_2^2 + B_3^2)/2 \tag{8}$$

$$A^2 + C^2 = 2B_2B_3. \tag{9}$$

It is not difficult to show that these equations can be satisfied by taking

$$\begin{aligned} A &= \tfrac{1}{2}B_2(1 + i) + \tfrac{1}{2}B_3(1 - i), \\ C &= \tfrac{1}{2}B_2(1 - i) + \tfrac{1}{2}B_3(1 + i). \end{aligned} \tag{10}$$

This completes the proof that (1) can be transformed into the general quartic (3). It is then well known that a transformation can be set up leading in the opposite direction, that is, from the general quartic (3) to the binomial form (1). This transformation will not be valid, however, when (3) has a double root; it clearly could not be, since (1) does not have a double root. Except for this case, the explicit transformation leading from (3) back to (1) may be set up by a method given by Dickson[4], or by other known methods. It does not seem necessary to do this here.

For the sake of completeness, it may be noted that while a quartic with a double root cannot be transformed into (1), it can be transformed into a binomial quartic whose constant term also vanishes, that is, $y^4=0$. For in this case

[4] Loc. cit., p. 211, exercise 5.

the left-hand side of (3) will have a third degree factor, say $P(x)$, which vanishes for all four roots of (3). Hence if we set $y=P(x)$, the transformed equation will have all of its roots equal to zero. Thus any quartic can be transformed into a binomial quartic.

Finally, if $B=B_1$ and the values of A and C from (10) are put in (2), and if the roots of (1), 1, -1, i, $-i$, are substituted successively for y, we obtain the four roots of (3). These can easily be put in the form:

$$\begin{aligned} x_1 &= B_1 + B_2 + B_3, & x_3 &= -B_1 + B_2 - B_3, \\ x_2 &= B_1 - B_2 - B_3, & x_4 &= -B_1 - B_2 + B_3. \end{aligned} \tag{11}$$

These values are, of course, well known, but this method of obtaining them is, perhaps, new. B_1 may be any root of (7), B_2 may be any other root except $-B_1$, and then B_3 must be chosen so that $B_1B_2B_3 = -a_3/8$.

QUESTIONS AND DISCUSSIONS

Edited by R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems especially new problems, which are reserved for the department of Problems and Solutions.

A TABLE OF SINES AND COSINES

By J. P. BALLANTINE, University of Washington.

Any table of sines and cosines may be regarded as a tabulation of the values of $\sin n\theta$ and $\cos n\theta$ for integral values of n and for some particular value of θ. Usually θ is chosen some integral divisor of a degree or a radian, or some other convenient unit. The size of the unit, as far as we know, is never determined with the convenience of tabulation in mind. In the present paper we introduce certain new units which give the tables of sines and cosines interesting and useful properties.

From trigonometry we have

$$\begin{aligned} \sin(n+1)\theta - \sin(n-1)\theta &= 2\cos n\theta \sin\theta, \\ \cos(n+1)\theta - \cos(n-1)\theta &= -2\sin n\theta \sin\theta. \end{aligned} \tag{1}$$

Let us, therefore, choose θ so that

$$2\sin\theta = (.1)^k, \tag{2}$$

where k is some such convenient number as 1, 2, or 3. Equations (1) then reduce to

$$\begin{aligned} \sin(n+1)\theta &= \sin(n-1)\theta + (.1)^k \cos n\theta, \\ \cos(n+1)\theta &= \cos(n-1)\theta - (.1)^k \sin n\theta. \end{aligned} \tag{3}$$

For $k=0$, $2\theta=60°$, a rough approximation to a radian. As k increases, $2\theta\cdot 10^k$ approaches the radian very rapidly. This is not surprising, since a radian is so chosen that the derivative of the sine is the cosine, while 2θ is so chosen that the differences of the sine are simple multiples of the cosine. For finely tabulated functions differences and derivative are nearly the same.

Note by the Editor

What unit of angular measurement would it be best to use if all trigonometric computations had to be carried out to twenty (20) significant figures? This question is suggested by the preceding article of Professor Ballantine's which may or may not therefore be regarded as furnishing an answer to it. Would one of Professor Ballantine's θ angles for some suitable value of k be a better unit for such extended calculation than any now in common use?

R. E. G.

A Peculiar Function

By J. P. Ballantine, University of Washington

I sometimes show my beginners in calculus the following function:

A pie reposes on a plate of radius R. A piece of central angle θ is cut and put on a separate plate of radius r. How large must r be? Obviously r is a function of θ. It turns out to have the following formula:

$$\begin{array}{ll} r = 0 & \theta = 0 \\ r = \frac{1}{2}R\sec\frac{1}{2}\theta & 0 < \theta \leqq 90° \\ r = R\cos\frac{1}{2}\theta & 90° \leqq \theta \leqq 180° \\ r = R & 180° \leqq \theta \leqq 360°. \end{array}$$

The function has one discontinuity at $\theta=0$, and its second derivative has various discontinuities.

Note by the Editor

The preceding example by Professor Ballantine seems an uncommonly good one to use when introducing the notion of discontinuous functions to an elementary class as illustrating how naturally such functions arise. It might be of considerable interest and value to make a collection of some more examples of this sort. Can anyone suggest another one equally simple and equally free from an appearance of artificiality?

R. E. G.

REVIEWS

Elements of the Differential and Integral Calculus. By W. A. Granville, P. F. Smith, W. R. Longley. Ginn and Company, Boston, 1929. xi+516 pages. $3.20.

The reviewer of this book approached his task with great pleasure. He had studied Granville's calculus in his undergraduate days and had taught the book in later years. We students liked the old book because of the many worked-out examples in the text, the answers that were given with the exercises, the many fine figures and curves, the systematic rules of procedure (such as the famous four step rule for differentiation), the many formulas that were quoted from earlier courses in mathematics, the large and clear print, the way important definitions and theorems stood out on the pages, the ease with which topics could be found in the book. In revising this book Professors Percey F. Smith and William R. Longley have retained all these good points of the older edition, have corrected some faults, and have added many improvements. They have practically rewritten the whole book and yet they have not been too radical in their changes. For one thing they are Yale men, just as was Professor Granville, and they therefore have followed out his purpose more closely perhaps than an outsider would have done.

We teachers of the old Granville's *Calculus* liked the book for the same reasons as did the students, and also because of the large number of topics to choose from, because of the logical order in which the text proceeded from topic to topic, because of the multiplicity of problems, because the book almost taught itself in that a student could read any section of the text and then study the illustrative examples and work the assigned problems himself. Even in these particulars the new revision is an improvement on the old text inasmuch as the arrangement of the topics is greatly altered for the better. Most of the unnecessarily difficult problems have been omitted, more problems have been introduced from D. D. Leib's *Problems in the Calculus*, and the sets of exercises have been better graded as to difficulty. We quote from the preface of this new edition: ". . . . Clearness and simplicity in the text, numerous illustrative examples solved in the text, a wealth of problems both to acquire practice in technique and to stimulate interest—these distinctive features have been retained. A change in arrangement is to be noted. The calculus for functions of one variable, both differential and integral, is developed completely. This is followed by the calculus for functions of more than one variable. . . . . Some topics in the old edition have been omitted. But this is more than compensated for by the addition of several topics of importance and interest."

In this latest edition the print is even clearer and the formulas stand out better than in former editions. The numbering of the sections is different, some of them being thrown together and others changed in position. The formulas for derivatives are divided now into two parts, algebraic and transcendental. First all the derivatives of algebraic functions are taken up (with their applications to

rates and to maxima and minima); they are followed by successive differentiation and its application to velocities, accelerations, and curve tracing. Next the derivatives of transcendental functions are studied. (In this connection we might ask why the use of trigonometric functions in solving maxima and minima problems is not stressed more.)

There follows a chapter (Chapter VIII) on "Applications to Parametric Equations, Polar Equations, and Roots" in which are collected discussions that are scattered in the earlier editions of the book. Here is introduced Newton's method for finding the roots of an equation. Chapter IX deals with differentials; in it the treatment of infinitesimals is much improved. Chapter X treats of curvature, radius and circle of curvature. The final chapter discusses the theorem of the mean value. This is an excellent place in the book for this discussion, because Chapter XII begins the study of the integral calculus in which this theorem plays such an important role. Less space is given to change of variables for derivatives. Fewer formulas are stressed in the new book; this makes for a less formal knowledge of the subject by the student. Asymptotes and singular points of curves are omitted from the book. Also t is replaced by θ. The authors postpone partial differentiation and multiple integrals until after differential equations; also series and expansions of functions are taken up after integration and its applications.

When we come to integration, we note that on page 194 the authors collect together the solved problems instead of leaving them scattered. Integration by parts is put on page 224 and not further on with the reduction formulas. The topic of integration as a process of summation is placed right after the definite integral, these two topics not being separated from each other by a discussion of rational fractions, rationalization, and the reduction formulas. On page 234 more emphasis is put on initial conditions and more examples are given under this heading than in the older editions of the book. More work and figures are given in the discussion of areas. The mean value of a function is put later with centroids. Simpson's rule and the trapezoidal rule for the approximate evaluation of definite integrals are given right away instead of at the end of the book. Chapter XVIII is an excellent discussion of centroids, fluid pressure, and other allied topics, using one variable. In the older book such work was done mostly by multiple integration. The authors have added, on page 362, integration and differentiation of power series; also they give more space to approximations from power series. They give less space to integration by rationalization and by the reduction formulas, but more space to training in the use of the tables of integrals. In the case of series the authors introduce binomial and other series; also they start the discussion from the standpoint of sequences and go from there to series. On page 354 they give Maclaurin's series first and operations with series, then Taylor's series. This in an improvement from the viewpoint of the teacher.

The new treatment of differential equations is a great improvement over the old; the subject matter is entirely rearranged, the discussion is less formal,

more stress is placed on applications to mechanics. Also the discussion of second order and special equations precedes that of the general equation of the nth order. The topic of partial differentiation is better treated than of old. Total differentials are discussed before total derivatives. The proof that $\partial^2u/\partial x\partial y =\partial^2u/\partial y\partial x$ follows the formula. The authors prove the law of the mean for functions of more than one variable; also they put here Taylor's and Maclaurin's series for functions of several variables. The notation $|\partial F/\partial x|_1$ is used instead of $\partial F_1/\partial x_1$. Not much change is noticeable in the treatment of multiple integration, except a rearrangement of the material.

The above review of the changes in the new edition will show that the book has been brought up-to-date and its weaknesses corrected. The original book was written when mathematics was still looked upon as good mental discipline. This attitude is not evident in the new edition. The emphasis of the older editions was too much on the geometrical applications of the calculus. This fault has been largely corrected. Students will not gather from the new book the false impressions we gathered from the old, namely that dy/dx was a derivative, but not ds/dt or $dr/d\theta$; that partial derivatives are not like ordinary derivatives; that the words increment, infinitesimal, and differential mean the same thing; that definite integrals are approximations; that every algebraic fraction can be split into partial fractions for the purposes of integration. The reviewer fears that no calculus book will ever get around the students' troubles with dy/dx considered as a quotient of differentials until derivatives are represented by D_x, y_x', etc. as in the English books; also that definite integrals will always be confusing when treated after indefinite integrals instead of vice versa; and that similarly multiple integration will always be the weakest spot in a calculus book until it is treated less formally. The new revision has removed the causes of the teacher's criticisms of the older book, that it was a drill-book that got results just as cramming helps a student to pass an examination, but after all the student did not really understand the calculus even though he could work the problems with facility.

It is very refreshing to come back to this conservative calculus book brought up-to-date by Professors Smith and Longley, after using newer books that have been experimenting with other ways of teaching the subject. If the students can be persuaded or compelled to read the text and follow through the steps in the illustrative problems, the Granville-Smith-Longley Calculus will demand less of a teacher than a book with a briefer text and with a mixture of differentiation and integration such as many new books have. This book does not lead the reader up to the various ideas of the calculus in perhaps the most approved modern way with much fine writing; rather it uses the old fashioned methods of laying down definitions and proving theorems like a book on plane and solid geometry. It is strongly urged however that every teacher try such a beautifully written book as this where the ideas of differentiation and integration, of ordinary derivatives and partial derivatives, and other such pairs of

ideas, are kept clearly distinct, and compare the results with those achieved when teaching a book where all these ideas are closely interwoven.

ALAN D. CAMPBELL

The Mathematics of Investment. By W. L. Hart. Revised Edition. New York, D. C. Heath and Co., 1929. xii+253+11 pp.

The revision pertains to Part I, relating to compound interest and annuities certain, which has been entirely rewritten. The new arrangement—in line with the rest of the text—lends itself especially well to a choice between a "minimum course" and a "maximum course" (or even other "courses"). In the minimum course the work in annuities is restricted to those annuities where the payment interval is the same as the interest period. The maximum course includes supplementary material both in the form of exercises and chapters. The general treatment of annuities is postponed to a fairly late stage. A chapter involving a second or "re-investment" rate of interest has been added.

Answers to the odd-numbered problems are provided with the text while answers to the even-numbered problems can be obtained separately. This text is notable for the large number of exercises included.

C. H. FORSYTH

Curve piane speciali algebriche e trascendenti. Teoria e storia. Volume I.—Curve algebriche con 122 figure illustrative intercalate nel testo. Prima edizione Italiana. By Gino Loria. Ulrico Hoepli, Milano, 1930. xvi+574 pp.

The German editions (1902, 1910) of Loria's treatise are well-known, and the second edition is one of the most widely used books of reference in the entire field of geometry. Although we are under great obligations to Teixeira and to Wieleitner for treatises of somewhat similar type, yet the usefulness of Loria's work has not been impaired by these later publications.

It may not be well known that the Teubner 1902 German edition of Loria's treatise was the first edition, and that no Italian edition has been brought out until this year. This edition must be a source of great satisfaction to Professor Loria.

This first volume is divided into five books, bearing the titles:

Libro I. Luoghi piani e luoghi solidi.

Libro II. Curve del terz'ordine.

Libro III. Curve del quart'ordine.

Libro IV. Curve algebriche particolari di un ordine determinato superiore a quattro.

Libro V. Curve algebriche particolari di ordine qualunque.

In general, the text bears no striking departure from the second German edition. The five books are the same, and the chapter headings are also the same except in book IV where there is a slight rearrangement and compression from seven chapters to five. As noted in the title, the figures now appear in

the text, a great improvement. The results of many recent investigations appear in this edition, including many from relatively obscure journals. In the reviewer's judgment the book is thoroughly up to date. There are approximately 75 additions to the index of names, bringing this to the surprising total of 650. The volume lacks a subject index, presumably a temporary omission which the second volume will remedy.

At the risk of seeming ungracious we must note the fact that the proofreading is badly done. This does not appear in the text to the extent that it does in the foot-notes, where the errors are even worse than those in the German editions. The numerical references seem to be accurate, but the titles, particularly the English and German titles, are very inaccurate and at times grotesque. The user of this volume will find it is unsafe to use Loria's references without checking them. This seems to us unfortunate.

Despite this, we prefer to end the review with deep appreciation to the author for his scholarly production, and for enriching and bringing up to date the most useful treatise of its kind. May the second volume soon follow.

B. H. BROWN

Plane Analytic Geometry. By N. J. Lennes and A. S. Merrill. Harper & Brothers, New York, 1929. xv+305 pages. \$2.50.

In this day of the multiplication of books on every subject, when, indeed, the aphorism of the Hebrew essayist in the biblical Ecclesiastes about the making of many books becomes significant, it is a considerable task to pick out the really superior books and say: "These are the very best available." There are so many good productions, and inferior, too, each unfortunately having some blemish, that one wonders why it should seem necessary to continue to write and to print new works on old subjects. Why not do as was done with Euclid's *Elements*: issue the same work decade after decade until the number of editions exceeded one thousand, and drill rising youth through the same work?

But these are days of progress, wonderful progress, and better expression. Verily he that followeth the plow knoweth more than the bishops.

Professors Lennes and Merrill of the University of Montana have written a commendable book under the title, "Plane Analytic Geometry." Two quotations from the prefaces (permission to quote being assumed) will be in order. The co-operating editor, Professor Slaught, says:

"The leaders of tomorrow in the mathematical world must come from the small group of exceptional pupils in our classes of today. Hence, any pedagogical procedure that best promotes the scientific stimulation of this group, while by no means overlooking the welfare of the larger group of average pupils, is to be commended and encouraged."

And the authors themselves say:

"Much has been said and written in deprecation of 'mere memory work.' But what is deprecated is not the remembering, but memorizing without under-

courses better established, we shall be getting closer to the English university system—and it is a pretty good system. The problems are copious and fine.

F. M. McGaw

Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie. By Edmund Landau. 2nd Edition. Published by Julius Springer, Berlin, 1929. 122 pp.

This monograph, which appears now in a second edition, is a veritable mine of important results in the modern theory of functions of a complex variable. In eight chapters, it treats successively bounded power series, summability of higher order, converses of Abel's theorem, peculiarities of power series on the circle of convergence, relations between the coefficients of a power series and its singularities, maxima and mean values of the modulus along circles, Picard's theorem, functions with one-valued inverses.

One of the most interesting changes in the new edition will be found in the chapter on Picard's theorem. The older elementary proof of Borel is replaced by one based on the highly novel and extremely simple method recently given by Bloch. Other new material is Fatou's theorem to the effect that an analytic function which is bounded within a circle approaches a limit radially except, perhaps, on a set of directions of zero measure, and Fabry's gap-theorem.

The monograph is composed in the individual, really inimitable, style which characterizes Landau's publications. A minimum of initial knowledge is assumed of the reader. Every proof is perfectly polished and no detail of reasoning or calculation is left to be filled in. No statement is made in ordinary language which can be made more briefly in terms of symbols. All remarks of motivation are made concisely, in a purely objective manner and with no attempt at literary effect. In this last connection, many will recall, however, that the pages of Landau's *Zahlentheorie* are interspersed with classroom jokes.

On starting to read anything written by Landau, one knows in advance that one will be able to read it successfully. There are never any snags. But it frequently results, from the absolute completeness of detail, that the central ideas of a proof do not stand out conspicuously.

To students of analysis, or of number theory, one cannot recommend too highly the study of Landau's writings, both original and expository. Their richness of substance, depth of scholarship and technical perfection give them an outstanding position in the current literature.

J. F. Ritt

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEM FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3425. *Proposed by Pauline Sperry, University of California.*

Cauchy's linear equation $\sum_{i=0}^{n} c_i x^{n-i} D^{n-i} y = X(x)$, where the c_i are constants, is reduced to an equation with constant coefficients by the substitution $x = e^z$. Then $x^k D^k y = \sum_{i=0}^{k-1} a_i^{(k)} D^{k-i} y$, where $Dy = dy/dz$. Show that the $a_i^{(k)}$ build a sort of Pascal's triangle with k elements in the k th row, in which any element of the $k+1$st row may be obtained from those in the kth row by taking the two numbers diagonally above it to the left and right and subtracting k times the former from the latter with the understanding that if either of the diagonals takes one outside the triangle the corresponding term is zero.

3426. *Proposed by B. C. Wong, Berkeley, California.*

Prove or disprove:
$$\sum_{i=0}^{t} (-1)^i \binom{r}{i} \frac{(2r-2i-1)!}{(r-2i)!} = (r-1)!$$

where $t = r/2$ if r is even and $t = (r-1)/2$ if r is odd.

3427. *Proposed by L. S. Johnston, University of Detroit.*

Let
$$S_n \equiv \tfrac{1}{2}n(n+1), \quad C_{n-k} \equiv (-1)^{k+1} S_{n+1-|k|};$$

show that the equation
$$\sum_{k=-n}^{k=n} C_{n-k} x^{n+k} = 0$$

has no negative roots, and that the number of positive roots is two or zero according as n is odd or even.

3428. *Proposed by Wm. B. Campbell, Judson College, Rangoon, India.*

A closed vessel in the form of a right circular cylinder, of radius a and height $2h$, contains liquid of volume $\pi a^2 k$. It is rotated about a horizontal axis per-

pendicular to the geometrical axis at the center. With a view to ascertaining the maximum torque, find the maximum displacement of the center of gravity of the liquid.

3429. *Proposed by Paul Wernicke, Washington, D. C.*

Given a tetrahedron $ABCD$ and a point P not on one of its edges. Draw through P three lines, each meeting two opposite edges. Express the ratios in which the latter will be divided by these points of intersection in terms of four quantities a, b, c, d, the edge AB being divided in the ratio $a:b$, etc.

3430. *Proposed by Richard M. Sutton, California Institute of Technology.*

It is physically possible, given a large number of unit resistances, to make any resistance p/q between two points A and B, where p and q are integers. The result may be accomplished by connecting in parallel q groups of p resistances each, requiring (pq) resistances. However, it is usually possible to accomplish the same result by a fewer number of unit resistances. The problem is: "Find the minimum number of unit resistances necessary to make a resistance p/q between two points A and B in an electric circuit, p and q being both integers."

3431. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

The letters a and b designate respectively two fixed, non-complanar lines, the angle between which is α, and the perpendicular distance between which is l; the letters P and Q designate two points on a third line c such that the distance PQ is constant; moreover, points P and Q of line c move on lines a and b, respectively. Show: (I) that the locus (relative to the frame formed by lines a and b) of any third point S of line c is an ellipse whose plane is parallel to lines a and b and whose semi-major and semi-minor axes are

$$(1)\qquad \left\{\frac{(n-m)^2-l^2}{4\sin^2\alpha}+nm\frac{(n-m)^2-l^2}{(n-m)^2}\right\}^{1/2}+\frac{\left\{(n-m)^2-l^2\right\}^{1/2}}{2\sin\alpha}$$

and

$$(2)\qquad \left\{\frac{(n-m)^2-l^2}{4\sin^2\alpha}+nm\frac{(n-m)^2-l^2}{(n-m)^2}\right\}^{1/2}-\frac{\left\{(n-m)^2-l^2\right\}^{1/2}}{2\sin\alpha},$$

wherein n and m represent the linear segments QS and PS, respectively; (II) that the angles which lines a and b make with the semi-major and the semi-minor axes, respectively, are

$$(3)\qquad \gamma = 45^\circ - \tfrac{1}{2}(\alpha+\beta),$$

and

$$(4)\qquad \delta = 45^\circ - \tfrac{1}{2}(\alpha-\beta),$$

wherein β is given by the relation,

$$(5) \qquad \tan\beta = (n - m)\cot\alpha/(n + m).$$

Note: The linear segments n and l of expressions (1) and (2) are regarded as positive for all configurations; m in these expressions is positive or negative according as point S divides segment PQ externally or internally; angles α and β, in equations (3), (4), and (5), are regarded as positive for all configurations; γ and δ may be either positive or negative, depending upon the position of point S on line c.

SOLUTIONS

3387 [1929, 397]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

With the same vertex of a given triangle as center, two circles are drawn respectively orthogonal to the nine-point circle and to the conjugate circle of the triangle. Show that the ratio of the squares of the radii of the two circles is 1:2.

Solution by Laurence Hampton, Alabama Polytechnic Institute

Let the vertices of the given triangle be $O(0,0)$; $A(a,0)$; $B(b,c)$; let H denote the orthocenter, which is the center of the conjugate circle; and let BH intersect the opposite side of the triangle in E. The coordinates of H are found to be $[b, bc^{-1}(a-b)]$; the radius R of the conjugate circle is defined by $R^2 = HB\cdot HE$, and this product is found to be $b(b-a)+b^2c^{-2}(b-a)^2$. Hence the equation of the conjugate circle is

$$cC_h \equiv c(x^2+y^2) - 2bcx - 2b(a-b)y + abc = 0.$$

The equations of the nine point circle and the circum-circle are, respectively,

$$2cC_n \equiv 2c(x^2+y^2) - c(a+2b)x - (ab+c^2-b^2)y + abc = 0,$$

$$cC_c \equiv c(x^2+y^2) - acx - (b^2+c^2-ab)y = 0.$$

Hence

$$2C_n - C_h \equiv C_c,$$

and this says that the circum-circle is one of the coaxial system of circles determined by $C_n=0$ and $C_h=0$, and that the square of the tangent from any point of the circum-circle to the conjugate circle is equal to twice the square of the tangent from the same point to the nine point circle. There are exceptional cases where no real tangent can be drawn.

Note by the Editors: The exceptional cases for which there exist no real tangents may be provided for in the statement of the problem by replacing the words "square of the tangent" by "power of the point with respect to the circle." If the equation of the circle is $x^2+y^2+Ax+By+C=0$, where A, B, C are real, then for any point x, y of the plane the left hand side of the equation is called the power of this point with respect to the circle. If $A^2+B^2-4C>0$, the circle is real and the power of the point with respect to it is the product of the two

segments on any secant drawn through the point, the segments being measured from the point to the intersections of the secant with the circle. If the point is outside the circle, it is also the square of the tangent to the circle, since the two segments then coincide. If $A^2+B^2-4C<0$ the circle is imaginary. In this case the power of the point is the square of the hypotenuse of the right triangle with the right angle at the center $(-A/2, -B/2)$ formed by the segment from this center to the point as one side and the segment of length $2^{-1}(4C-A^2-B^2)^{1/2}$ as the other side. The case for which the expression is zero is easily interpreted.

With this interpretation of the problem the proof is easy by geometry. If ABC is the triangle, and if A_h, B_h, C_h are the feet of the altitudes meeting in H, then by definition the conjugate circle has the center H and the radius R such that $R^2=HA\cdot HA_h=HC\cdot HC_h=HB\cdot HB_h$. If we denote by P_{nb} the power of B with respect to the nine point circle, then from certain right triangles in the figure we find $2P_{nb}=BA_h\cdot BC=BC_h\cdot BA=BH\cdot BB_h$, and similar relations for A and C. These equations are true in sign and magnitude for all cases. If the triangle has only acute angles, H lies within the triangle and R^2 is negative. In this case if a circle be drawn with BB_h as diameter and the chord TT' be drawn perpendicular to this diameter at H, then $R^2=HB\cdot HB_h=HT\cdot HT'=-HT^2$. Hence $P_{hb}=BT^2=BH\cdot BB_h=2P_{nb}$. The reasoning is similar for A and C.

If the angle at C is greater than 90° then H lies outside the triangle and R^2 is positive. In this case real tangents may be drawn from A and B. If S is the point of tangency of a tangent from A to the conjugate circle, then $P_{ha}=AS^2=AH\cdot AA_h=AB_h\cdot AC=2P_{na}$. Similarly $P_{hb}=2P_{nb}$. On the other hand C is within the circle (H) and $CH\cdot CC_h$ is negative. If the chord TCT' is perpendicular to CH at C, then $P_{hc}=T'C\cdot CT=-CT^2=CH\cdot CC_h=CB_h\cdot CA=2P_{nc}$. If C is a right angle then (H) is a null circle and $P_{hc}=2P_{nc}=0$.

Since the three circles have their centers on the same straight line we have more than enough to infer their linear dependence.

Also solved by J. W. Clawson, A. Pelletier, O. J. Ramler, and William Hoover.

3388 [1929, 397] (*Corrected*). *Proposed by S. A. Corey, Des Moines, Iowa.*

Prove that, if the tenth and higher derivatives of f vanish identically, $f(i)+f(-i)=120f(0)+30f(1)+30\mathrm{f}(-1)+640f(r)+640f(-r)-405f(s)-405f(-s)-324f(t)-324f(-t)$, where $i=\sqrt{(-1)}$, $r=\sqrt{\frac{1}{2}}$, $s=\sqrt{\frac{1}{3}}$, $t=\sqrt{\frac{2}{3}}$, and where f is analytic. It follows that when the tenth and higher derivatives are negligibly small the indicated relation holds approximately.

Solution by the Proposer.

The function f being analytic, may be written thus:
$f(x)=x^9+ax^8+bx^7+cx^6+dx^5+ex^4+fx^3+gx^2+hx^1+k$. Whence, we get

(1) $$f(i)+f(-i)=2(a-c+e-g+k),$$

(2) $$30f(1)+30f(-1)=60(a+c+e+g+k),$$

$$(3)\qquad 640f(r) + 640f(-r) = 80a + 160c + 320e + 640g + 1280k,$$

$$(4)\qquad 405f(s) + 405f(-s) = 10a + 30c + 90e + 270g + 810k,$$

$$(5)\qquad 324f(t) + 324f(-t) = 128a + 192c + 288e + 432g + 648k,$$

$$(6)\qquad 120f(0) = 120k,$$

whence $(1)=(2)+(3)+(6)-(4)-(5)$, as required.

From the given relation we may deduce formulas such as the following: $405\log(3x+5)+324\log(3x+4)+\log(x+3)+640\log 2=60\log(x+2)+30\log(x+1)+640\log(2x+3)+729\log 3$, approximately, (obtained by taking $f(z)=\log[\sqrt{(x+2)}+z]$, and $\log(x+12)=30\log x+640\log(x+3)+60\log(x+6)-324\log(x+2)-405\log(x+4)$, (obtained by taking $f(z)=\log[\sqrt{(x+6)}+z\sqrt{6}]$, approximately, x being large enough that the sum of the omitted derivatives becomes small.

3390 [1929, 448]. *Proposed by Paul Wernicke, Washington, D. C.*

With the altitudes of a triangle $t_i=A_iB_iC_i$ as sides construct a consecutive triangle t_{i+1} in the series of $t_i(\cdots -1, 0, 1, \cdots)$. Compare the areas of the triangle t_i in the series. Under what condition does the construction become impossible?

Solution by Otto J. Ramler, the Catholic University of America.

Denote the sides of the ith triangle by a_i, b_i, c_i, and its area by T_i. Then, since its altitudes are the sides of the $(i+1)$ th triangle, we have $2T_i=a_ia_{i+1}=b_ib_{i+1}=c_ic_{i+1}$; $2T_{i+1}=a_{i+1}a_{i+2}=b_{i+1}b_{i+2}=c_{i+1}c_{i+2}$. Hence $T_{i+1}/T_i=a_{i+2}/a_i=b_{i+2}/b_i=c_{i+2}/c_i$, and so alternate triangles of the sequence are similar. It then follows that $a_{i+2}^2/a_i^2=T_{i+2}/T_i$. Using one of the above equations to eliminate the a's from this last, we find that $T_{i+1}/T_i=T_{i+2}/T_{i+1}$. Hence when the construction is possible the areas form a geometric progression, and alternate triangles of the sequence are similar.

Let c_i be the side of the ith triangle which is as small or smaller than either of the other two sides of this triangle. The sides of the $(i+1)$th triangle are proportional to the reciprocals of the sides of the ith triangle, and, hence, we must have $1/a_i+1/b_i>1/c_i$ in order to have an $(i+1)$th triangle and an $(i-1)$th triangle. The construction is possible when and only when the smallest side of a triangle of the sequence is greater than half the harmonic mean of the other two sides of that triangle.

Also solved by the Proposer.

3392 [1929, 448]. *Proposed by V. M. Spunar, Chicago, Ill.*

Two points M and N are taken on the sides AB and AC, respectively, of the triangle ABC; and then the point P is taken on the line MN. If these points are chosen so that $BM/MA=AN/NC=MP/PN$, find the locus of P.

Solution by Irene Ramler, Washington, D. C.

Let the common ratio $BM:MA=AN:NC=MP:PN$ equal $t:1$. Then letting ABC be the triangle of reference in a system of areal coordinates, we find the coordinates of M, N, and P to be, respectively, $(t:1:0)$, $(1:0:t)$, $(2t/(1+t):1/(1+t):t^2/(1+t))$.

Hence the parametric equations of the locus of P are

$$x_1 = 2t, \quad x_2 = 1, \quad x_3 = t^2.$$

Eliminating t, we find the equation of the locus to be $x_1^2=4x_2x_3$, a parabola tangent to AB at B and to AC at C. The median through A is a diameter, and the line MPN is a tangent at P.

Thus the parabola is not only the locus of the point P, but it is also the envelope of the line MPN as well.

Note by the Editors: This system of coordinates has been used in the solution of other problems in this Monthly, see, for example, the solution of 3335 [1929, 401]. It is sometimes called the barycentric system of coordinates. Following this order of ideas the coordinates of P are proportional to masses which placed at the vertices A, B, C of the triangle of reference have P as their center of mass. Thus the masses $t(t+1)$ at N and $(t+1)$ at M have P as their center of mass. Now replace $t(t+1)$ at N by two masses at A and C whose ratio is $1:t$ and whose sum is $t(t+1)$: they are easily seen to be t and t^2. In the same way $(t+1)$ at M may be replaced by t at A and 1 at B. Hence we may write the coordinates of P as $2t$, 1, t^2.

The final conclusion of the above solution suggests the idea of first seeking the envelope of MN. This line cuts two projective ranges on AB and AC, respectively, such that A, B, M, ∞ is projective with C, A, N, ∞. Thus the envelope is a conic tangent to AB and AC at B and C, respectively. Since it is also tangent to the line at infinity it is a parabola. Conversely, if AB and AC are any two tangents to a parabola at B and C, any other tangent as MN cuts AB and AC so that $A, B, M, \infty \barwedge C, A, N, \infty$. Hence $AM:MB=CN:NA$. Now use MN and AB as the initial tangents of the same parabola, and let MN be tangent at P. Then, AC being tangent at C, we have as before $M, B, A, \infty \barwedge P, M, N, \infty$; and hence $AM:MB=NP:PM$. Since the three ratios are equal, P the point of contact of MN with the parabola is the point of the problem.

This property of the parabola is useful in elementary instruction, since it gives a simple illustration of the construction of a curve by means of its tangents. The line AC is divided into n equal segments and the points of division are numbered consecutively from A to C. Similarly, BA is divided into n equal parts and the points of division are numbered from B to A. The straight lines joining the points with the same numbers are the line elements of the parabola.

Also solved by W. E. Buker, P. S. Dwyer, L. W. Johnson, J. H. Neelley, A. Pelletier, H. E. Schoonmaker and Paul Wernicke.

3395 [1929, 492]. *Proposed by J. Rosenbaum, Milford, Conn.*

Given an n-gon, $A_1A_2 \cdots A_n$, show how to locate a point X such that the vectors $XA_1, XA_2, \cdots, XA_n$ formed a closed n-gon.

Solution

Let V be the origin of vectors and X any other point. Then $VX+XA_i=VA_i$ and hence $nVX+\sum_{i=1}^{n}XA_i=\sum_{i=1}^{n}VA_i$. A necessary and sufficient condition for $\sum_{i=1}^{n}XA_i=0$ is $nVX=\sum_{i=1}^{n}VA_i$.

Set $mVB_m=\sum_{i=1}^{m}VA_i$, where $B_1=A_1$ and $B_n=X$. Then

$$VA_{m+1}=(m+1)VB_{m+1}-mVB_m=(m+1)[VB_{m+1}-VB_m]+VB_m.$$

Hence $B_mA_{m+1}=(m+1)B_mB_{m+1}$, $m=1, 2, \cdots, m-1$. If we suppose that equal masses are placed at the A-points, then it is clear from the formula for the construction of the B-points that B_m is the center of mass of the first m masses $(m=1, 2, \cdots, n-1)$ and X is the center of mass of all the n masses.

Also solved by A. Pelletier.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The *Annals of Mathematical Statistics* is the name of a new periodical published by the American Statistical Association under the editorship of Professor Harry C. Carver, of the University of Michigan. It is a quarterly journal devoted to the theory and applications of mathematical statistics. The first issue appeared in February. The *Journal of the American Statistical Association* will continue to be the official journal of that organization, but articles involving advanced mathematics will be published in the *Annals* rather than in the *Journal.* The *Annals* will deal not only with the mathematical technique of statistics but also with the applications of such technique to the fields of astronomy, physics, psychology, biology, medicine, education, business, and economics. An announcement in the first issue says: "The editorial policy will be to select articles that will best meet the needs of the time. There can be no questioning the statement that at the present time there are in this country many more who need stimulation in the fundamentals of mathematical statistics than there are individuals whose prime interest is the advancement of statistical theory. Therefore particular stress will be laid on articles of a fundamental nature during the first few years of the life of the *Annals*."

The Franklin Institute has awarded a medal to Sir William Bragg, director of the Davy-Faraday Research Laboratory of the Royal Institution of Great Britain, in recognition of his contributions to the knowledge of atomic structure.

The Rockefeller Institute has given funds to endow a new institute of mathematics at the University of Göttingen. It will be directed by Professor R. Courant.

Professor R. W. Babcock, head of the department of mathematics at DePauw University, has been appointed dean of the division of general science of the Kansas State Agricultural College.

Assistant Professor F. R. Bamforth, of Cornell University, has been appointed assistant professor of mathematics at the Ohio State University.

Professor J. W. Barker, of Lehigh University, has been appointed dean of the faculty of Engineering of Columbia University.

Dr. O. E. Brown has been appointed associate professor of mathematics at Lawrence College.

Professor H. M. Dadourian, of Trinity College, is on leave of absence during the present semester. He is spending the time in Europe.

Dr. Lincoln LaPaz, of the University of Chicago, has been appointed assistant professor of mathematics at the Ohio State University.

Professor Cassius J. Keyser, of Columbia University, delivered a lecture at the New Jersey College for Women, Rutgers University on January 7, 1930. His lecture dealt with the meanings of the terms Mathematics and Science.

Associate Professor J. H. Neeley has been promoted to a professorship in mathematics at the Carnegie Institute of Technology.

Dr. Tibor Radó, of the University at Szeged, Hungary, has been appointed professor of mathematics at the Ohio State University. Dr. Radó was visiting lecturer at Harvard University during the first semester of this year, and is lecturing at Rice Institute during the present semester.

The following courses in mathematics are announced for the summer of 1930:

Stanford University, June 19 to August 30. In addition to the usual courses in calculus and differential equations, the following advanced courses will be given: By Professor Harald Bohr (University of Copenhagen): Theory of functions of a complex variable (and another course to be decided upon later). By Associate Professor Harold Hotelling: Probability and statistics.

University of Texas, first term, June 10 to July 19; second term, July 21 to August 29. In addition to freshman courses, both terms, the following courses

are offered: First term—By Professor R. L. Moore: Foundations of geometry; Theory of sets. By Professor E. L. Dodd: Mathematical statistics; Almost periodic functions and generalized Fourier series. By Professor A. E. Cooper: Advanced calculus. By Professor P. M. Batchelder: Teaching problems in mathematics; Difference equations. By Professor Mary E. Decherd: Calculus. Second term—By Professor H. J. Ettlinger: Differential equations; Ruler and compass constructions. By Professor R. G. Lubben: Non-Euclidean geometry; Functions of real variables. By Professor H. V. Craig: Advanced calculus; Vector analysis. By Mr. C. W. Vickery: Calculus.

University of Vermont. By Professors Bullard, Butterfield, and Millington: Courses in algebra, plane trigonometry, solid geometry, differential and integral calculus, and differential equations.

Professor G. W. McCoard, of the department of mathematics of the Ohio State University, died on March 19, 1930 at the age of 80. He had taught in this department for forty-eight years.

Associate Professor E. A. Pattengill, of Iowa State College, died on February 10, 1930 at the age of 55.

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn; Ohio State University, Columbus, Ohio.

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EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.
INDIANA, Earlham College, May 2-3.
IOWA, Ames, Iowa, May 2-3.
KANSAS, February 15.
KENTUCKY, Lexington, Ky., April 5.
LOUISIANA-MISSISSIPPI, Cleveland, Miss., March 7-8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA.
MICHIGAN, Ann Arbor, Mich., March 22.
MINNESOTA, Carleton College, May 17.
MISSOURI.
NEBRASKA, Peru, Neb., May 9.
OHIO, Columbus, Ohio, April 3.
PHILADELPHIA, Philadelphia, Pa., November 29.
ROCKY MOUNTAIN, Denver, Colo., April 11-12.
SOUTHEASTERN, Atlanta, Ga., May 2-3.
SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, Calif., March 8.
TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
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(INCORPORATED)

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THE MATHEMATICAL ASSOCIATION OF AMERICA

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To Institutional Membership

NORTHEAST MISSOURI STATE TEACHERS COLLEGE, Kirksville, Mo.
UNIVERSITY OF KENTUCKY, Lexington, Ky.

FIFTEENTH ANNUAL MEETING OF THE OHIO SECTION

The fifteenth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 3, 1930, the day preceding the meetings of the Ohio College Association, with an afternoon session, dinner, and evening session. The Chairman, Professor S. A. Rowland, presided at both sessions.

Seventy-seven persons registered attendance, the largest attendance since ten years ago. Among these were the following fifty-two members of the Association: R B. Allen, L. C. Bagby, Grace M. Bareis, I. A. Barnett, P. E. Baur, H. M. Beatty, L. T. Black, H. Blumberg, M. G. Boyce, L.Brand, J. B. Brandeberry, C. T. Bumer, R. S. Burington, W. D. Cairns, F. E. Carr, R. Crane, W. Dancer, O. L. Dustheimer, P. S. Dwyer, T. M. Focke, B. C. Getchell, B. C. Glover, H. Hancock, R. C. Hildner, E. J. Hirschler, H. W. Kuhn, E. M. Justin, A. C. Ladner, Anna D. Lewis, C. C. MacDuffee, J. A. McLaughlin, Florentina Mathias, G. M. Merriman, C. N. Moore, C. C. Morris, J. R. Musselman, J. J. Nassau, R. L. Newlin, S. E. Rasor, G. Y. Rainich, P. L. Rea, B. H. Redditt, F. W. Reed, C. E. Rhodes, Hortense Rickard, S. A. Rowland, W. G. Simon, G. W. Spenceley, C. F. Thomas, M. O. Tripp, F. B. Wiley, C. O. Williamson.

The following officers were elected for the coming year: Chairman, W. G. Simon; Secretary-Treasurer, Rufus Crane; Member of Executive Committee, C. C. Morris; Member of Program Committee, F. E. Carr. It is expected that the next meeting will he held at the Ohio State University on Thursday, April 2, 1931.

The following ten papers were presented:

1. "The solutions of a system of linear homogeneous differential equations with Laurent coefficients" by the Chairman, Professor S. A. Rowland, Ohio Wesleyan University.

2. "Iterated integrals and difference equations" by Professor Louis Brand, University of Cincinnati.

3. "Some theoretical considerations of the solar motion as determined by ether drift" (illustrated) by Professor J. J. Nassau, Case School of Applied Science.

4. "A curve associated with a triangle" by Professor J. H. Weaver, Ohio State University.

5. "The solution of linear equations by means of a computing machine" by Professor G. W. Spenceley, Miami University.

6. "A topic in the theory of finite groups" by Professor B. H. Redditt, Kenyon College.

7. "The mathematics club, its function and its vogue" by Professor Mary E. Sinclair, Oberlin College; with discussion opened by Professor F. B. Wiley, Denison University.

8. "A non-technical survey course in mathematics" by Professor F. B. Wiley, Denison University.

9. "Radio lessons in arithmetic" by Professor Ida M. Baker, Western Reserve University (introduced by Professor Simon).

10. "Linear vector functions, their applications and generalizations" by Professor G. Y. Rainich, University of Michigan, (by invitation).

On account of the absence of Professor Sinclair, and of Professor Weaver at the time his paper was scheduled, the fourth paper was presented by Professor Crane and the seventh paper by Professor Carr.

Abstracts of these papers follow:

1. The method of successive integrations is used to establish a particular fundamental set of solutions of the system

$$x_i' = \sum_{j=1}^{n} \theta_{ij}(t) \cdot x_j \qquad (i = 1, 2, \cdots n),$$

in which the coefficients $\theta_{ij}(t)$ are expansible in Laurent series, and the fundamental equation is obtained. By using suitable transformations and making circuits about the origin, a canonical fundamental set of solutions is established

whose form depends upon the number and multiplicity of the multiple roots of the fundamental equation, and upon the rank of its matrix when a particular root is substituted in it. The functions in terms of which this canonical set is expressed are expansible in Laurent series whose coefficients, in turn, are expressed by means of iterated integrals involving the particular fundamental set first employed.

2. In this paper it is shown that certain n-fold iterated integrals over regions whose defining inequalities are symmetric in the variables, and whose integrands are likewise symmetric, may be evaluated by one quadrature and the solution of a difference equation. The method was exemplified by computing the volume of an n-dimensional sphere.

3. Some of the difficulties of the optical theory of the Michelson interferometer were first discussed. This included the development of formulae for the reflection of moving mirrors and their effect in producing interference. Also the study of the width of fringes was given, and the possible first order effect. The main part of the paper dealt with the determination of solar motion (Astrophysical Journal, March, 1927, Nassau and Morse) assuming the second order effect given. In closing, the difficulties involved in reconciling Professor Miller's observations with present theories were pointed out.

4. Professor Weaver reported on his investigation of the properties of a curve which is associated with a triangle in the following manner. Let there be a triangle $A_1A_2A_3$ and let a point P in the plane of the triangle satisfy the equation

$$\angle A_iPA_j = \angle(m\pi + kA_n) \qquad (i, j, n = 1, 2, 3; i \neq j \neq n).$$

Let $A_i = \pi/3 - A_i'$, $(i=1, 2, 3)$. If $|A_1'|:|A_2'|:|A_3'| = p_1:p_2:p_3$ (p_i relatively prime integers), the locus of P is an algebraic curve with multiple points of order p_i at the points A_i. If the p_i are not commensurable the curve will not be algebraic.

5. A Monroe computing machine is used. Transcription of numbers from machine to paper and back (a prolific source of error and delay) is reduced to the equivalent of recopying once the coefficients of the original set of equations. Checking at the end of each step is convenient The method indicated a combination of computing machines that would make the whole process automatic.

6. This paper consisted of a discussion of the groups of movements by which a regular solid may be brought into coincidence with itself, special attention being given to the principles involved in Klein's treatise on the Icosahedron. The ultimate purpose was a study of the group of degree six, and in this connection special use was made of the six lines joining the six pairs of opposite vertices of the icosahedron.

7. This paper presented a resumé of the college experiance with mathematics clubs, especially as reflected in the Monthly and as developed in Ohio. The purpose of the club was discussed, and the means used to accomplish results.

Mention was made of such points as the composition of clubs, frequency and character of meetings, social features, and the use of outside speakers. Serious undergraduate interest and effort are a real reward to the promoters of clubs.

8. This paper was in the nature of a report of a course given at Denison University, as an experiment, to non-mathematics students with the purpose of giving them some appreciation of the historical development and significance of mathematics, its philosophical background, and the logic it involves; also of showing, as far as non-technical methods allow, with what the different fields of mathematics are concerned. From the reactions of the students and the instructor of the course, Professor Wiley feels that courses of this general nature might well receive more attention in our colleges.

9. Is it possible to teach arithmetic by means of radio in such a manner that the learning process as well as the learning material is controlled and directed by the broadcaster? A year of experimentation in the public schools in Cleveland has proved that this is possible. This experiment also points out that radio lessons can be built so as to eliminate waste by securing concentration and by furnishing carefully planned lessons and drill material; to employ efficient teaching methods that create and develop power to reason and ability to compute; to include useful and interesting learning material. Radio has a distinct contribution to make to education.

10. The purpose of this paper was to point out the important role of the linear vector function as one of the unifying principles in mathematics. The idea was first introduced by several simple examples. Then the different forms in which the linear vector function appears (quadratic form, matrix, tensor, etc.) and the fundamental problems that arise were briefly sketched and applications to various fields of mathematics were mentioned. The applications to mathematical physics were given more in detail, and the development in that field was shown to be characterized by the increasing use of tensors. Finally, the conceptions of vector and linear vector function were generalized to a space of infinitely many dimensions where they appear as a function and a functional operator respectively, and where the same fundamental problems were shown to arise as in the finite case.

RUFUS CRANE, *Secretary*

INTEGRAL EQUATIONS AS A METHOD OF THEORETICAL PHYSICS.

By DIMITRY E. OLSHEVSKY, Yale University

If the beginning of the development of integral equations is identified with the work of Volterra and Fredholm in 1897 and 1903, respectively, this chapter of analysis is only one generation old. As a method of theoretical physics it is considerably younger, thus naturally lagging in development behind the older methods. As an unfortunate rule the presentation of integral equations is purely formal and corresponds to a point of view adapted from the methods of differential equations.

The popular opinion that any problem of integral equations can be reduced to a problem of differential equations is not only wrong, but neglects the fact that in most of those possible cases a simple integral equation reduces to a differential equation with a boundary problem. These boundary problems are, however, far away from the simplicity of the classical problems of differential equations.

A deeper study of integral equations reveals a method of supreme possibilities, great simplicity, and direct adaptability to physical problems. The auxiliary conditions are most naturally included in the integral equations. Thus the regions of both classical and boundary problems of ordinary differential equations are joined and included in the integral equations. Also, problems ordinarily formulated by different orders of differential equations are taken care of by the same standard type of integral equations. Problems of one or many independent variables can be treated from essentially the same point of view, thus bridging the domains of ordinary and partial differential equations. Also, as will be shown in this paper, the treatment of physical problems by means of integral equations can be regarded as a method quite distinct from that of the differential equations. The resulting possibilities of generalization beyond the scope of the methods of differential equations combined with adaptability of integral equations to the methods of successive approximation, and to graphical and numerical solution, raises the method of integral equations to one of the most hopeful achievements of *modern analysis.*

FORMAL CONSIDERATIONS

Definition: An equation involving unknown functions under the sign of integral operators is called an integral equation.

Classification: Integral equations are broadly classified according to the type of integral operators entering in the equations. An important case is represented by the linear integral operator of the first order:

$$L[y(x)] = \int_a^b K(x,\xi)y(\xi)d\xi.$$

The operator is distributive with respect to $y(x)$. There exists a great variety of integral equations, new types being suggested by analytical considerations as well as by actual problems of theoretical physics and other domains of science.

The linear integral equation: A linear integral equation involves only linear integral operators, as well as linearity with respect to the unknown function. For many reasons the linear integral equations have been so far the most extensively studied. Linear integral equations may be regarded as first approximations to integral equations involving more general forms of integral operators. Also the possibility of regarding them as limiting cases of systems of ordinary equations (linear algebraic), the theory of which is already highly developed, yields fundamental methods of solution.

The linear integral equations are classified according to the presence of:

(1) a known function, not under the sign of an integral operator;

(2) the unknown function, outside of the sign of the integral operator;

(3) the type of limits of the integral operator;

(4) the highest order of the integral operator.

Thus we have the following standard types of linear integral equations (with a single dependent and a single independent variable):

$$f(x) = \lambda \int_a^x (K(x,\xi)y(\xi)d\xi.$$

The Volterra equation of the first kind,

$$f(x) = \lambda \int_a^b K(x,\xi)y(\xi)d\xi.$$

The Fredholm equation of the first kind,

$$y(x) = \lambda \int_a^x K(x,\xi)y(\xi)d\xi.$$

The Volterra equation, homogeneous, of the second kind,

$$y(x) = \lambda \int_a^b K(x,\xi)y(\xi)d\xi.$$

The Fredholm equation, homogeneous, of the second kind,

$$y(x) = \lambda \int_a^x K(x,\xi)y(\xi)d\xi + f(x).$$

The Volterra equation, inhomogeneous, of the second kind,

$$y(x) = \lambda \int_a^b K(x,\xi)y\xi d\xi + f(x).$$

The Fredholm equation, inhomogeneous, of the second kind.
The equation,

$$h(x)\cdot y(x) = f(x) + \lambda \int_a^b K(x,\xi)y(\xi)d\xi,$$

sometimes called[1] a Fredholm equation of the third kind, may be reduced to the form,

$$y(x) = \frac{f(x)}{h(x)} + \lambda \int_a^b \frac{K(x,\xi)}{f(x)}\cdot y(\xi)d\xi,$$

which is the Fredholm equation of the second kind. In the case when $h(x)$ vanishes in the interval (a, b), generally (unless f and K do not vanish at the same points) singularities are introduced. Recently it has been proposed[2] to call this equation a singular Fredholm equation of the second kind.

Restrictions and concluding remarks: We assume the constants a, b, and the variables (x, ξ) to be real; also, $K(x, \xi)$ real and continuous for $a \leqq x, \xi \leqq b$. The known function $K(x, \xi)$ is called the kernel of the integral equation.

It may be noted that Volterra's equations can be considered as particular cases of Fredholm equations. It is sufficient to substitute in the latter the expression, $\theta\cdot K(x, \xi)$, where $\theta=1$ for $\xi \leqq x$, and $\theta=0$ for $\xi > x$, in the place of $K(x, \xi)$ to prove the statement. This standpoint brings however only formal advantages; the physical meaning of a Volterra equation is quite distinct.

RELATION OF INTEGRAL EQUATIONS TO THEORETICAL PHYSICS

General considerations: The subject will be approached by considering:

(a) Physical interpretation of integral equations.

(b) Mathematical interpretation, by means of integral equations, of certain physical problems.

The physical interpretation of integral equations: Physical interpretation of integral equations is closely connected with that of integral operators. This will be discussed in some detail.

Integral operators of the general form,

$$I_1(f) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} K(x_1x_2 \cdots x_n,\xi_1\xi_2 \cdots \xi_n)\cdot f(\xi_1\xi_2 \cdots \xi_n) \prod_{i=1}^{n} d\xi$$

have a definite and important meaning. Consider two points in an n dimensional space. Let us suppose that an observer, situated at the (variable) point $P(x_1x_2 \cdots x_n)$ studies the effect produced by a unit quantity (*the cause*) at another variable point $Q(\xi_1\xi_2 \cdots \xi_n)$. As a result of his observations he will obtain the function

$$G(1,P,Q) = K(x_1x_2 \cdots x_n\, \xi_1\xi_2 \cdots \xi_n),$$

---

[1] D. Hilbert, *Grundzüge einer allgemeinen Theorie d. Linearen Integralgleichungen*, 1912.

[2] H. T. Davis, Indiana University Studies, vol. 70 (1926).

which is the so-called Green's Function for the n-dimensional space. Now let us assume the validity of two well known physical principles: (1) the principle of the superposition of the effects due to several arbitrary values of the quantity acting at the same point,

$$G(c,P,Q) = c\cdot G(1,P,Q); \tag{1}$$

and (2) the principle of superposition of the effects of two "unit causes" acting at any two different points in our space, within the specified region at the point of observation. This may be mathematically expressed as follows:

$$G_1(1,P,Q_1;\ 1,P,Q_2) = G(1,P,Q_1) + G(1,P,Q_2). \tag{2}$$

The knowledge of (2), (3), (4) gives us the possibility of finding the effect at any point $(x_1x_2 \cdots x_n)$ due to the action of the quantity "cause," distributed with a known density;

$$f(\xi_1\xi_2 \cdots \xi_n) \tag{3}$$

in our "n"-dimensional space. The quantity comprised in an element of space at the point $(\xi_1\xi_2 \cdots \xi_n)$ will be represented by

$$c = f(\xi_1\xi_2 \cdots \xi_n) \prod_{i=1}^{n} d\xi_i \tag{4}$$

and the effect at the point $(x_1x_2 \cdots x_n)$ will be, according to (1),

$$G(c,P,Q) = K(x_1x_2 \cdots x_n;\xi_1\xi_2 \cdots \xi_n)\cdot f(\xi_1\xi_2 \cdots \xi_n) \prod_{i=1}^{n} d\xi_i.$$

(2) gives us the right to make a summation over all points of our specified region. The total effect at the point $(x_1x_2 \cdots x_n)$ will be represented by the integral operator:

$$E = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \cdot K(x_1x_2 \cdots x_n\ \xi_1\xi_2 \cdots \xi_n)\cdot f(\xi_1\xi_2 \cdots \xi_n) \prod_{i=1}^{n} d\xi_i. \tag{5}$$

It is a function of the x_i only. The auxiliary variables "ξ_i" "disappeared."

In the case of one-dimensional space, (5) represents the well known expression,

$$\int_a^b K(x,\xi)f(\xi)d\xi.$$

The solutions of a linear integral equation have thus the following physical meaning: they represent functions which (with or without addition of a known function) are proportional to the "effect" they produce when regarded themselves as "causes" in a space with properties described by the kernel. Thus the equation,

$$y(x) - f(x) = \lambda \int_a^b K(x,\xi)y(\xi)d\xi,$$

with proper form of the kernel, describes the vibrations of an un-uniform string under a general periodic loading. The deflection $y(x)$ of the string with the addition of a known function $f(x)$ (dependent on the kernel, loading and mass distribution of the string), is proportional to the integral operator. This integral operator,

$$\int_a^b K(x,\xi)y(\xi)\cdot d\xi$$

represents physically the effect produced by $y(x)$ in a one-dimensional space. The properties of this space are given the kernel $K(x, \xi)$. It may be noted here that the solution is found to exist only for certain discrete values of the proportionality factor λ. Since physically the proportionality factor represents in this case the square of the frequency, the problem may be interpreted as determination of proper values of energy for the vibrating system together with the proper functions describing the corresponding shapes of deflections of the string.

Mathematical interpretation of physical problems. The phenomenon of hysteresis: Let us start by considering the phenomenon of stretching of a wire. In the case of ordinary causal dependence, the strain, y would be expected to be a certain function of the stress p; let us say, $y=f(p)$. As a simple approximation we shall expect Hooke's law with all the advantages of simplicity in linear phenomena,

$$y = a\cdot p. \tag{6}$$

It was certainly a surprise to early investigators to find that the phenomenon simply cannot be described by such elementary mathematical methods. The stress strain "curves" were "not reproducible" with different specimens of the same material and also were dependent on the previous history in case of the same specimen.

Volterra[3] was first to realize that new, more powerful mathematical methods should be developed to describe phenomena with dependence on previous states. Later Painlevé pointed out that philosophically we are not forced to abandon the causality principle as such; it must be however abandoned formally when, as in phenomena of hysteresis, we reduce a problem of a great number of molecular variables (microscopic world) to a small number of macroscopical, "average" variables, capable of direct observation.

Assuming for simplicity the validity of the superposition principle,[4] the strain at time "t" will be a sum of two effects. First, the "classical" elongation due to the stress $p(t)$ according to (6); secondly, the sum of all the effects due to

[3] Clark University lectures, 1918; *Fonctions des lignes*, Paris, 1913.

[4] Dropping of this restriction will lead to integral equations involving multiple integrals.

previous existence of stress at the moments ξ_i running continuously from $-\infty$, up to the present moment, t. Each individual effect will generally depend upon both the time of its past activity, ξ, and the time of observation t. For unit stress each effect can be represented by the function, $K(\xi, t)$; and if the past values of stress were given by a known function, $p(\xi)$, the individual effects would be

$$K(\xi, t) \cdot p(\xi).$$

Now the sum of those individual effects will be represented by the integral,

$$\int_{-\infty}^{t} K(\xi, t) \cdot p(\xi);$$

and the strain under consideration is

$$y(t) = ap(t) + \int_{-\infty}^{t} K(\xi, t) \cdot p(\xi).$$

A linear integral equation is thus obtained when, physically, the variation of stress $p(t)$ with time must be obtained from observations on the variation of strain, $y(t)$.

Numerous other problems of vibrations, heat and hydrodynamics, optics and potential theory can be very naturally formulated and solved by integral equations. Their importance in the treatment of modern wave mechanics is now generally recognized.[5]

APPENDIX

Solution of Integral Equations with Bilinear Kernel.

Consider a Fredholm integral equation, inhomogeneous, of the second kind, with bilinear kernel:

$$y(x) = f(x) + \lambda \int_a^b K(x, \xi) y(\xi) d\xi, \tag{7}$$

$$K(x, \xi) = \sum_{i=1}^{n} p\,(x) \cdot q\,(\xi). \tag{8}$$

The integral equation is supposed to be regular, i.e., the kernel K is supposed to represent a continuous, bounded function, finite between (a, b); n may be finite or infinite; in the second case, however, the series (8) is supposed to be absolutely convergent.

Substituting (8) into (7) we obtain:

$$y(x) = f(x) + \lambda \int_a^b \sum p_i(x) \cdot q_i(\xi) \cdot y(\xi) d\xi \tag{9}$$

[5] C. G. Darwin, Proceedings of the Royal Society, vol. 117A, (1928), p. 267.

or

$$y(x) = f(x) + \lambda \sum p_i(x) \cdot Q_i,$$

where

$$Q_i = \int_a^b q_i(\xi) \cdot y(\xi) d\xi$$

are certain (unknown) coefficients.

The problem of solving the equation (7) is thus reduced to the determination of the coefficients Q_i.

Referring to (9) we see that

$$y(\xi) = f(\xi) + \lambda \sum p_i(\xi) \cdot Q_i; \tag{10}$$

for Q_i are constants. Now in the expression for Q_i,

$$Q_i = \int_a^b q_i(\xi) \cdot y(\xi) \cdot d\xi,$$

make the substitution (10).

This yields a set of equations:

$$Q_i = \int_a^b q_i(\xi) \cdot f(\xi) \cdot d\xi + \lambda \sum_{k=1}^{n} Q_k \int_a^b p_k(\xi) \cdot q_i(\xi) d\xi$$

or

$$Q_i = B_i + \lambda \sum Q_k \cdot A_{ik}, \quad \text{where} \quad B_i = \int_a^b q_i(\xi) \cdot f(\xi) d\xi, \tag{11}$$

$$A_{ik} = \int_a^b p_k(\xi) \cdot q_k(\xi) \cdot d\xi.$$

The reader will see that (11) represents a set of linear equations with reference to Q_i. It may be written in the form:

$$\begin{gathered}
Q_1(1 - \lambda A_{11}) - \lambda A_{12} Q_2 - \cdots - \lambda A_{1n} Q_n = B_1 \\
- \lambda A_{21} Q_1 + (1 - \lambda A_{22}) Q_2 - \cdots - \lambda A_{2n} Q_n = B_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
- \lambda A_{n1} Q_1 - \lambda A_{n2} Q_2 - \cdots + (1 - \lambda A_{nn}) Q_n = B_n .
\end{gathered}$$

The solution is $Q_p = \Delta_p / \Delta$,
where, as usual,

$$\Delta = \begin{vmatrix} (1 - \lambda A_{11}) & - \lambda A_{12} & \cdots & - \lambda A_{1n} \\ - \lambda A_{21} & (1 - \lambda A_{22}) & \cdots & - \lambda A_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ - \lambda A_{n1} & - \lambda A_{n2} & \cdots & (1 - \lambda A_{nn}) \end{vmatrix} \tag{12}$$

and

$$\Delta p = \begin{vmatrix} (1-\lambda A_{11}) & -\lambda A_{12} & \cdots & -\lambda A_{1,p-1} & B_1 & -\lambda A_{1,p+1} & \cdots & -\lambda A_{1n} \\ -\lambda A_{21} & (1-\lambda A_{22}) & \cdots & -\lambda A_{2,p-1} & B_2 & -\lambda A_{2,p+1} & \cdots & -\lambda A_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\lambda A_{n1} & -\lambda A_{n2} & \cdots & -\lambda A_{n,p-1} & B_n & -\lambda A_{n,p+1} & \cdots & (1-\lambda A_{nn}) \end{vmatrix}$$

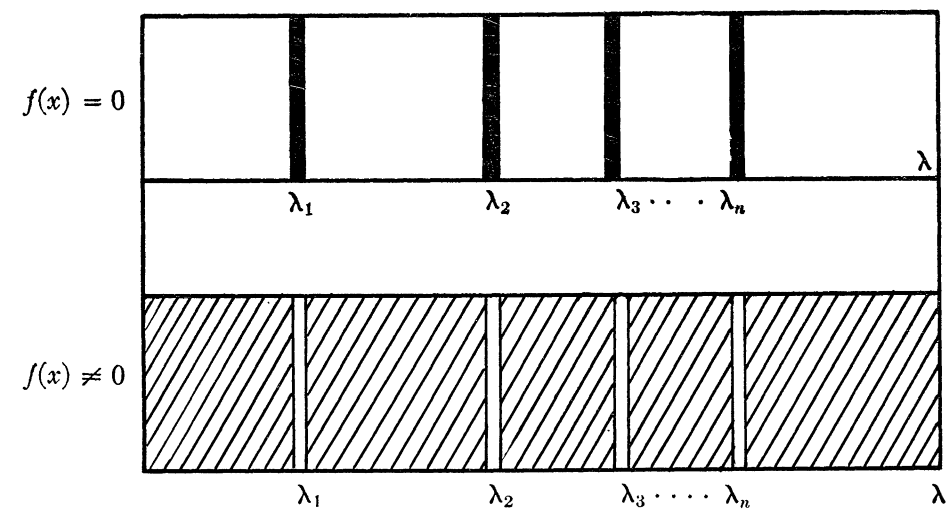
We observe that the question of the existence of a solution of an integral equation becomes thus closely related to the problem of the existence of a solution of a system of linear equations. The system is infinite if this happens to be the case with "n."

We observe, further, that the determinant of the system, (12), depends on the kernel and the parameter, λ, only.

When the determinant of the system is equal to zero—this can be expected for not more than n different values of the parameter, λ,—there exists no solution for any $f(x)=0$ and n solutions (some of them identical) in the case when $f(x)=0$. The equation has thus a line spectrum of solutions, corresponding to the roots of (12). When the determinant (12) is not equal to zero, a continuous set of solutions exists for any function, f, not equal to 0. This band of solutions includes the trivial solution,

$$y(x) = 0 \quad \text{for} \quad f(x) = 0.$$

If we denote by a line perpendicular to the λ-axis the existence of a solution, the results of this discussion may be represented by the following graphical scheme:



EVALUATION OF THE MODIFIED BESSEL FUNCTION OF THE FIRST KIND AND ZEROTH ORDER[1]

C. C. FURNAS[2]

A Bessel function is a solution of the differential equation,

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}+\left(1-\frac{n^2}{x^2}\right)u = 0. \tag{1}$$

For real, positive integers the solution of this equation is usually represented by the symbol $J_n(x)$, where n stands for the "order" of the solution. The formula of this integral is

$$J_n(x) = \frac{x^n}{2^n\cdot n!}\left(1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2\cdot 4(2n+2)(2n+4)}-\cdots\right). \tag{2}$$

The values of this integral have been determined for wide ranges of n and x and may be found in mathematical tables.

If the argument is imaginary, then the substitution $x=it$, where $i=\sqrt{(-1)}$, transforms Bessel's equation into

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}-(n^2+t^2)y = 0. \tag{3}$$

One solution of this equation, for positive integral values of n, is

$$J_n(it) = i^n\frac{t^n}{2^n\Pi(n)}\left(1+\frac{t^2}{2(2n+2)}+\frac{t^4}{2\cdot 4(2n+2)(2n+4)}+\cdots\right). \tag{4}$$

Π represents the Gauss function and is equal to the factorial for integral values of n.[3]

The function ordinarily used, however, is

$$I_n(t) = i^{-n}J_n(it) = \sum_{s=0}^{\infty}\frac{1}{\Pi(s)\Pi(n+s)}\left(\frac{t}{2}\right)^{n+2s} \tag{5}$$

which is known as the modified Bessel function of the first kind.

If $n=0$,

$$I_0(t) = \sum_{s=0}^{\infty}\frac{1}{[\Pi(s)]^2}\left(\frac{t}{2}\right)^{2s}. \tag{6}$$

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[1] Published by permission of the Director of the U. S. Bureau of Mines. (Not subject to copyright.)

[2] Physical chemist, North Central Experiment Station, Bureau of Mines, Minneapolis, Minn.

[3] Gray, Andrew, Mathews, G. B., and MacRobert, T. M., *A Treatise on Bessel Functions*, Second Edition, Macmillan & Co., 1922, page 254.

This is the modified Bessel function of the first kind and zeroth order and is very important in the study of certain types of heat transfer. However, the value of the function increases very rapidly with increasing values of t and as far as the author knows has never been computed for values[4,5] greater than $t=11.0$ because of the very large numbers involved.

It so happens in this case that there is an application which furnishes a means of empirical evaluation for any values of t and to any desired degree of accuracy.

Consider a fluid stream flowing uniformly through a porous prism of broken solid particles when the initial constant temperature of the solid phase is different from the initial constant temperature of the liquid phase. Heat will be transferred from the liquid to the solid until the solid particles have acquired the temperature of the liquid. The equations that determine the temperature history of any point in the fluid or solid system have been developed and may be found in the literature.[6]

From paper just referred to,

$$\frac{\partial T_s}{\partial z} = T_g - T_s, \tag{7}$$

$$\frac{\partial T_g}{\partial y} = T_s - T_g, \tag{8}$$

$$T_s = T_0(U - V)e^{-y-z}, \tag{9}$$

$$T_g = T_0(U + V)e^{-y-z}, \tag{10}$$

$$V = \tfrac{1}{2}I_0(2i\sqrt{yz}), \tag{11}$$

where T_s = temperature of the solid at any time,
T_g = temperature of the fluid at any time,
T_0 = initial temperature of fluid,
y = a function of position in the prism,
z = a function of the time.

From the above equations it can be shown by simple algebra that

$$\frac{T_g - T_s}{T_0} = 2Ve^{-y-z} = I_0(2\sqrt{yz})e^{-y-z}. \tag{12}$$

From equations (7) and (12), if y is constant, and if initial temperature of solid equals 0,

$$\int_0^\infty \frac{\partial T_s}{T_0} = \int_0^\infty I_0(2\sqrt{yz})e^{-y-z}dz. \tag{13}$$

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[4] Reference 3, page 309.

[5] Jahnke, E., and Emde, F., *Funktiontafeln mit Formeln und Kurven*, B. G. Teubner, Berlin, 1909, page 130.

[6] Schumann, T. E. W., *Heat Transfer: A Liquid Flowing Through a Porous Prism:* Journal of the Franklin Institute, *208*, (1929), p. 405.

Equation (13) is the key to the solution of values of $I_0\,(2\sqrt{yz})$ *for the integral of* $\partial T_s/T_0$ *from zero time to infinite time must be equal to unity* for the total possible change in solid temperature, T_s, is equal to T_0. Therefore

$$(14) \qquad \int_0^{\infty} I_0(2\sqrt{yz})e^{-y-z}dz = 1.$$

A solution may be assumed for the Bessel function, and then its validity may be tested by the above criterion. The work may be done graphically by measuring the area under the curve of $I_0\,(2\,\sqrt{yz})\,e^{-y-z}$ plotted against z. If the assumed solution is correct, the area will be unity. The value of this function approaches zero so rapidly for large values of z that for practical purposes it is

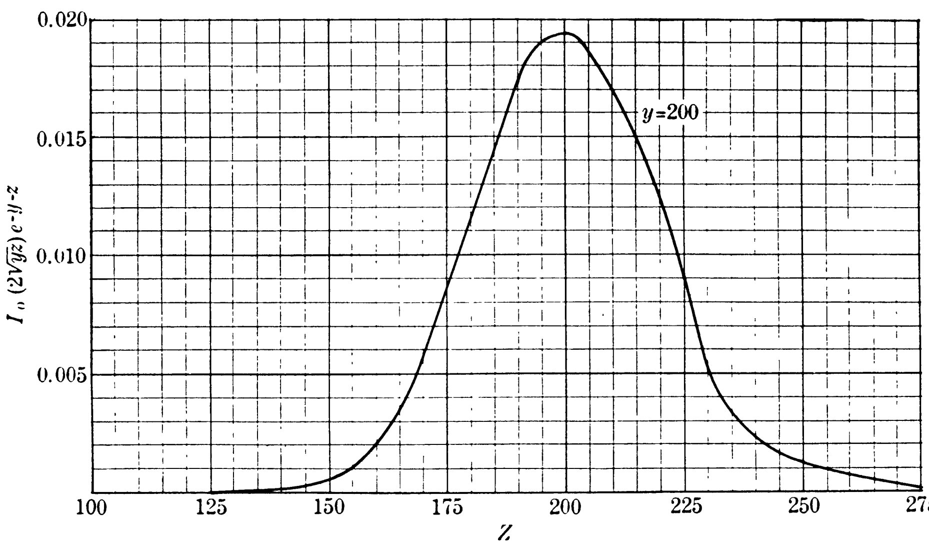


Figure 1.- Typical curve of the function $I_0(2\sqrt{yz})\,e^{-y-z}$ against Z. Second approximation.

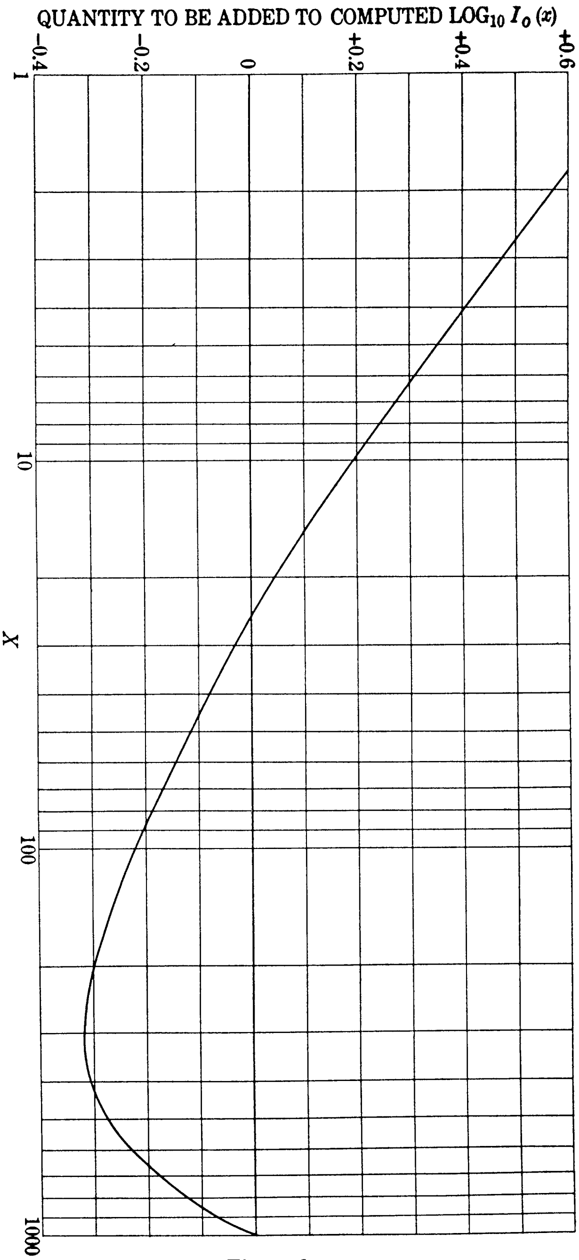
not necessary to consider the function for large values of z. The rapid convergence of the function is shown by the curves of Figure 1. Since the function of e^{-y-x} can be evaluated to any required degree of accuracy, approximate values of $I_0(2\sqrt{yz})$ can be obtained when the trial solution has been shown to be approximately correct. If a planimeter is used in measuring areas, the degree of accuracy is limited by the possible experimental error in the use of the instrument. In the present instance the largest error was 0.4 per cent. This was taken as the limiting accuracy of the determination. Greater accuracy could be obtained by using larger areas.

In the solution, y of equation (14) was considered constant and the integral evaluated for the specified limits of z.

For simplicity it will be considered that

$$2\sqrt{yz} = x. \tag{15}$$

For values of x greater than 10, $I_0(x)$ is closely approximated by the simple formula,



First correction. Quantity to be added to $\log_{10} I_0(x)$ as computed by equation 16.

Figure 2

$$\log I_0(x) = ax + b.$$

The first trial solution, obtained graphically, is

$$\log_{10} I_0(x) = 0.433490x - 1.0840. \tag{16}$$

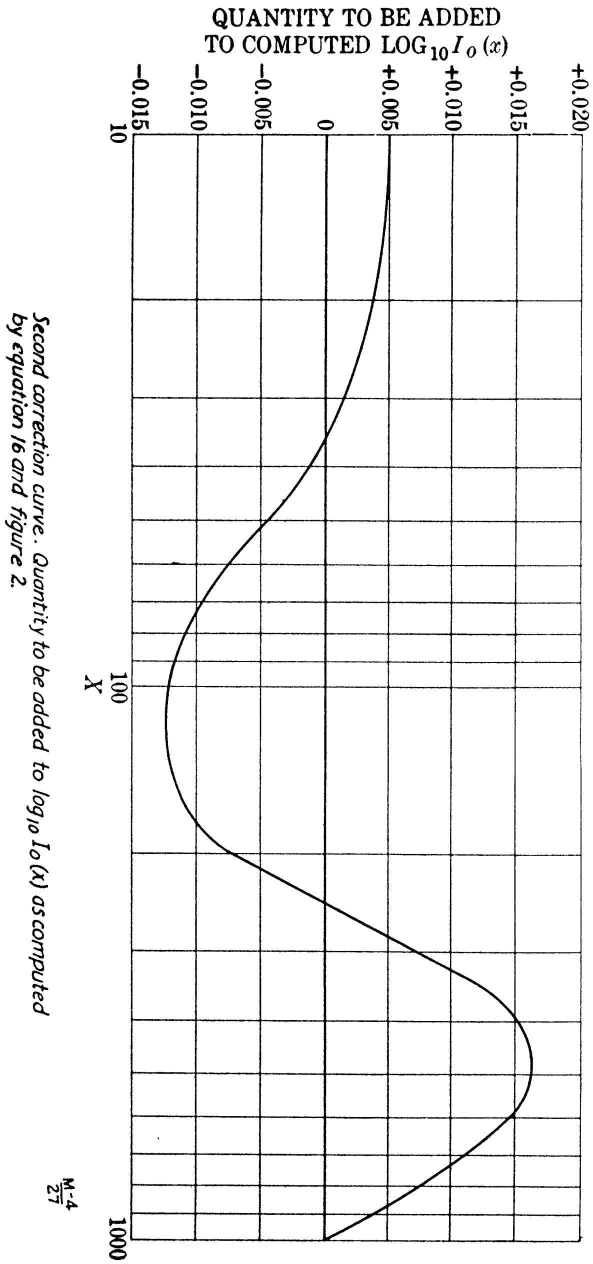


Figure 3

This solution has been tested for error by the criterion given above and the quantity to be subtracted from the computed value of $\log_{10} I_0(x)$ for an accurate second approximation is shown in Figure 2.

The correction terms represent the cumulative error in the integral from zero to infinity for the particular value of y considered. As an aid in operation it was considered that this error all occurs at the maximum point of the curve. This maximum comes at a point where z is approximately equal to y (y, constant). Therefore the correction term for each value of y is applied to the assumed solution when $x = 2y$.

It was found that it was possible to obtain a second correction curve (see Figure 3) within the prescribed range of accuracy thus giving a third approximation.

The computed values of $I_0(x)$ for values of x up to 1,000 are given in Table 1.

TABLE 1.
Values of the Modified Bessel Function $I_0(x)$
(The last significant figure is doubtful)

x	$I_0(x)$	x	$I_0(x)$
12	1.901×10^{4}	90	5.177×10^{37}
14	1.285×10^{5}	100	1.084×10^{42}
16	8.867×10^{5}	200	2.037×10^{85}
18	6.107×10^{6}	300	4.467×10^{128}
20	4.274×10^{7}	400	1.052×10^{172}
30	7.757×10^{11}	500	2.541×10^{215}
40	1.498×10^{16}	600	6.295×10^{258}
50	2.955×10^{20}	700	1.545×10^{302}
60	5.948×10^{24}	800	3.846×10^{345}
70	1.211×10^{29}	900	9.840×10^{388}
80	2.496×10^{33}	1000	2.547×10^{432}

LARGE-NUMBER DIVISION BY CALCULATING MACHINE.[1]

By HARRY H. LAUGHLIN[2]

In many laboratories which use calculating machines, there frequently arises the necessity for the division of numbers greater than the direct capacity of the particular available machine. Thus far no accurate and practical machine-method for doing this has been worked out. In making such computations therefore, recourse is had to the usual and laborious "hand-method."

[1] A new and practical method of accurate division in case the dividend and the divisor each comprises any number of digits beyond the direct machine-capacity.

[2] Eugenics Record Office, Carnegie Institution of Washington, Cold Spring Harbor, L. I., N. Y.

In the development of a useful machine-method for such division the following specifications must be fulfilled:

1. The method must be accurate to any desired degree, which degree can be readily provided for and judged.

2. The method must accommodate dividends and divisors of any number of digits.

3. The principle must be direct and simple.

4. Processes of trial and error, other than as in ordinary division to determine the next digit of the quotient, must not be required. That is, more complicated processes of trial and error involving at one time a considerable block of the whole process must not be used.

5. The process must be such that the probability of making mistakes will be relatively low.

6. The amount of time and labor required must represent a great saving compared with the usual "hand-method."

Division: Case A.

The division of large numbers by the calculating machine falls logically into two cases. The first of these cases is presented as follows.

Case A: Division by calculating machine in problems in which the dividend (significant digits) consists of any number of digits *greater* than the direct machine-capacity, and the divisor (significant digits) is *within* the direct machine-capacity.

1. *Method: Dividend Sectioning.*

2. *Principle:* Based on the fact that in the dividend the left-hand digits are first used up, while the right-hand digits (still beyond the number of digits of the divisor plus those consumed from the left of the dividend) are not yet reached. Consequently, at any stage before the first digit to the right of the particular dividend-block is required, the division thus far is correct; when carried to such a stage in the first dividend-block the quotient can be set down so far; the remainder of the first block of the dividend moved to the left; and additional digits of the original dividend added to the remainder's right. This process can be repeated block by block as the divisor moves to the right.

By this method a dividend of any number of digits can be handled by a machine of any direct dividend-capacity and the quotient can be carried to any desired number of digits.

3. *Procedure.*

(1) Set up on carriage the left-hand digits of the dividend, to the machine-capacity.

(2) Work through with the complete divisor as far as possible, *i. e.*, to the capacity of the machine.

(3) Set down on paper the digits of the quotient thus far found. Add, to the right of such quotient, ciphers equal to the number of digits of the divisor minus number of digits of the remainder.

(4) Set up on key-board the digits of the unused remainder of the dividend as shown on the carriage. Clear carriage.

(5) Set up on the dividend carriage, to the extreme left, the remainder (4); and add on the next block of unused digits of the original dividend, until the dividend-capacity is again reached.

(6) Continue the division with the original complete divisor.

(7) Add on the new digits of the quotient to the right of the partial quotient (3) already found. Caution: If in any sectional division following the first, the first digit of the dividend $<$ the first digit of the divisor, a zero will appear as the first digit of the quotient. Such an initial zero should be ignored, *i e.*, not included in the quotient.

(8) Continue this "dividend sectioning" process until the quotient is carried to the required number of digits.

(9) Point off decimal places in the quotient by the usual rule.

4. *Example*: On a calculating machine with a dividend-capacity of 15 digits, a divisor-capacity of 8 digits, and a quotient-capacity of 8 digits divide

493, 725, 812, 976, 932, 814 by 85, 675, 492,

correct to the number of digits necessary to prove the work by restoring the dividend, by multiplying the quotient by the divisor, such product being correct to the number of digits in the original dividend—in this case 18.

First sectional division.

Dividend = 493, 725, 812, 976, 932 = first 15 digits of the whole dividend.
Divisor = 85, 675, 492 = whole divisor.
Remainder = 56, 857, 868.

Second sectional division.

Dividend = 56, 857, 868, 814 = remainder of first sectional division plus added unused digits of the original whole dividend.
Divisor = 85, 675, 492 = whole divisor.
Remainder = 53, 845, 868.

Third sectional division.

Dividend = 53,845,868 = remainder of second sectional division plus added unused digits (if there be any) of the original whole dividend.
Divisor = 85, 675, 492 = whole divisor.
Remainder = 35, 997, 896.

Fourth sectional division · · · and so on.

Quotient = 5, 762, 742,	663.642,1	62, 85 x, xx
	Digits from 1st. sectional division.	Digits from 2nd. sectional division.	Digits from 3rd. sectional division.

Carry to 8 decimal places. There being 8 integral digits in the divisor, the error in the product (quotient $\times$ divisor = dividend) is less than 1, as demanded.

5. *Proof*: Quotient × divisor
   = 5, 762, 742, 663. 642, 162, 85 × 85, 675, 492
   = 493, 725, 812, 976, 932, 814.117, 872, 20
   = the original dividend with an error <1. Q E. I.

Division: Case B.

The second type, here referred to as *Case B*, comprises those problems in which the dividend may be either greater or less than the direct machine-capacity, but in which the divisor comprises any number of digits greater than the direct divisor-capacity of the machine. The problem in which both the dividend, "a," and the divisor "$b+c$" are greater than the direct machine-capacity presents the greatest difficulty in machine-division. The method here worked out consists in converting the problem from one of division into one of multiplication. It requires finding the reciprocal of the final divisor which is first transformed so that such reciprocal can be found without using a divisor larger than the direct divisor-capacity of the machine. This transformation of the process from division to multiplication[1] is the essential feature of the present method, and it enables the computer, with accuracy and ease, to carry the quotient to any desired number of digits.

One significant feature of the present method consists in the fact that no matter how many digits there may be in the divisor it is only necessary to separate the divisor into two sections—section "b" comprising the left-hand digits of the divisor to the direct divisor-capacity of the machine, and section "c" the remainder of the divisor-digits regardless of their number.

The method, principle, procedure, example, and proof of this type of machine-division are as follows:

Case B. Division by calculating machine in problems in which the dividend (significant digits) consists of any number of digits *greater than* the direct machine-capacity, and the divisor (significant digits) consists of any number of digits *greater than* the direct machine-capacity.

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[1] *Note:* Because the simplified method of division of large quantities by the use of the calculating machine as presented in this article involves the multiplication of large numbers, the following well-known method of multiplying numbers in excess of the direct machine-capacity for multiplication is here presented:

Principle: $(a+b\cdots+n)\times(p+q\cdots+n')=ap+aq\cdots+an'+bp+bq\cdots+bn'+np+nq\cdots+nn'$

Example: Multiply 394,783,521,743 by 831,624,631,955 on a machine with a multiplication-capacity of 6 digits by 6 digits.

a = 394,783,*xxx,xxx* $\quad$ p = 831,624,*xxx,xxx*
b = 521,743 $\quad$ q = 631,955

ap =	328,311,017,592,000,000,000,000
aq =	249,485,090,765,000,000
bp =	433,894,000,632,000,000
bq =	329,718,097,565
Product =	328,311,700,971,421,115,097,565

1. *Method:* Transformation to multiplication by divisor-reciprocal.

2. *Principle*: Based on the principle that if in *all* required divisions the divisor will fit on the key board (i.e., is within the divisor-capacity) the computation can be completed on the particular machine.

The task then is to find a method which will call for no divisions not within the direct divisor-capacity of the machine.

$$(1)\qquad \frac{a}{(b+c)} = \frac{a/b}{1+(c/b)} = \left(\frac{a}{b}\right)\cdot\left(\frac{1}{1+(c/b)}\right)$$

$$(2)\qquad \frac{1}{1+(c/b)} = \left[1-\left(\frac{c}{b}\right)+\left(\frac{c}{b}\right)^2-\left(\frac{c}{b}\right)^3\cdots\pm\left(\frac{c}{b}\right)^\infty\right].$$

(3) Substituting (2) in (1),

$$\frac{a}{b+c} = \left(\frac{a}{b}\right)\cdot\left[1-\left(\frac{c}{b}\right)+\left(\frac{c}{b}\right)^2-\left(\frac{c}{b}\right)^3\cdots\pm\left(\frac{c}{b}\right)^\infty\right].$$

(3) calls for no divisor greater than b, i.e., the divisor-capacity. It being previously demonstrated (see footnote 1) that any two numbers regardless of size can be multiplied by a calculating machine of any limited capacity, and that (by *Case A*) a machine can divide any dividend howsoever large, by any divisor which is within the divisor-capacity of the particular machine, therefore the present computation can be readily carried out on any machine.

3. *Procedure*:

(1) Separate the divisor into two sections, "b" and "c".

(2) Let the first section "b" consist of the left-hand digits to the divisor-capacity of the machine (or fewer if more convenient), followed by ciphers equal to the number of remaining integral divisor-digits.

(3) Let the second section "c," consist of the remaining digits of the divisor so that the entire divisor equals $b+c$.

(4) If the particular section of the divisor be a decimal then of course the significant digits of such section should be preceded (instead of followed) by the correct number of ciphers, and a decimal point.

4. *Example*: On a calculating machine with a dividend-capacity of 15 digits, a divisor-capacity of 8 digits, and a quotient-capacity of 8 digits, divide

493, 725, 812, 976, 932, 814 by 856, 754, 923, 849

correct to the number of digits necessary to prove the work by restoring the dividend, by multiplying the quotient by the divisor, such product being correct to the number of digits in the original dividend,—in this case 18.

Let $a = 493, 725, 812, 976, 932, 814$
$b = \quad 856, 754, 92x, xxx$
$c = \quad 3{,}849$

(1) $(a/b) = 576, 274.\ 26 \underset{l}{|} 6, 364, 216, 2 \underset{m}{|} 85, x.$ $\quad n$

Divide by the *Case A* method. Carry to 12 decimal places (i.e., the number of integral digits in the divisor), so that, when "quotient × divisor = dividend" is found, the error, as required, will be less than 1. Section into 8-digit places for multiplication.

$$\text{Find } [1-(c/b)+(c/b)^2-(c/b)^3 \cdots].$$

(2) $(c/b)=.000{,}000{,}004{,}492{,}533{,}291+$

Divide by the *Case A* method. Carry to 18 decimal digits, i.e., the number of integral and decimal digits in (a/b).

(3) $(c/b)^2=.000{,}000{,}000{,}000{,}000{,}020+$

Use enough left-hand significant digits of (c/b) to make $(c/b)^2$ correct to 18 decimal digits.

(4) $(c/b)^3=\ldots\ldots\ldots\ldots\ldots,$

beyond range of accuracy required in present computation. Therefore neither $(c/b)^3$ or any higher power of (c/b) in this series is required.

(5) $1-(c/b)=.999{,}999{,}995{,}507{,}466{,}709$

(6) $1-(c/b)+(c/b)^2=.\underbrace{999{,}999{,}99}_{r}|\underbrace{5{,}\ 507{,}466{,}\ 7}_{s}|\underbrace{29}_{t}$

Section into 8-digit blocks for multiplication.

In $a/(b+c)=(a/b)\cdot[1-(c/b)+(c/b)^2-(c/b)^3\cdots\pm(c/b)^\infty]$ find quotient, Q. Σ cross-products of blocks l, m, n, in (a/b), and of blocks r, s, t, in

$$\left[1-\left(\frac{c}{l}\right)+\left(\frac{c}{b}\right)^2-\left(\frac{c}{b}\right)^3\cdots\pm\left(\frac{c}{b}\right)^\infty\right]$$

equals Q.

l×r=	576,274·254,237,257,400,*xxx*,*xxx*
l×s=	·003,173,811,297,*xxx*,*xxx*
l×t=	·000,000,000,017,*xxx*,*xxx*
m×r=	·006,364,216,136,*xxx*,*xxx*
m×s=	·000,000,000,035,*xxx*,*xxx*
m×t=	·000,000,000,000,*xxx*,*xxx*
n×r=	·000,000,000,085,*xxx*,*xxx*
n×s=	·000,000,000,000,*xxx*,*xxx*
n×t=	·000,000,000,000,*xxx*,*xxx*

Quotient = Σ = 576,274.263,775,284,970,

which is correct to 12 decimal places. Cutting off right-hand digits, beyond 12 decimal places, limits the accuracy of Q to 12 decimal digits, and therefore, limits the accuracy to the restored dividend ($Q\times$divisor), since there are 12 integral digits in the divisor, to $12-12=0$ decimal places, i.e., to <1 unit.

5. *Proof*: Quotient × divisor
= 576,274.263,775,284,970 × 856,754,923,849.
= 493,725,812,976,932,813.720,624,249,530
= the original dividend, with an error < 1. Q.E.I.

Systematic Arrangement of Work

Sheets or cards printed or drawn in the following manner[1] enable the computer to arrange the work of division of large numbers in an orderly and convenient fashion:

To divide a dividend by a divisor, each too large for direct machine-capacity

Each ruled space is for three digits.

The first block of a should comprise digits equal to the dividend-capacity of the machine. Subsequent blocks of a generally will comprise digits equal to the divisor-capacity, but in special cases will comprise more. (See (5) under *Case A*, *Procedure.*)

Block b should comprise digits equal to the divisor-capacity of the machine; block c the remainder of the divisor.

Blocks l, m, n, and o; and r, s, t, and u each comprise digits equal to the multiplier-capacity of the machine.

Divide			a								
by		b			c						
(1) $\left(\frac{a}{b}\right) =$		l		m			n			o	
(2) $\left(\frac{c}{b}\right) =$											
(3) $\left(\frac{c}{b}\right)\cdot =$											
(4) (2)+(3)=											
(5) $\left(\frac{c}{b}\right)^2 =$											
(6) (4)−(5)=											
(7) 1−(6)=		r		s			t			u	
1. $r\times l =$											
2. $r\times m =$											
3. $r\times n =$											
4. $r\times o =$											
5. $s\times l =$											
6. $s\times m =$											
7. $s\times n =$											
8. $s\times o =$											
9. $t\times l =$											
10. $t\times m =$											
11. $t\times n =$											
12. $t\times o =$											
13. $u\times l =$											
14. $u\times m =$											
15. $u\times n =$											
16. $u\times o =$											
Quotient $=\sum$ 1 to 16=											

Error in $Q<\left(\frac{c}{b}\right)^{n+1}\cdot\left(\frac{a}{b}\right) =$

$n+1$ = next unused power of $\left(\frac{c}{b}\right)$.

[1] This particular sheet is ruled for a machine-capacity of 15 dividend digits and 8 divisor digits. In practice, sheets 7 by 9 inches in size have proved generally satisfactory.

THE p DIMENSIONAL ANALOGUE OF SMITH'S DETERMINANT

D. H. LEHMER, Brown University

Discussions of determinants of more than three dimensions are, as a rule, necessarily beset with such typographical difficulties that they are quite formidable to the casual reader. It is possible however, to give a simple treatment of certain determinants of any number, p, of dimensions. The determinant discussed in this note is the outgrowth of the remarkable plane determinant of H. J. S. Smith

$$(1) \qquad |a_{ij}| = |F(i,j)| = \prod_{\nu=1}^{n} f(\nu)$$

where $F(i,j)$ is an arbitrary function of the greatest common divisor (i,j) of i and j, and where

$$(2) \qquad \sum_{\delta/n} f(\delta) = F(n),$$

the summation, as indicated, extending over the divisors δ of n. According to the well known Dedekind inversion,

$$(3) \qquad f(n) = \sum_{\delta/n} F(\delta)\mu(n/\delta),$$

where μ is the familiar Merten's inversion function.[1] This plane determinant has been the subject of a number of papers by Smith, Cesaro, Mansion, Catalan and others.[2] Gegenbauer[3] has considered some highly general determinants whose elements are complicated functions of a general set of integers. He states that Smith's determinant is a very special case of one of these. It is doubtless true that at least one of Gegenbauer's determinants can be sufficiently specialized to produce the determinant discussed below. After several genuine efforts the present writer admits his inability to perform the necessary simplifications. Inasmuch as the result of the present note is simple and the proof not more difficult than that given by Cesaro in the plane case, it would seem of interest to bring to light the simplest extension of Smith's determinant to p dimensions.

Consider the determinant $\Delta_n^{(p)}$ of dimensionality or class p and order n whose general element is

$$(4) \qquad a_{i_1 i_2 \cdots i_p} = F(i_1, i_2, \cdots, i_p),$$

where F is an arbitrary function of the greatest common divisor $(i_1, i_2, \cdots, i_p)$ of the indices $i_1, i_2, \cdots, i_p$.

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[1] Dickson, *History of the Theory of Numbers*, Vol. 1, Ch. 19.

[2] Dickson, loc. cit., Chap. 5.

[3] Sitzungsberichte, Akademie Wien, IIa, vol. 101 (1892), pp. 425–84, See also Lecat, *Leçons sur la Théorie des Déterminants à n Dimensions*, Gand, 1910, Book 2, Chap. 3.

Theorem: The value of $\Delta_n{}^{(p)}$ *is independent of* p *and is*

$$\prod_{\nu=1}^{n} f(\nu) \quad where \quad \sum_{\delta/n} f(\delta) = F(n).$$

Proof: If we fix one of the variables, say i_p, at a certain value Δ, then $a_{i_1 i_2 \cdots i_{p-1}\Delta}$ will range over a matrix S_Δ of class $p-1$ and order n. Let us add to each element of the matrix $S_n(\Delta=n)$ a linear combination of the corresponding elements of the matrices S_δ where δ ranges over the divisors (less than n) of n. This will not alter the value of the determinant.[4] Let us choose that linear combination in which all the elements of S_Δ are multiplied by $\mu(n/\Delta)$. Then the elements of the matrix S_n become

$$\sum_{\delta/n} F(i_1, i_2, \cdots, i_{p-1}, \delta)\mu(n/\delta). \tag{5}$$

To evaluate this sum let us consider the function $\epsilon(x)$ which is 1 or 0 according as x is or is not an integer. Then in view of (2)

$$\sum_{\delta/n} f(\delta)\epsilon(\nu/\delta) = F(n, \nu). \tag{6}$$

In fact the only terms which contribute to the sum are those for which δ divides ν as well as n. Inverting (6) we get

$$\sum_{\delta/n} F(\nu, \delta)\mu(n/\delta) = f(n)\epsilon(\nu/n). \tag{7}$$

To obtain (5) we set $\nu=(i_1, i_2, \cdots, i_{p-1})$ and since $((a, b), c)=(a, b, c)$ we have

$$\sum_{\delta/n} F(i_1, i_2, \cdots, i_{p-1}, \delta)\mu(n/\delta) = f(n)\epsilon\left(\frac{(i_1, i_2, \cdots, i_{p-1})}{n}\right).$$

This shows that all the elements of S_n are zero except the corner element for which $i_1=i_2=\cdots=i_{p-1}=i_p=n$, whose value is $f(n)$. Expanding the determinant by minors over the elements of S_n and noting that the cofactor of the corner element is precisely $\Delta_{n-1}{}^{(p)}$, we have

$$\Delta_n^{(p)} = f(n)\Delta_{n-1}^{(p)}.$$

But $\Delta_1{}^{(p)} = F(1) = f(1)$. Consequently

$$\Delta_n^{(p)} = \prod_{\nu=1}^{n} f(\nu),$$

which is the theorem.

Gegenbauer's method consists in factoring the given determinant of class $r+s-2$ into the product of two known determinants of class r and s. His

[4] See for example, Rice, Amer. Journal of Mathematics, vol. 40 (1918), p. 247. In case p is odd certain precautions are necessary in the general determinant on account of asymmetry. This does not affect the argument, however, because our determinant is p-way symmetric.

method is considerably simplified, at least in the present case, if we let $r=2$ and $s=p$. Our determinant can then be shown to be the product of the following two determinants:

$$\Delta_n^{(p)} = \left| f(j)\epsilon\left(\frac{i}{j}\right)\right|_n \cdot \left|\epsilon\left(\frac{i_1, i_2, \cdots, i_{p-1})}{i_p}\right)\right|_n = D_1 \cdot D_2.$$

One finds without difficulty that $D_2 = 1$ and $D_1 = \Pi_{\nu=1}^n f(\nu)$. To carry out the details of this proof requires more of the theory of the general p-dimensional determinant than is necessary in the proof we have actually given.

THE LAGRANGE SOLUTIONS OF THE PROBLEM OF THREE BODIES

By W. MARKOWITZ, The University of Chicago

In section 169 of his *An Introduction to Celestial Mechanics*, F. R. Moulton discusses the solutions of the problem of three bodies in which the ratios of the mutual distances are constants. He assumes that the plane containing the three bodies remains fixed in space or else moves parallel to itself. Let the masses of the three bodies be m_1, m_2, and m_3, respectively, and let the ξ, η-plane be the plane of motion. Then the differential equations of motion of the first body with respect to the center of gravity are

$$(1)\qquad \begin{aligned} \frac{d^2\xi_1}{dt^2} &= -\frac{m_2(\xi_1-\xi_2)}{r^3_{1,2}} - \frac{m_3(\xi_1-\xi_3)}{r^3_{1,3}}, \\ \frac{d^2\eta_1}{dt^2} &= -\frac{m_2(\eta_1-\eta_2)}{r^3_{1,2}} - \frac{m_3(\eta_1-\eta_3)}{r^3_{1,3}}. \end{aligned}$$

The equations of motion for m_2, and m_3 are obtained by cyclical permutations of the subscripts.

Suppose the coördinates of m_1, m_2, and m_3 at $t=t_0$ are, respectively, (x_1,y_1), (x_2, y_2), and (x_3, y_3). Let the respective distances from the origin be $r_1^{(0)}$, $r_2^{(0)}$, and $r_3^{(0)}$. Suppose the angles that $r_1^{(0)}$, $r_2^{(0)}$, and $r_3^{(0)}$ make with the ξ-axis are ϕ_1, ϕ_2, and ϕ_3; then, as Moulton shows,

$$(2)\qquad \begin{aligned} x_1 &= r_1^{(0)}\cos\phi_1, & x_2 &= r_2^{(0)}\cos\phi_2, & x_3 &= r_3^{(0)}\cos\phi_3, \\ y_1 &= r_1^{(0)}\sin\phi_1, & y_2 &= r_2^{(0)}\sin\phi_2, & y_3 &= r_3^{(0)}\sin\phi_3. \end{aligned}$$

The coordinates of the bodies at time t are (ξ_1, η_1), (ξ_2, η_2), and (ξ_3, η_3). The mutual distances at t are

$$\rho r_{1,2}, \quad \rho r_{2,3}, \quad \rho r_{1,3},$$

ρ being the proportionality factor, and the r's being the values of the mutual distances at t_0. Since the shape of triangle formed by the three bodies does not vary with the time, it follows that the radii vectores r_1, r_2, and r_3 have the following values:

$$r_1 = r_1^{(0)}\rho, \quad r_2 = r_2^{(0)}\rho, \quad r_3 = r_3^{(0)}\rho. \tag{3}$$

Moreover, the radii r_1, r_2, and r_3 will have turned through the same angle θ. Hence

$$\begin{aligned} \xi_1 &= r_1^{(0)}\rho \cos(\theta + \phi_1) = (x_1 \cos\theta - y_1 \sin\theta)\rho, \\ \eta_1 &= r_1^{(0)}\rho \sin(\theta + \phi_1) = (x_1 \sin\theta + y_1 \cos\theta)\rho. \end{aligned} \tag{4}$$

Similar expressions for ξ_2, η_2, ξ_3, and η_3 are obtained by permuting the subscripts in (4) cyclicly.

If we transform the equations of motion, (1), by (4), and make use of a substitution employed by Moulton,

$$\psi = \rho^2(d\theta/dt), \tag{5}$$

then the following set of equations is obtained:

$$\begin{aligned}
\frac{d^2\rho}{dt^2} - \frac{y_1}{x_1\rho}\frac{d\psi}{dt} - \frac{\psi^2}{\rho^3} &= -\frac{1}{x_1}\left\{\frac{m_2(x_1 - x_2)}{r^3_{1,2}} + \frac{m_3(x_1 - x_3)}{r^3_{1,3}}\right\}\frac{1}{\rho^2}, \\
\frac{d^2\rho}{dt^2} - \frac{x_1}{y_1\rho}\frac{d\psi}{dt} - \frac{\psi^2}{\rho^3} &= -\frac{1}{y_1}\left\{\frac{m_2(y_1 - y_2)}{r^3_{1,2}} + \frac{m_3(y_1 - y_3)}{r^3_{1,3}}\right\}\frac{1}{\rho^2}, \\
\frac{d^2\rho}{dt^2} - \frac{y_2}{x_2\rho}\frac{d\psi}{dt} - \frac{\psi^2}{\rho^3} &= -\frac{1}{x_2}\left\{\frac{m_3(x_2 - x_3)}{r^3_{2,3}} + \frac{m_1(x_2 - x_1)}{r^3_{2,1}}\right\}\frac{1}{\rho^2}, \\
\frac{d^2\rho}{dt^2} - \frac{x_2}{y_2\rho}\frac{d\psi}{dt} - \frac{\psi^2}{\rho^3} &= -\frac{1}{y_2}\left\{\frac{m_3(y_2 - y_3)}{r^3_{2,3}} + \frac{m_1(y_2 - y_1)}{r^3_{2,1}}\right\}\frac{1}{\rho^2}, \\
\frac{d^2\rho}{dt^2} - \frac{y_3}{x_3\rho}\frac{d\psi}{dt} - \frac{\psi^2}{\rho^3} &= -\frac{1}{x_3}\left\{\frac{m_1(x_3 - x_1)}{r^3_{1,3}} + \frac{m_2(x_3 - x_2)}{r^3_{2,3}}\right\}\frac{1}{\rho^2}, \\
\frac{d^2\rho}{dt^2} - \frac{x_3}{y_3\rho}\frac{d\psi}{dt} - \frac{\psi^2}{\rho^3} &= -\frac{1}{y_3}\left\{\frac{m_1(y_3 - y_1)}{r^3_{1,3}} + \frac{m_2(y_3 - y_2)}{r^3_{2,3}}\right\}\frac{1}{\rho^2}.
\end{aligned} \tag{6}$$

Equations (6) are the necessary conditions for the existence of solutions in which the ratios of the mutual distances are constants. Inasmuch as only two variables, ρ and ψ, are to be determined by these six equations, it follows that all the constants entering in them cannot be independent. It is evident that solutions of (6) will exist if the coefficients of corresponding terms in ρ and ψ are the same in the different equations. That is, if

$$
(7)\qquad
\begin{aligned}
&\frac{m_2(x_1 - x_2)}{r^3_{1,2}} + \frac{m_3(x_1 - x_3)}{r^3_{1,3}} = n^2 x_1,\\
&\frac{m_2(y_1 - y_2)}{r^3_{1,2}} + \frac{m_3(y_1 - y_3)}{r^3_{1,3}} = n^2 y_1,\\
&\frac{m_3(x_2 - x_3)}{r^3_{2,3}} + \frac{m_1(x_2 - x_1)}{r^3_{2,1}} = n^2 x_2,\\
&\frac{m_3(y_2 - y_3)}{r^3_{2,3}} + \frac{m_1(y_2 - y_1)}{r^3_{2,1}} = n^2 y_2,\\
&\frac{m_1(x_3 - x_1)}{r^3_{3,1}} + \frac{m_2(x_3 - x_2)}{r^3_{3,2}} = n^2 x_3,\\
&\frac{m_1(y_3 - y_1)}{r^3_{3,1}} + \frac{m_2(y_3 - y_2)}{r^3_{3,2}} = n^2 y_3,
\end{aligned}
$$

where n^2 is the common constant value of the bracket in the right side of (6), and

$$(8)\qquad d\psi/dt = 0.$$

As an alternative to (8), Moulton gives

$$(9)\qquad y_1/x_1 = y_2/x_2 = y_3/x_3.$$

Moulton states that these equations, (7) and (8) or (7) and (9), could be shown to be necessary by expanding the solutions of (6) as power series in $t-t_0$ and then equating the coefficients of the various power of $t-t_0$. This process, however, is difficult to carry out in practise, and another method will be given to show the necessity of equations (7) and (8).

Let us multiply equations (6) by $-m_1 x_1 y_1$, $+m_1 x_1 y_1$, $-m_2 x_2 y_2$, $+m_2 x_2 y_2$, $-m_3 x_3 y_3$, and $+m_3 x_3 y_3$, respectively, and add. The result is

$$(10)\qquad \frac{1}{\rho}\left[m_1(x_1^2 + y_1^2) + m_2(x_2^2 + y_2^2) + m_3(x_3^2 + y_3^2)\right]\frac{d\psi}{dt} = 0.$$

Hence, the condition

$$(11)\qquad \frac{d\psi}{dt} = 0$$

is a consequence of equations (6).

By substituting this equation in (6), the set (7) is obtained. It follows, therefore, that equations (7) and (8) are both necessary and sufficient conditions for the existence of solutions of (6).

A SPECIAL BIRATIONAL CUBIC TRANSFORMATION BETWEEN TWO 3-SPACES IN 4-SPACE

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A general birational cubic transformation between two 3-spaces S_3 and S_3' transforms a line or a plane in either 3-space into a twisted cubic curve or a cubic surface in the other 3-space. In S_3 there is a sextic curve, J^6, of singular points which transform into lines in S_3' forming an octavic surface j'^8 trisecant to a sextic curve J'^6 of singular points in S_3'. The points of J'^6 have for images in S_3 lines trisecant to J^6 and forming an octavic surface j^8. The two curves J^6, J'^6 and the two surfaces j^8, j'^8 are all of deficiency 3.

It is the purpose of this note to notice a special case of this transformation in which the curves J^6 and J'^6 degenerate in a certain manner. This case is not unknown but the manner of obtaining it seems new and interesting.

To obtain this special transformation we assume in S_4 a quartic surface F which is the projection of a Veronese surface in S_5 and whose projection upon any S_3 in S_4 is a Steiner's quartic surface. This surface F has one apparent triple point. From any point P in a given 3-space in S_4 let a line be drawn incident with F three times and let this line meet another given 3-space S_3' in S_4 in a point P'. P and P' are said to be corresponding points of the transformation. From the very nature of the construction we see that the transformation is birational. To show that it is of order 3 we notice that the lines meeting a plane once and F three times form a cubic hypersurface, for any 3-space through the plane meets the hypersurface in the plane itself and a quadric surface whose rulings are trisecants of the unicursal quartic curve in which F is met by the cutting 3-space.

Let K^4 and K'^4 denote the unicursal quartic curves in which S_3 and S_3' intersect F respectively; Q the quadric surface of trisecants to K^4 and Q' that of trisecants to K'^4; s the plane common to S_3 and S'_3; and K^2 and K'^2 the conics in which s meets Q and Q' respectively.

It is not difficult to see that the sextic curve $J^6(J'^6)$ of singular points in $S_3(S_3')$ is made up of K^4 (K'^4) and K'^2 (K^2) and that the ruled octavic surface j^8 (j'^8) in S_3 (S_3') is made up of the quadric surface Q (Q') and a sextic surface W (W') whose rulings meet K^2 (K'^2) once and K^4 (K'^4) twice. The images of the points on K'^2 (K^2) considered as points in S_3 (S_3') are generators of Q' (Q) in S_3' (S_3) and those of the points on K^4 (K'^4) are the generators of W' (W). The points of s not on K^2 or K'^2 are their own images.

A plane f in S_3 (or f' in S_3') has for image in S_3' (S_3) a cubic surface F'^3 in S'_3 (or F^3 in S_3). Since f meets every line of Q, its image F'^3 passes through K^2. The plane s meets F'^3 in K^2 and hence in a line besides. This line is the intersection of s with f. The image of the plane s is a degenerate cubic surface consisting of s itself and a quadric surface. This quadric surface is Q' if s is considered as belonging to S_3 and Q if s is considered as belonging to S_3'.

Now consider some further special cases. First, let S_3' remain general and S_3 pass through a tangent plane of F, i.e., meet F in a pair of conics L, M having a point in common. L meets the plane v_m of M in a further point P_m and M meets the plane v_l of L in a further point P_l. The lines of v_m and v_l passing through P_l and P_m respectively are to be considered as trisecants of the quartic curve $K^4 \equiv L+M$ and hence the quadric surface Q consists now of the two planes v_l, v_m and the conic K^2 of the two lines k_l, k_m in which s meets v_l and v_m, respectively. The sextic curve J^6 in S_3 is now composed of three conics, L, M, and K'^2, the last conic being the intersection of s with Q' in S'_3, and the octavic surface j^8 of the two planes v_l, v_m and the sextic surface whose rulings meet the conic K'^2 once and L, M also each once. The J'^6 in S'_3 is composed of K^4 and the two lines k_l, k_m and the j'^8 has degenerated into the quadric surface Q' and two cubic surfaces. The generators of one of the cubic surfaces cut across k_l and those of the other cut across k_m but the generators of both are bisecants of K^4.

If both S_3 and S_3' are tangent to F, then both J^6 and J'^6 are each composed of two conics having one point in common and two intersecting lines lying in s and each meeting a conic twice. Both j^8 and j'^8 are each composed of the planes of the conics and two cubic surfaces.

Now if the quartic surface F in S_4 degenerates, we obtain further special cases. F can degenerate only in two ways, either into a quadric surface and two non-incident planes intersecting the quadric surface in two non-intersecting generators or into four planes one of which meets each of the other three which are non-incident in a line.

Consider the first case. The curve K^4 (K'^4) in which S_3 (S_3') assumed general meets the degenerate F is now composed of a conic C^2 (C'^2) and two skew lines l, m (l', m') each intersecting C^2 (C'^2) once. The lines incident with C^2, l, m, (C'^2, l,' m') form the quadric surface Q in S_3 (Q' in S_3') whose intersection K^2 (K'^2) with s common to S_3 and S_3' forms with C'^2, l', m' (C^2, l, m) the sextic curve J'^6 in S_3' (J^6 in S_3). The ruled octavic surface j^8 in S_3 (j'^8 in S_3') is now composed of four quadric surfaces.

The other case may be obtained by assuming three general planes u, v, w in S_4. These three planes and the plane t determined by the three points in which they intersect two by two may be considered as a degenerate F. An S_3 meets the configuration of these four planes in four lines l_u, l_v, l_w, l_t of which the first three are skew and the last is incident with each of them. The quadric Q is formed by the lines cutting across l_u, l_v, l_w and the j^8 is now composed of Q and three cones with vertices on the line l_t where it is met by the lines l_u, l_v, l_t. The configuration in S_3' is similar.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A NOTE ON GEOMETRICAL APPLICATIONS OF COMPLEX NUMBERS

By S. A. SCHELKUNOFF, Bell Telephone Laboratories Inc.

The writer was greatly pleased with Professor L. L. Smail's suggestion of geometric problems for illustrating the use of complex numbers.[1] On several occasions in his teaching (at the State College of Washington) the present writer used similar illustrations, although he favored a somewhat more direct application of the algebra of complex quantities. The purpose of this note is partly to point out the difference in the two methods of approach but mainly to say another word for the cause. To make the comparison of the two methods effective, we shall discuss some of the problems treated by Professor Smail. It is hoped that in this way we can illustrate the effectiveness of the algebra of complex numbers in certain types of geometric problems. We wish to emphasize the fact that in the following proofs we do not separate the real and imaginary parts of complex numbers and that thereby we obtain simple and elegant demonstrations. It is readily granted that in some problems such separation, evidently preferred by Professor Smail, is advantageous. However, even in treating problems involving distances, it is often convenient to use conjugate complex numbers to avoid the separation into real and imaginary parts.

In this article we shall make use of the following simple propositions:

(a) *If k is real, then kz and z represent parallel vectors whose lengths are in the ratio $k:1$;*

(b) *If z_1 and z_2 are the complex numbers representing respectively points A and B, then $\frac{1}{2}(z_1+z_2)$ represents the mid-point of the line-segment AB.*

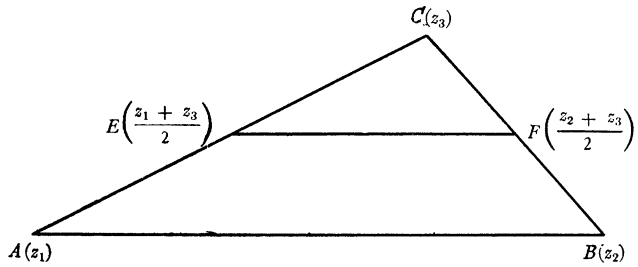

FIG. 1

[1] Smail, Lloyd L., *Some geometrical applications of complex numbers*, in this Monthly, vol. 36 (1929), pp. 504–511.

We shall now prove a few geometric theorems:

Theorem 1: *The line-segment joining the mid-points of two sides of a triangle is equal to half the third side and is parallel to it.*

Proof: Let the complex numbers corresponding to the vertices of any triangle in reference to some origin be as shown in Fig. 1. Then, the complex numbers representing the mid-points E and F are immediately found, and we have: $EF=\frac{1}{2}(z_2-z_1)$. But, we have also: $AB=z_2-z_1$. Hence, the theorem follows.

Theorem 2: *The line-segments joining the mid-points of opposite sides of any quadrilateral bisect each other.*

Proof: The procedure is similar to the above. Thus, in the notation of Fig. 2, the mid-point of EF is represented by the complex numbers,

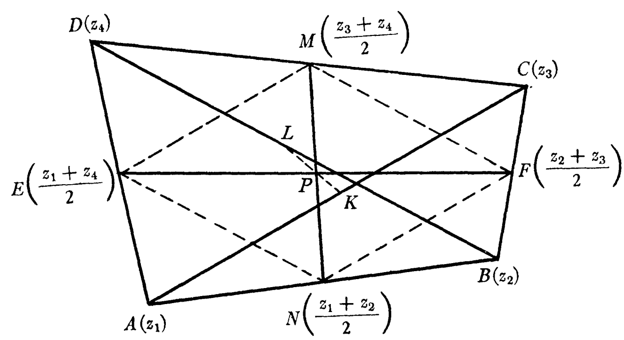

Fig. 2

$$\tfrac{1}{2}\left\{\tfrac{1}{2}(z_1 + z_4) + \tfrac{1}{2}(z_2 + z_3)\right\} = \tfrac{1}{4}(z_1 + z_2 + z_3 + z_4),$$

and the mid-point of MN is represented by

$$\tfrac{1}{2}\left\{\tfrac{1}{2}(z_1 + z_2) + \tfrac{1}{2}(z_3 + z_4)\right\} = \tfrac{1}{4}(z_1 + z_2 + z_3 + z_4),$$

which is equal to the above. Hence the theorem is proved.

The complex number representing P can be also written as follows:

$$\tfrac{1}{4}(z_1 + z_2 + z_3 + z_4) = \tfrac{1}{2}\left\{\tfrac{1}{2}(z_1 + z_3) + \tfrac{1}{2}(z_2 + z_4)\right\},$$

where $\frac{1}{2}(z_1+z_3)$ evidently represents the mid-point K of the diagonal AC and $\frac{1}{2}(z_2+z_4)$ the mid-point L of BD; hence P is also the mid-point of KL. Thus, we have the following:

Theorem 3: *The mid-point of the line-segment joining the mid-points of the diagonals of a quadrilateral coincides with the point of intersection of the line-segments joining the mid-points of opposite sides of the quadrilateral.*

Theorem 4: *The lines joining mid-points of adjacent sides of a quadrilateral form a parallelogram.*

Again, referring to Fig. 2, we have: $EM=\frac{1}{2}(z_3-z_1)$, and $NF=\frac{1}{2}(z_3-z_1)$, which proves that EM and NF are parallel and equal in length. Hence, the theorem follows.

Enrique Cruchaga's Solution of the Quartic Equation

By Raymond Garver, University of California at Los Angeles

In the last paragraph of a short paper which was published recently in this Monthly,[1] I mentioned a solution of the quartic equation which seemed to me to have some advantages over the usual solutions, at least from the standpoint of presenting the matter to a class in the theory of algebraic equations. The details, which I did not give at that time, may be summarized as follows.

If we consider the general quartic in the reduced form

$$x^4 + a_2x^2 + a_3x + a_4 = 0, \tag{1}$$

with roots, x_1, x_2, x_4, x_4, the following identities may be written:

$$\begin{aligned} 2x_1 &= (x_1 + x_2) + (x_1 + x_3) + (x_1 + x_4), \\ 2x_2 &= (x_2 + x_1) + (x_2 + x_3) + (x_2 + x_4), \\ 2x_3 &= (x_3 + x_1) + (x_3 + x_2) + (x_3 + x_4), \\ 2x_4 &= (x_4 + x_1) + (x_4 + x_2) + (x_4 + x_3), \end{aligned} \tag{2}$$

or, putting $x_1+x_2=k_1$, $x_1+x_3=k_2$, $x_1+x_4=k_3$,

$$\begin{aligned} 2x_1 &= k_1 + k_2 + k_3, \\ 2x_2 &= k_1 - k_2 - k_3, \\ 2x_3 &= -k_1 + k_2 - k_3, \\ 2x_4 &\quad -k_1 - k_2 + k_3. \end{aligned} \tag{3}$$

The four roots are thus expressed in terms of three quantities which are, in turn, square roots of the three roots of a cubic equation which is easily set up. For, knowing the values of the symmetric functions of the roots of (1), it

[1] Vol. 35 (Dec. 1928), pp. 558–560.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College of the City of New York.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

Foundations of Geometry and Induction. By Jean Nicod; translated by Philip Wiener. Harcourt, Brace, and Company, New York, 1930. 286 pages. $5.00.

This book, which lies in the no-man's-land between philosophy and mathematics, consists of two independent treatises: "Geometry in the Sensible World" and "The Logical Problem of Induction." In his introduction to the former, M. Nicod states that he is concerned with the rôle of physics as an expression of sensory experience, and proposes "to ascertain in what way geometry is an aid to physics; how its propositions are applied to the order of the perceived world; how knowledge of them helps us in the formulation of experiments and laws." He effects this examination by studying the geometries, or, indeed, the systems of physics, which might be constructed by hypothetical organisms variously simplified with respect to sensory faculties, and by then applying the results of this study to the more complicated human situation.

He opens by acquainting the reader with the nature of pure geometry as the study of certain logical systems any one of which may admit of an indefinite variety of applications, or *solutions*, in the concrete world. He then discusses various relations between geometries, notably that of *inseparability*, by which he means the relation obtaining between two geometries such that entities satisfying the axioms of the first geometry can be defined by purely logical means in terms of entities satisfying the axioms of the second geometry, and *vice versa*. He then discusses Whitehead's geometry of volumes, and shows it to lie in the relation of inseparability, in the above sense, with the ordinary point geometry.

The next five chapters constitute a thesis on the relations of sense data, the purpose of which is to justify the applying "of the categories of logic to the flux of sensation." It is here that Nicod makes the crucial point in his reconciliation of logic with perception. He shows that a sense datum can be *interior* to another without forming one of the latter's essential *parts*. Interiority is a simple relation between sense data as such, and does not presuppose more fundamental reality in referent than in relatum. This paves the way philosophically for the construction of fictions, such as points, lines and planes, out of sensed objects occupying volumes, and endows such abstractions with the same validity enjoyed by classes.

He now turns to his artificial universes, the first and simplest of which is that perceived by a being capable of locomotion along an open curve, and endowed with but one sense, audition. Every location on the curve is characterized by a sound. The organism, by means of the relations of succession and

global (complete) resemblance, contrives a complete one-dimensional geometry. The idea of *local identity* is derived from the matter of a symmetric sequence of sounds, indicative of complete retracing of steps. Here we have *thinghood* in its minimum essentials. Nicod then complicates the situation by locus nodes and bifurcations, and finally extending it to two and three dimensions.

The next monster dwells in a plane and is capable of no sensation except kinaesthesis. He constructs, quite ingeniously, a complete plane geometry identical in form with the Euclidean. Liberated to space, he meets with similar success. But of especial significance is the fact that in all of the above situations, from the simplest on, it has been possible for the organism to construct additional geometries, identical in form with the first, but concerned with secondary logical constructions out of sense data instead of with the more obvious entities of perception.

Thus Nicod proceeds through universes of increasing complexity, ending with a geometry of perspectives constructed by an animal who sees while at rest but not while in motion. He leaves it to the reader's imagination to multiply the complexities sufficiently to derive therefrom the world we live in. By the device of illustrative simplification he has thrown philosophical light on the manner in which a geometry may admit of multifarious applications in the same universe. Above all, he has made a significant step in the task of relating pure geometry to pure experience.

The second part of the book, *The Logical Problem of Induction*, is a quite rigorous criticism and extension of prevalent theories of induction, chiefly that of Keynes.[1] The bulk of the work is divided into a discussion of (1) induction by invalidation (elimination of alternative laws) and (2) induction by confirmation. For the operation of induction of the first type, Nicod finds it necessary to postulate determinism; i.e., for any character A, that "any case of A is the case of some other character X every case of which is a case of A." This assumed, every case of A furnishes a group of potential laws, the true one of which may be found by invalidating all the rest. He does not add explicitly what the ensuing matter reveals to be intended; viz., that the class of potential "causes" of A—the range of X in the above—must in some wise be limited so as to include only a finite number of eliminands. Otherwise, elimination would make no progress. Such a restriction of X to the class of "feasible causes," however, would seem to the reviewer to introduce an extra-logical element of common sense.

The author then examines the complications introduced by plurality and complexity of causes, in the presence of neither of which can elimination be complete. He devises principles, however, to act against each of these tendencies. The first, directed against complexity, assumes that the improbability of p-fold complexity increases with p. The second, directed against plurality,

[1] *A Treatise on Probability*, London, 1921.

is analogous. Multiplying of instances thus, through the joint operation of invalidation and of increased protection against disruptive complications, increases probability indefinitely, but never beyond the joint probabilities of the truth of the two above laws and that of determinism. Still further complications arise in nature, he shows, inasmuch as our knowledge of the characters attending an instance is necessarily incomplete. Hence an additional law would have to be assumed here, namely that, as instances are multiplied, the existence of a single character common to the unobserved parts of all those instances becomes increasingly improbable.

Turning to induction by confirmation, he begins, like Keynes, with the fundamental axiom that the probability of pq is equal to the product of the probability of p into the probability of $p . \supset . q$. With this as sole basis, he shows that the necessary and sufficient conditions, in order that the probability of a law be increased by a verification, are that (1) the law possess *some* initial probability, and (2) the *verification* possess *some* initial improbability. He goes on to demonstrate that one more condition, besides that of existent initial probability, is necessary and sufficient in order that the probability approach certainty as the instances of successive verification are multiplied without limit; this condition is that the verification of a false law in an infinite number of successive instances is infinitely improbable. Then comes destruction of various of Keynes' deductions, and a conclusion pointing out that induction by elimination cannot surpass a mediocre probability, and that even induction by confirmation has not been demonstrated to yield infinite probability except on the basis of certain assumptions.

At occasional passages in the book, the manner of exposition is lacking in clarity and preciseness. Confusion of "any" with "every," of "includes" with "is included by," and similar inaccuracies, whether due to author or to translator, render the perusal labored at points. The organization of the treatise on geometry is not all that might be desired, for, although all three sections thereof vitally concern the same general problem, yet their real mutual significance does not enter the structure of the treatise in the closely knit fashion that one would expect. One misses the skilled pen of a Russell. The thought behind Nicod's work, however, is rigorous and painstaking, and the matter presented is of considerable interest and unquestionable importance.

W. V. QUINE

Geometrische Konfigurationen. By Friedrich Levi. S. Hirzel, Leipzig, 1929. 310 pages.

The bulk of the energy of the American graduate student of mathematics is used in learning the essentials of algebra, analysis, and geometry. After he gets settled in his life-work, he seldom feels the urge to go through a lot of memoirs in the periodical literature to find out what a certain branch of his study is about. In the absence of a comprehensive, unified theory, there will

be few texts to guide his path, and probably he will forego exploration in that direction altogether.

Until the publication of Dr. Levi's text, this has been the case with reference to configurations. The author, in his preface, expresses the hope that the few topics he has selected will beguile more mathematicians into a study of configurations. He bears in mind the necessity of assuming more maturity than knowledge in his readers, and promises to unveil the hidden connections between group theory and analysis situs, combinatorial topology and geometric figures. He plans a second book, to cover gaps in the present text.

Without precedents to follow in his choice of subjects to be included, the author has consulted his personal tastes. The fifty odd-pages devoted to the Pascal diagram seem disproportionate, unless it be regarded as a case study exhibiting the rapidity and smoothness with which the classical results are unfolded by the notation and methods of the text.

Chapters I and II offer a rapid introduction to group theory and to the foundations of combinatorial topology. In his desire for conciseness, Levi may perplex a neophyte. At the high-school age of mathematical development, it may be permissible to define a rectangle as a parallelogram possessing four right angles, but when an author compresses three distinct postulates and a bit more into a single assumption, as in Levi's definition of group, it is questionable whether the average student benefits by the ellipsis. On pages 25 and 26 are two important notational conventions, which might be overlooked by a student eager to plunge into the material of the text proper.

After the two introductory chapters come a pair on the "simplest projective configurations," and "polyhedral configurations." Among the suggestive sub-heads we find Möbius tetrahedra, and three dealing with various aspects of the Desargues configuration. To the fifth chapter, "The Pascal Figure," we have already alluded. The last chapter discusses regular polyhedra, rotation groups, etc., in Euclidean, and non-Euclidean spaces.

There are 58 well drawn figures in the text.

The French have a proverb, "C'est le premier pas qui coûte!" The mathematical world is indebted to Dr. Levi for making this first text upon a fascinating topic.

CHARLES A. RUPP

Sir Isaac Newton, 1727–1927. Edited by Frederick E. Brasch. Williams and Wilkins Company, Baltimore, 1928. ix+351 pp.

The two hundredth anniversary of the death of Sir Isaac Newton was the occasion for the publication of a series of essays which were first presented at a joint meeting of the History of Science Society, the American Mathematical Society, the Mathematical Association of America, and other organizations on November 25 and 26, 1927, the actual anniversary having occurred on March 20 of that year. The purpose of the volume is expressed in its sub-title "A

Bicentenary Evaluation." This appears again and again in the titles of the essays. For example, Professor Birkhoff writes on "Newton's philosophy of gravitation with special reference to modern relativity ideas;" Professor W. W. Campbell on "Newton's influence upon the development of astrophysics"; Professor Ernest W. Brown on "Developments following from Newton's work."

It was decided that each essay should be sufficiently popular to appeal to educated readers who are not highly trained in the particular topic under discussion. This was a necessary provision. Otherwise, Newton's influence in such fields as economics and theology would have been ignored or else the majority of readers would have been unable to profit from each of these discussions of Newton's contributions. It inevitably follows that these essays are uneven in this respect, some being much more popular in character than others. But in no case has the general policy been disregarded.

There is an occasional repetition of incident or of quotation as, for example, Newton's characterization of himself as a child playing on the sea-shore, perhaps finding a prettier shell than ordinary, while the great ocean of truth lay all undiscovered before him. These repetitions may best be explained by the circumstances of the preparation of these papers, and the deletion of these passages by the editor would doubtless have done greater harm to the individual contributions than it would have done good to the volume as a whole.

In the first of these essays, Professor David Eugene Smith raises a number of questions regarding Newton and his appraisal by posterity, thus stimulating the reader to a more lively interest in Newton's achievements than he would have had had he been allowed to continue in the usual placid and perhaps unthinking acquiescence to Newton's preëminence. These queries, substantiated by the lack of a definitive edition of Newton's works although England has so honored lesser men, and backed by comments by various biographers, whet one's curiosity to know of Newton's work in optics as given by Professor Dayton C. Miller, to read of Newton's dynamics as discussed by Professor Pupin, to hear Professor Newell's comment on Newton's work in alchemy and chemistry, and Dr. Heyl's description of Newton as an experimental philosopher, and to have Newton's often condoned career in London evaluated by George F. Roberts, a banker and sometime director of the United States Mint.

To the student of the history of mathematics, perhaps the most interesting articles are those of Professor Cajori, the one dealing with fluxions, and the other studying Newton's delay of twenty years in announcing the law of gravitation, and producing evidence that this delay was not due to the traditional reason of the need for a more accurate determination of the radius of the earth.

The scientist will perhaps be surprised to find that Newton had a place in the history of religious thought. Here, Professor George S. Brett of Toronto writes that "in bulk Newton's theological writings are as big as his scientific

writings, that his interest was as keen, and that he felt no shadow of doubt about the importance of the subjects he undertook to elucidate."

Further, those who are apt to overlook the beginnings of scientific work on this side of the Atlantic will be interested in Dr. Brasch's account of Newton's first critical disciple in the American colonies.

The work closes with an appendix which lists the principal materials relating to Newton and his contemporaries that were exhibited in the Museum of Natural History in New York at the time of the meeting at which these essays were presented. A valuable feature of this list is the grouping of the items according to the collections from which they were gathered: the Babson Institute, Mr. Plimpton's library, Professor Smith's collection, and Yale University. Only a brief note appears regarding the material concerning Newton's predecessors in England and on the continent, exhibited in a corridor leading to the room where the meetings were held. It is perhaps not out of place, however, to repeat the comment of one speaker who quoted Newton's remark that if he had seen farther than most, it was because he had stood on the shoulders of the giants, and who then dubbed this passage the "Giants' Causeway."

VERA SANFORD

A Source Book in Mathematics. By David Eugene Smith. McGraw-Hill Book Co., New York, 1929. 701 pages.

A series of source books in the history of the sciences under the general editorship of Gregory D. Walcott was made possible by a grant of $10,000 to the American Philosophical Association in 1927 by the Carnegie Corporation of New York. These volumes when completed are "to present the most significant passages from the works of the most important contributors to the major sciences during the last threeor four centuries." The volume dealing with mathematics, compiled by a special committee under the supervision of David Eugene Smith, has recently been published.

The material in the "Source Book in Mathematics" covers such an extensive range that the book will be of use to teachers and students in preparatory schools as well as in colleges. The selections illustrate in a most illuminating way the theory and the processes that lie at the foundation of many of our most important mathematical concepts. In the earlier excerpts we find the style quaint and frequently amusing. Indeed several articles are so non-technical in character that they will be enjoyed by the general reader. A group of the most entertaining include: several pages from two famous arithmetics; the Treviso arithmetic, the first one to be printed, and one written by Robert Recorde as a dialogue called "The Ground of Arts"; passages from Berkeley's "Analyst" addressed to an "Infidel Mathematician" wherein he denounces certain alleged inconsistencies of the calculus; and Bernoulli's celebrated verses on infinite series.

It may be well to consider first the distribution of the material among various

"material that is not so ultra-technical as to serve no useful purpose for any considerable number of readers," and also the question of whether the "source material is stated succinctly enough for purposes of quotation has to be considered." Nevertheless the reviewer ventures to suggest that in view of the fact that this collection contains five articles on hyper-geometry, three on non-Euclidean geometry and two each on the following: binomial theorem, slide rule, law of sines, and the nine point circle, it might have been well to sacrifice a few of the duplications in order to include some of the topics that are omitted, such as; the metric system, hyperbolic functions, elliptic functions, Fourier's series, invariants, modular analysis, postulates, calculus of variations, differential geometry, and point sets. In addition to this duplication of subject matter, certain mathematicians are favored with several quotations, while others who are also among the foremost contributors to the science of mathematics are not mentioned. Among these omissions we note the following: Cantor, Dirichlet, Gibbs, Gordan, Jacobi, Klein, Kronecker, Lagrange, Lambert, Maxwell, Peano, Plücker, Ruffini, Von Staudt, and Weierstrass.

Certain items concerning the general plan of the book should be noted. The articles, more than ninety in number, are about equally divided among the fields of number, algebra, geometry, calculus, and function theory, with the exception that the geometry of the last century is emphasized somewhat, although this perhaps is not surprising since the term geometry covers a large variety of quite different phases of the subject. Classified by centuries, two mathematicians of the fifteenth century are quoted; seven of the sixteenth century; thirteen of the seventeenth century; nine of the eighteenth century; and thirty-two of the nineteenth century. Historical notes at the beginning of each article give the principal facts concerning the author's life and explain the important contribution to mathematics contained in the selection quoted. The footnotes are frequent, elucidating the topic still further and providing many references for a more detailed study of the subject. Moreover the excerpts have been chosen and translated by prominent mathematical scholars and hence may be relied on as unquestionably accurate and authentic. The book contains eight full page portraits and many illustrations and diagrams. Very few misprints have been noted. A volume as attractive as this in form and in content is certainly destined to arouse interest in mathematics and its history and hence deserves an honored place in every mathematical library.

L. P. COPELAND

Compléments et Exercices sur la Mécanique des Solides. By Georges Bouligand. Paris, Librairie Vuibert, 1929. viii+132 pages.

The aim of this book is to fill in a gap between the author's tome I, *Précis de Mécanique Rationnelle* (Paris, Vuibert, 1925) and the forthcoming tome II, and to aid students preparing for the *agrégation.*

The book is divided into two parts, the first dealing with kinematics, and

the second with the theory of friction and impact with friction. This part is supplemented by a series of rather complicated problems taken from recent *agrégation* examinations.

In the first part we find the elements of the theory of the motion of a plane on a plane, followed by an application to a definite problem. The more complicated problem of the motion of a solid is not treated in general, but is illustrated by exercises on rolling without sliding and rolling with sliding.

In the second part, which takes up most of the book, the author insists upon the convenience of the use of the Lagrangian equations in general, along with the possibility of obtaining the forces of constraint without the use of Lagrange multipliers in many problems. The treatment of the theory of friction, while classic, is yet interesting because Professor Bouligand states clearly the fundamental hypotheses in the special cases of sliding, non-sliding, and in the general case of sliding, rolling, and pivoting. Many problems are given to illustrate various possibilities, and to bring to light the shortcomings of the theory. A geometric form is given to the discussion of impact with friction, along the line of two articles by Pérès in the *Nouvelles Annales*, tome II, series 5, pages 98–108, 216–231, and the assumptions are, again, clearly put into evidence.

The theory of friction and impact with friction is followed by a note on unilateral constraints, in which the method of Delassus is developed, both analytically and geometrically, for the case of a holonomic system with two or more unilateral constraints. The book is completed by 14 problems with brief indications of their solutions.

In checking over the problems incorporated in the main text the reviewer found a few typographical errors, as follows:

On page 63, formula (2) should read $\psi' = b/\sin^2\theta$ instead of $\psi^1 = b/\sin\theta$.

On page 67, in line 1, $0'0$ should replace $00'$.

On page 68, in line 7, $1+l^2/B+a^2/A$ should replace $1+l/B+a/A$.

On page 75, in line 30, $R_y = \mathrm{Mg}\cos\alpha$ should replace $R_y = f\mathrm{Mg}\cos\alpha$.

On page 82, in line 28, $(-ab/mk^2)\int Yd\tau$ should replace $(-2ab/mk^2)\int Yd\tau$.

On page 86, in the last line, $Adp/d\tau = cY - bZ$ should replace $AdP/d\tau = cY - bX$.

On page 112, in line 4, $d^2f/dt^2 \geqq 0$ should replace $d^2f/dt^2 > 0$.

On page 43, the auxiliary variable z introduced in the process of integration is not the z coordinate of page 42.

This book, like the others of Professor Bouligand, is very carefully written. The judicious use of vector analysis and geometric discussions should make the book appeal to those who give graduate courses in mechanics, as well as to those who wish to make a technical use of the theory. The author is to be commended for stating very clearly his assumptions and stressing the danger of following too blindly those who "invoquent l'intuition mécanique (la meilleure et la pire des choses!)." We shall look forward with interest to the publication of tome II on the mechanics of continuous systems.

W. E. BYRNE

Standard Table of Square Roots. By L. M. Milne-Thomson. London, G. Bell & Sons, 1929. $1.75.

This is a table of "the square roots to eight significant figures of all four-figure numbers, with printed differences," to quote the sub-title. From the preface: "The table is arranged with the square roots of x and $10\,x$ side by side, four hundred roots at each opening. . . . . A printed decimal point has been given to make the values perfectly definite. . . . . Each value has been printed in full so that it may be transcribed with the minimum of effort."

The explanations give directions for interpolation; it appears that in most cases simple interpolation gives eight correct figures, and in the cases where second differences must be used, a simple table of corrections is available.

In the preface the statement is made that the table was originally prepared for the author's own use, and was used in manuscript form in preparing tables of elliptic functions which are to be published in the near future. For this work it was necessary to determine square roots to fourteen figures, and the explanation shows how these may easily be found from the tabulated eight-figure values, by a simple and well-known rule. Namely, if r is a close approximation to $x^{1/2}$, then a much closer approximation is $r'=\frac{1}{2}(r+x/r)$; if the error in r is e, that of r' is $e^2/2r$. Hence if the table yields a square root correct to eight places, the operation of this formula yields fourteen correct figures. With a calculating machine, the work is easy.

R. A. J.

PROBLEMS AND SOLUTIONS

Edited by B. F. Finkel, Otto Dunkel, and H. L. Olson

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3432. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

Given n points, $A_1, A_2, A_3, \cdots, A_n$ in a plane. Arbitrary lines, A_iA_j, are drawn (joining pairs of these points) provided that they do not intersect each other. Prove that, regardless of the relative positions of the given points, the possible number of lines is $3(n-2)$ if $n\geqq 3$.

3433. *Proposed by Thurman Andrew, Jamaica Plain, Mass.*

Consider three elastic rods of equal lengths and cross sectional areas.

The cross sections have the forms of a circle, an equilateral triangle and a rectangle whose long side is twice the diameter of the circle. What will be the ratio of the torques required to turn them through a given angle about their axes? What is the shape of the solid rod which requires the greatest torque? The elastic limit is not to be exceeded.

3434. *Proposed by Harry Gwinner, Baltimore, Maryland.*
Evaluate

$$\lim_{x\to 0} (1+x)^{\log x}.$$

3435. *Proposed by H. K. Hughes, University of Michigan.*
The function $F(z)$ defined by the series

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma^2(n+k)}$$

where k is any constant, real or complex, satisfies the differential equation,

$$z^2 \cdot F''(z) + (2k-1)\cdot z \cdot F'(z) + (k^2 - 3k - z)F(z) = \Gamma^{-2}(n+k).$$

Find the general solution of this equation and determine the arbitrary constants so as to obtain the particular solution given by the above series.

3436. *Proposed by Eugene M. Berry, Lynchburg College.*
Let p, q, and r be concurrent lines. Through any point on p draw lines p', a, b, c, d, such that p' is perpendicular to p and that angles ab and cd are bisected by p. Show that the line aq-dr meets the line cr-bq on p' and meets the line cq-br on p.

3437. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*
A variable circle passes through a fixed point and is tangent to a fixed circle. Prove that the diametric opposite of the fixed point on the variable circle describes a central conic of which the fixed point is a focus and the fixed circle is the auxiliary circle.
State and prove the converse of this proposition.
State the corresponding propositions for the parabola.

3438. *Proposed by the late F. P. Matz.*
Solve

$$\int_0^{dy/dx} \frac{\cos w\,dw}{16 + 9\sin^2 w} = \frac{1}{12}\tan^{-1}(x).$$

3439. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*
In the *Pecqueur Bibasal Sextic Spur-Gear Train*, one of the 105 angular velocity ratios which are associated with it is

$$N_{31}/N_{61} = (z_2 z_4 z_5 z_6)/(z_2 z_4 z_4' z_5' - z_3 z_4 z_5 z_6 - z_2 z_3' z_4' z_5'), \tag{1}$$

in which the eight z's designate the numbers of teeth on the eight gears of the train. In designing such a train the z's must be chosen in accordance with the following fundamental requirements:

(A) All of the z's must be positive integers.

(B) No z may be less than a certain minimum, say 12, nor greater than a certain maximum, say 60.

Four questions now present themselves, in answering which it must be remembered that whatever special conditions may be imposed on equation (1), general conditions (A) and (B) must always be satisfied. The four questions are:

(I) Between what limits lie the possible numerical values of the ratio N_{31}/N_{61}?

(II) For an assigned, allowable value of the ratio N_{31}/N_{61},—say for the value 1,000,000, if that should prove to be an allowable value,—select the z's.

(III) If possible, so select the z's that the special condition

$$N_{31}/N_{61} = \pm z_2z_4z_5z_6$$

will be satisfied.

(IV) If possible, so select the z's that the following special requirements will be simultaneously fulfilled: (a) the ratio N_{31}/N_{61} is to have an assigned numerical value,—say 3,373,785 ($=1809\times 1865$) if that should prove to be an allowable value; (b) the sum $z_2+z_3+z_3'+z_4+z_4'+z_5+z_5'+z_6$ is to be a minimum.

3440. *Proposed by A. Pelletier, Montreal, Canada.*

A triangle is circumscribed about a circle. Prove that the following three lines are concurrent: (1) the line joining the points of contact of any two sides; (2) the line joining the points of intersection of these sides with the bisectors of the opposite angles; (3) the line joining the feet of the altitudes on these sides.

SOLUTIONS

2954 [1922, 81; 1930, 94]. *Proposed by C. N. Mills.*

A machine-gun is placed on an armored train which is moving with a velocity v feet per second along a straight horizontal track. The muzzle velocity of the bullets is v feet per second. Find the greatest range, (1) in front and (2) behind the train.

Solution by Frank L. Wilmer, Odebolt, Iowa.

Let s denote the horizontal, and h, the vertical distance travelled by a bullet in the elapsed time t. Let α denote the angle of elevation of the gun, and T the time for the range R. Then $s=v(1\pm\cos\alpha)\ t$, $h=v\ \sin\alpha t-\frac{1}{2}gt^2$, and $v\sin\alpha-\frac{1}{2}gT=0$. Hence

$$R = 2v^2g^{-1}(1 \pm \cos\alpha)\sin\alpha.$$

For the maximum range we have

$$dR/d\alpha = 2v^2g^{-1}[\cos\alpha \pm (2\cos^2\alpha - 1)] = 0.$$

Then, for the maximum range, in front: $\cos\alpha = \frac{1}{2}$, $\alpha = 60°$, and $R = 3^{3/2}v^2/2g$; in the rear: $\cos\alpha = 1$, $\alpha = 0$, and $R = 0$.

Also solved by B. W. Carrie and Paul Wernicke.

2963 [1922, 129]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Through a given point to draw a line so that the sum of the squares constructed on the two segments cut off by it on the sides of a given angle should be equivalent to a given square.

II. *Solution by the Proposer.*

Let $AC = b$, $AB = c$ be the segments cut off by the required line BCD, through the given point D, on the sides AC, AB of the given angle, A. Denoting AD by d and considering the areas of the triangles ABD, ACD, ABC, we have:

$$cd \sin BAD + bd \sin DAC = bc \sin BAC.$$

We also have, by assumption, $b^2 + c^2 = s^2$, where s is a given quantity. If we divide both sides of the first equation by $bc \sin BAC$, both sides of the second equation by d^2, and put $b/d = x$, $c/d = y$, $\sin BAD/\sin BAC = p$, $\sin DAC/\sin BAC = q$, $s/d = r$, these two equations become:

$$px^{-1} + qy^{-1} = 1; \qquad x^2 + y^2 = r^2,$$

where p, q, r are known quantities. Solving these two equations simultaneously we shall obtain the values of the ratios x, y of the required segments b, c to the given segment d.

The elimination of one of the unknowns between the two equations leads to an irreducible biquadratic equation. It follows that the proposed problem cannot, in general, be solved by ruler and compasses. Such a solution may, however, be possible, in some special cases, as for instance, when $p = q$, i.e., when the point D lies on one of the bisectors of A.

See also the solution on p. 172 of vol. 36 (1929) of this Monthly.

3034 (1923, 276; 1930, 94). *Proposed by the late J. L. Riley, Stephenville, Texas.*

If every root of the equation $f'(x) = 0$ be subtracted from every root of the equation $f(x) = 0$, find the sum of the reciprocals of the differences.

Solution by Andrew G. Clark, Colorado Agricultural College.

Assuming that $f(x)$ is a polynomial of degree n, we may write

$$f(x) = \prod_{i=1}^{i=n}(x - a_i), \quad f'(x) = n\prod_{j=1}^{j=n-1}(x - b_j),$$

where a_i and b_j are the several roots of $f(x) = 0$ and $f'(x) = 0$ respectively. Differentiating $f(x)$, we have

$$f'(x) = f(x)\sum_{i=1}^{i=n}(x - a_i)^{-1}.$$

Now, if b_j is any given root of $f'(x)=0$, and if it is assumed that all the roots of $f(x)=0$ are distinct so that $f(b_j)\neq 0$, it is clear that when $x=b_j$ in the above identity, then $\sum_{i=1}^{n}(b_j-a_i)^{-1}=0$, and therefore

$$\sum_{i=1}^{i=n}\sum_{j=1}^{j=n-1}(b_j - a_i)^{-1}=0.$$

Of course if $f(x)=0$ has one or more repeated roots, it is obvious to begin with that our result is infinity, and the fact that $f'(x)/f(x)$ for x equal b_j (a repeated root) evaluates to infinity merely adds confirmation.

Also solved by Raymond Garver, A. Pelletier, O. J. Ramler, Paul Wernicke, and F. L. Wilmer.

3251 [1927, 216]. *Proposed by J. V. Uspensky.*

Show that the system

$$\phi(x, y)=x+y+\sum A_{ik}x^iy^k=1,\ x\phi_x'-ay\phi_y'=0,$$

where $A_{ik}>0$, $a>0$, $2\leqq i+k\leqq n$.
has one and only one solution in positive numbers.[1]

Solution by the Proposer.

The sum $\sum A_{ik}x^iy^k$ is supposed to contain terms of the second and higher dimensions. The solution easily follows from the fact that the proposed system is entirely equivalent to the system,

$$\phi(x,y) = 0, \quad d(x^ay)/dx = 0,$$

supposing y defined as a function of x by the first equation.

To show that this system is satisfied by a couple of *positive* numbers, x, y consider a function of x, namely $z=x^ay$, where y is defined by $\phi(x, y)=0$. For $x=0$ the equation $\phi(0, y)=0$ has one root between 0 and 1; and for $y=0$ the equation $\phi(x, 0)=0$ has a root between 0 and 1. Denoting by ϵ the least positive root of this equation, we readily see that

$$z(0) = 0, \quad z(\epsilon) = 0,$$

while for very small positive values of x the equation $\phi(x, y)=0$ admits of a positive root y, so that $z(x)>0$ for $0<x<\epsilon$. Applying Rolle's theorem we see that there exists one couple, $x>0$, $y>0$ satisfying the proposed system. To show that this couple is unique one can start from the fact that $d^2(x^ay)/dx^2<0$ whenever $\phi(x, y)=0$, $d(x^ay)/dx=0$, which can be proved as follows:

The equation $\phi(x, y)=x+y+\sum A_{ik}x^iy^k-1=0$ is transformed, by the substitution $z=x^ay$ or $y=x^{-a}z$, into one of the form,

$$\theta(x,z) = x + zx^{-a} + \sum A_{ik}x^{i-ka}z^k - 1 = 0.$$

[1] This problem was proposed to Mr. Uspensky by a Russian chemist in connection with a certain problem in chemistry.

Here the right hand member can be arranged according to descending powers of x as follows:

$$P_1x^{\alpha_1} + P_2x^{\alpha_2} + \cdots + P_{j-1}x^{\alpha_{j-1}} + P_j + P_{j+1}x^{\alpha_{j+1}} + \cdots + P_mx^{\alpha_m} = \theta(x,z),$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_{j-1} > 0$; $0 > \alpha_{j+1} > \alpha_{j+2} \cdots > \alpha_m$. The P's are polynomials in z with positive coefficients, except P_j whose constant term is -1, the other coefficients, if there are any, being positive. The system $dz/dx=0$, $\phi(x, y)=0$ for positive x *and* y is equivalent to $\theta(x,z)=0$, $dz/dx=0$. But $(\partial\theta/\partial x)+(\partial\theta/\partial z)$ $(dz/dx)=0$; therefore $\partial\theta/\partial x=0$. To find d^2z/dx^2 we have the equation,

$$(\partial^2\theta/\partial x^2) + (\partial\theta/\partial z)(d^2z/dx^2) = 0.$$

But $\partial\theta/\partial z$ is positive for positive x and z; therefore to prove that $d^2z/dx^2<0$ we must prove that $\partial^2\theta/\partial x^2>0$. We have

$$\frac{\partial\theta}{\partial x} = \sum_{k=1}^{j-1} P_k\alpha_k x^{\alpha_k-1} + \sum_{k=j+1}^{m} P_k\alpha_k x^{\alpha_k-1} = 0$$

or

$$\sum_{k=1}^{j-1} P_k\alpha_k x^{\alpha_k} + \sum_{k=j+1}^{m} P_k\alpha_k x^{\alpha_k} = 0. \tag{1}$$

On the other hand,

$$\frac{\partial^2\theta}{\partial x^2} = \sum_{k=1}^{j-1} P_k\alpha_k(\alpha_k - 1)x^{\alpha_k-2} + \sum_{k=j+1}^{m} P_k\alpha_k(\alpha_k - 1)x^{\alpha_k-2}$$

and

$$x^2\frac{\partial^2\theta}{\partial x^2} = \sum_{k=1}^{j-1} P_k\alpha_k(\alpha_k - 1)x^{\alpha_k} + \sum_{k=j+1}^{m} P_k\alpha_k(\alpha_k - 1)x^{\alpha_k},$$

which by virtue of (1) reduces to

$$x^2\frac{\partial^2\theta}{\partial x^2} = \sum_{k=1}^{j-1} P_k\alpha_k^2\, x^{\alpha_k} + \sum_{k=j+1}^{m} P_k\alpha_k^2\, x^{\alpha_k} > 0$$

because, for $k\neq j$, P_k as a polynomial in z *with positive coefficients* is positive for positive z.

3363 [1929, 105]. *Proposed by Otto Dunkel, Washington University.*

In an urn there are $k+1$ counters of which one is a blank while the others are numbered from 1 to k. A single counter is withdrawn and then replaced, and this is continued until there are n such drawings. In how many ways may the n drawings be made so that in the first r or more drawings only blanks are obtained and in the rest of the n drawings no numbered counter is followed by as many as r consecutive blanks? Also determine in how many ways n drawings may be made so that r or more consecutive blanks are drawn.

Solution by the Proposer.

The result of a set of n drawings may be recorded on a straight line AB divided into n equal spaces. We shall first count the number of ways in which the first r consecutive spaces from A may be reserved for blanks, while in the remaining $n-r$ spaces there shall be at least j groups such that each group consists of a number chosen from the k numbers followed consecutively by r blanks. We can select j such groups in k^j ways, and the remaining $n-r-j(r+1)$ spaces can be filled in $(k+1)^{n-r-j(r+1)}$ different ways. There are j groups and $n-r-jr-j$ single spaces, and thus $n-r(j+1)$ things which are to be arranged in different positions. For a given selection of j places for the j groups we must disregard the order in which the groups are placed in these positions, for this has already been accounted for in k^j. The number of ways of selecting the j places is then ${}_{n-r(j+1)}C_j$. Thus

$$\phi_j = {}_{n-r(j+1)}C_j k^j (k+1)^{n-r-j(r+1)} \tag{1}$$

is the number of ways of having at least j such groups and at least r consecutive blanks at the start.

Now consider a given particular way of filling the n spaces in the required manner so that there will be precisely p groups of a given composition and position. The upper limit for p is the greatest integer in $(n-r)/(r+1)$ which we shall denote by $[(n-r)/(r+1)]$; the lower limit is zero. This particular case will be counted precisely once in ϕ_0, p times in $\phi_1, \cdots, {}_pC_j$ times in ϕ_j, where $j \leqq p$. We can eliminate the count of this particular case, and any similar case of precisely p groups, if we take the sum

$$\sum_{j=0}^{j=p} (-1)^j \phi_j; \tag{2}$$

for in this sum this particular arrangement is counted precisely a number of times equal to

$$\begin{aligned} \sum_{j=0}^{j=p} (-1)^j {}_pC_j &= (1-1)^p = 0, \quad & p > 0, \\ &= 1, & p = 0. \end{aligned} \tag{3}$$

If we denote the answer to the first part of the problem by $P_2(n)$, then an expression for $P_2(n)$ is obtained by assigning to p in (2) its upper limit, for then the count of every group will be eliminated, while each and every arrangement which does not contain a group will be counted precisely once. We shall now write the formula for $P_2(n)$ replacing j by $j-1$:

$$P_2(n) = \sum_{j=1}^{j=i} (-1)^{j-1} {}_{n-jr}C_{j-1} k^{j-1} (k+1)^{n-j(r+1)+1}, \quad i = [(n+1)/(r+1)]. \tag{4}$$

This gives the value of $P_2(n)$ for $n \geqq r$; $P_2(n) = 0$ for $0 \leqq n \leqq r-1$.

The expression for $P_2(n)$ may also be obtained by finding first its difference equation and then solving this equation by known methods. This difference equation will now be derived. If to each set of n drawings counted in $P_2(n)$ we adjoin at the end each of the k numbers in turn, we obtain $kP_2(n)$ cases counted in $P_2(n+1)$; and these must be all of the cases ending in a number. There is just one case in $P_2(n+1)$ in which all of the $(n+1)$ counters are blanks. The remaining cases ending in a blank may be obtained by adjoining at the end of each case counted in $P_2(n-j)$ a number followed by j consecutive blanks. There are then $kP_2(n-j)$ such cases of j blanks at the end, $1 \leqq j \leqq r-1$. Hence

$$P_2(n+1) = k[P_2(n) + P_2(n-1) + \cdots + P_2(n-r+1)] + 1, \tag{5}$$

where $P_2(n)=0$, $0 \leqq n \leqq r-1$. If we replace n by $n-1$ and subtract the two equations we obtain a homogeneous equation

$$P_2(n+1) = (k+1)P_2(n) - kP_2(n-r), \tag{6}$$

where the initial conditions are now $P_2(n)=0$, $0 \leqq n \leqq r-1$; $P_2(r)=1$. This equation may also be obtained by a different manner of counting. Adjoin at the end of each case counted in $P_2(n)$ each of the $k+1$ counters in turn and we obtain cases counted in $P_2(n+1)$, but also some cases which are to be excluded. The cases to be excluded are those cases of $P_2(n)$ ending in a number followed by $r-1$ consecutive blanks and to which a blank is adjoined. The number of such cases is $kP_2(n-r)$. This gives (6) very readily, but the first method is preferable since it gives an equation of lower order.

Consider now the meaning of $P_2(n+r)$: this gives the number of ways of making n drawings so that no number will be followed consecutively by as many as r blanks. The number of ways of making all kinds of n drawings is $(k+1)^n$. Hence $(k+1)^n - P_2(n+r)$ is the number of ways of making n drawings so that each will have at least one set of a number followed by r or more consecutive blanks. Call this number $P_1(n)$. Then

$$P_1(n) = (k+1)^n - P_2(n+r). \tag{7}$$

If we denote the answer to the last question of the problem by $P_3(n)$, we have at once:

$$P_3(n) = P_1(n) + P_2(n). \tag{8}$$

By the use of (4) together with the above relations we may obtain expressions for $P_1(n)$ and $P_3(n)$.

We shall now derive a second relation between $P_1(n)$ and $P_2(n)$. If to each case counted in $P_1(n)$ we adjoin in turn at the end one of the $k+1$ counters, we shall have cases counted in $P_1(n+1)$, but not all. The remaining cases of $P_1(n+1)$ are those ending in a number followed by precisely r blanks but con-

taining no other set of a number followed by as many as r consecutive blanks. The number of such cases is precisely $kP_2(n)$. Hence

$$kP_2(n) = P_1(n+1) - (k+1)P_1(n). \tag{9}$$

From (7) and (9) we may derive the difference equation for $P_1(n)$. It will then be easy to obtain the difference equation for $P_3(n)$. It will obviously be the same equation as for $P_1(n)$, but the initial conditions will be different in the two cases.

3399 [1929, 543]. *Proposed by B. C. Wong, Berkeley, California.*
Prove or disprove

$$\sum_{i=0}^{t}(-1)^i\binom{r+1}{i}\binom{2r-2i}{r} = r+1,$$

where $t=r/2$ if r is even and $t=(r-1)/2$ if r is odd.

I. *Solution by T. L. Smith, Carnegie Institute of Technology.*

The general term a_i of the summation may be written

$$a_i = (-1)^i\frac{(r+1)!}{i!(r+1-i)!}\,\frac{(2r-2i)!}{(r-2i)!r!}\,.$$

The Legendre polynomial is given by summation over the same range:

$$\begin{aligned}
P_r(x) &= \frac{(2r-1)\cdots 5\cdot 3\cdot 1}{r!}\times \\
&\quad \sum(-1)^i\frac{r(r-1)(r-2)\cdots(r-2i+1)}{2\cdot 4\cdot 6\cdots 2i(2r-1)(2r-3)\cdots(2r-2i+1)}x^{r-2i} \\
&= \sum(-1)^i\frac{1}{r!}\,\frac{1}{i!2^i}\,\frac{r!}{(r-2i)!}[(2r-2i-1)\cdots 5\cdot 3\cdot 1]x^{r-2i} \\
&= \sum(-1)^i\frac{r!(2r-2i)!}{i!r!(r-2i)!2^i2^{r-i}(r-i)!}x^{r-2i} \\
&= \frac{1}{2^r(r+1)}\sum a_i(r+1-i)x^{r-2i}.
\end{aligned}$$

But for the factor $(r+1-i)$, the given sum could be evaluated by merely setting $x=1$. This factor may be eliminated by putting $x^{1/2}$ in place of x, multiplying the equation by $x^{r/2}$, and then integrating. This gives

$$(r+1)2^r\int_0^1 x^{r/2}P_r(x^{1/2})dx = \sum a_i\int_0^1(r+1-i)x^{r-i}dx = \sum a_i.$$

The first integral is easily evaluated by using the relation

$$\sum_{n=0}^{\infty} t^n P_n(x) = (1 - 2xt + t^2)^{-1/2};$$

substitute $x^{1/2}$ for x, $tx^{1/2}$ for t, and integrate:

$$\sum t^n \int_0^1 x^{n/2} P_n(x^{1/2}) dx = \int_0^1 (1 - 2xt + xt^2)^{-1/2} dx$$

$$= \frac{2}{2 - t} = 1 + \frac{t}{2} + \frac{t^2}{2^2} + \cdots + \frac{t^n}{2^n} + \cdots ,$$

whence

$$\int_0^1 x^{r/2} P_r(x^{1/2}) dx = 1/2^r.$$

The insertion of this integral gives the result, $r+1 = \sum a_i$

II. *Solution by B. P. Gill, College of the City of New York.*

The stated relation follows from equating coefficients of x^r in the identity,

$$(1 + x)^{r+1} = (1 - x^2)^{r+1}(1 - x)^{-r-1};$$

thus,

$$\binom{r+1}{r} = \sum_{i=0} (-1)^i \binom{r+1}{i} \binom{-r-1}{r-2i} (-1)^{r-2i}$$

$$= \sum_{i=0} (-1)^i \binom{r+1}{i} \binom{2r-2i}{r-2i} = \sum_{i=0} (-1)^i \binom{r+1}{i} \binom{2r-2i}{r}$$

See Netto, *Lehrbuch der Combinatorik*, (Leipzig, 1927), page 252, formula 28.

III. *Solution by Otto Dunkel, Washington University.*

This solution is given to illustrate the method of symbols of operation which may be employed frequently to advantage; for this particular problem other methods may be briefer. Let the operator U be defined so that $U^j f(x) = f(x+j)$; then $U - 1 = \Delta$, the usual difference operator. Consider the sum

$$\sum_{i=0}^{r+1} {}_{r+1}C_i (-1)^i f(i), \tag{1}$$

where $f(x)$ is a polynomial of degree n in x. Then (1) may be written

$$\left[\sum_{i=0}^{r+1} {}_{r+1}C_i (-1)^i U^i \right] f(0) = (-1)^{r+1} \Delta^{r+1} f(0). \tag{2}$$

In the given problem $f(x)$ is of degree r and hence $\Delta^{r+1} f(0) = 0$; also in this case $f(i)$ in (1) is zero for all values of i from r to $[(r+2)/2]$ inclusive where $[m]$ is the greatest integer in m; but $f(r+1) = (-1)^r (r+1)$. Hence for this problem we have

$$(3) \qquad \sum_{i=0}^{t} {}_{r+1}C_i(-1)^i f(i) = r+1,$$

where $t=[r/2]$.

One of the solutions of this problem makes use of the Legendre Polynomial, which is written in the form

$$(4) \qquad P_r(x) = \frac{1}{2^r} \sum_{i=0}^{t} {}_rC_i\, {}_{2r-2i}C_r(-1)^i x^{r-2i}, \quad t = \left[\frac{r}{2}\right].$$

This furnishes a similar problem. With the aid of the above symbols we may write

$$(5) \qquad P_r(x) = \left(-\frac{1}{2}\right)^r \Delta^r{}_{2r-2\cdot 0}C_r x^{r-2\cdot 0},$$

where Δ^r operates only on j in the function ${}_{2r-2j}C_r\, x^{r-2j}$, and at the end of the operation j is set equal to zero. We may now evaluate $P_r(1)$ by observing that the function operated upon is of the rth degree and that the coefficient of j^r is $(-2)^r/r!$. Hence it follows at once that

$$P_r(1) = (-1)^r P_r(-1) = \frac{1}{2^r} \sum_{i=0}^{t} {}_rC_i\, {}_{2r-2i}\, C_r\, (-1)^i = 1.$$

Also solved by E. S. Knebelman.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Summer Meeting of the Mathematical Association of America

The summer meeting of the Mathematical Association of America will be held at Brown University on Monday afternoon and Tuesday morning, September 8–9, 1930, in conjunction with the summer meeting and colloquium of the American Mathematical Society. The program will consist of invited papers, among these being papers by Dr. W. A. Shewhart of the BellTelephone Laboratories on "Random Sampling," by Dr. J. J. Smith of the General Electric Company on "The Heaviside Operational Calculus," and by G. D. Birkhoff of Harvard University on "A Mathematical Theory of Harmony and Melody." The complete program and full details of the meeting, together with reservation cards will be sent to members of the Association the latter part of July.

The schedule of the Society, as tentatively arranged, provides for general or sectional sessions on Wednesday, Thursday, and Friday mornings; for the Colloquium Lectures by Professor Solomon Lefschetz of Princeton University on Tuesday and Thursday afternoons and Wednesday and Friday mornings; and for addresses on topics in the general theory of linear operations and integral

equations by Professors T. H. Hildebrandt and J. D. Tamarkin on Tuesday and Thursday afternoons respectively.

Pembroke College in Brown University will provide board and lodging for many of the visiting mathematicians in two of its dormitories. Provided the room is engaged for a minimum of three days, the charge for a single room will be \$1.50 per person per day and \$1.00 per person per day when a room is occupied by two. Meals will be served in the Pembroke College Cafeteria at a cost not to exceed \$2.00 per day. Rooms can also be secured at about the same price in boarding houses near the campus. Provision can be made for families with children at approximately the same rates. Other available hotels and apartments will be indicated in the full announcement.

Plans for entertainment include a walk around the old part of Providence on Monday afternoon to see the fine examples of early American architecture; a reception by the University officials on Monday evening; a joint dinner of the Society and Association on Tuesday evening; and an excursion to Newport on Wednesday afternoon. At Newport there will be opportunity for swimming, for taking the cliff walk, and for partaking of a shore dinner, which is one of Rhode Island's specialties. Tennis on the University courts will be available free of charge; the Wannamoisett Country Club will offer the use of its golf course at a reduced green fee. An excursion to Lexington and Concord is being tentatively planned for Friday afternoon.

The mathematical library will be open for the use of the members during the meetings. A cordial welcome will also be extended to mathematicians who may wish to work in the library either before or after the meetings; all its facilities will be placed at their disposal and board and room in the vicinity of the Brown University campus will be arranged for upon request.

Providence may be reached conveniently from New York by train or by boat, and from all parts of the country by good automobile roads. To those who may be interested in visiting this part of the country before the meetings, New England offers many and varied opportunities for a summer vacation. The transportation companies serving this district have booklets giving information concerning their territory, including lists of boarding houses and hotels, with their rates. Inquiries may be directed as follows: Concerning southern New England, to the New York, New Haven and Hartford Railroad Co., Passenger Traffic Dept., New Haven, Conn.; concerning Maine, New Hampshire, and Vermont, to the Boston and Maine Railroad, Publicity Dept., North Station, Boston, Mass.; concerning northern Maine, to the Maine Central Railroad Co., Passenger Traffic Department, Portland, Maine; and concerning the Maritime Provinces to the Dominion Atlantic Railway Co., 12 Milk Street, Boston, Mass. It has been announced that the yacht races for the American Cup will begin off Newport on Saturday, September 13.

Association members who expect to have a change of address in September should notify Secretary W. D. Cairns, Oberlin, Ohio, well in advance in order to be sure of receiving the August–September Monthly.

The Association is in need of copies of the Monthly for January, 1930. Cash, or credit toward future dues, will be given for a limited number of copies to the amount of forty-five cents for each copy. Address all communications to the Mathematical Association of America, W. D. Cairns, Secretary, Oberlin, Ohio.

Dr. Irving Langmuir, of the General Electric Company, has been awarded the Willard Gibbs gold medal of the Chicago Section of the American Chemical Society "for fundamental work on atomic hydrogen, on surface relations, and electrical discharge phenomena, for contributions to various branches of physical chemistry, and for his presentation of a theory of atomic structure."

The Society of Arts and Sciences has awarded one of its gold medals to Professor G. N. Lewis, of the department of physical chemistry of the University of California.

By arrangement with the proprietors of the Quarterly Journal of Pure and Applied Mathematics and the Messenger of Mathematics, these journals have been discontinued, and have been succeeded by the new Quarterly Journal of Mathematics (Oxford Series). The editors are T. W. Chaundy, E. G. C. Poole, and W. L. Ferrar; and the members of the advisory board are A. L. Dixon, E. B. Elliott, G. H. Hardy, A. E. H. Love, E. A. Milne, F. B. Pidduck and E. C. Titchmarsh. The first number, which was published in April, contained the following articles: *The motion of a fluid in a field of radiation*, by E. A. Milne; *Some problems connected with Fourier's work on transcendental equations*, by G. Polya; *Electrical notes*, by E. E. Pidduck; *The zeros of certain integral functions*, by M. L. Cartwright; *A generalization of the quadratic differential form*, by Oswald Veblen.

All enquiries relating to this new journal should be addressed to the publisher, Mr. Humphrey Milford, Oxford University Press, Amen House, Warwick Square, London, E. C. 4.

DOCTORATES IN 1929

The following sixty-three doctorates with mathematics or mathematical physics as major subject were conferred during 1929 in universities in the United States and Canada; the university, month in which the degree was conferred, minor subject (other than mathematics), the title of dissertation are given in each case if available.

Nola L. Anderson, Missouri, June, astronomy, *An extension of Maschke's symbolism.*

G. A. Baker, Illinois, May, economics, *Random sampling from non-homogeneous populations.*

M. A. Basoco, California Institute of Technology, June, mathematical physics, *Fourier expansions of doubly periodic functions of the third kind.*

W. D. Baten, Michigan, June, *Theorems concerning probability.*

A. C. Berry, Harvard, June, *Fourier representations.*

H. R. Beveridge, Illinois, May, astronomy, *An expansion problem connected with a system of partial differential equations.*

O. K. Bower, Illinois, May, physics, *Applications of an abstract existence theorem to both differential and difference equations.*

A. B. Brown, Harvard, June, *Relations between the critical points of a real analytic function of n independent variables.*

O. E. Brown, Chicago, August, *The equivalence of triples of bilinear forms.*

F. H. J. Burkett, New York University, June, physics, *Some properties of the sextic with a quadruple point.*

R. V. Churchill, Michigan, June, *On the geometry of the Riemann tensor.*

W. M. Coates, Michigan, June, engineering mechanics, *The state of stress in thin-walled pressure vessels.*

C. J. Coe, Harvard, June, *Exterior motion in the restricted problem of three bodies.*

J. B. Coleman, California, August, *Concerning the reducibility of the characteristic equation of a ternary continued fraction.*

A. J. Cook, Chicago, June, *Pairs of rectilinear congruences with generators in one-to-one correspondence.*

H. V. Craig, Wisconsin, October, *On the simultaneous differential invariants of two functions with an application to the calculus of variations.*

Mildred W. Dean, Johns Hopkins, February, *Studies in inversive geometry with reference to a special set of six points.*

L. A. B. DeCleene, Catholic, *On triangles circumscribed about a conic and inscribed in a cubic curve.*

H. P. Doole, Nebraska, June, physics, *Certain multiple-parameter expansions.*

H. H. Downing, Chicago, December, *Absolute minima for space problems of the calculus of variations in parametric form.*

Nat Edmonson, Jr., Rice, June, *Poisson's integral and plurisegments on the hypersphere.*

P. D. Edwards, Indiana, June, astronomy, *Functions possessing an addition theorem from the standpoint of a functional equation given by Abel.*

J. D. Elder, California Institute of Technology, June, mathematical physics, *Arithmetized trigonometrical expansions of doubly periodic functions of the third kind.*

H. T. Engstrom, Yale, June, *On the common index divisors of an algebraic field.*

H. P. Evans, Wisconsin, June, *The two dimensional boundary value problem for the transmission of alternating currents through a heterogeneous earth.*

E. C. Goldsworthy, California, May, Physics, *Curves autopolar with respect to a cyclic set of conics.*

Deborah M. Hickey, Rice, June, *A three dimensional treatment of groups of linear transformations.*

J. J. L. Hinricksen, Harvard, June, *The problem of n bodies.*

T. J. Jaramillo, Chicago, March, *A generalization of the energy function of elaticity theory*

C. I. Lubin, Harvard, June, *Singular points of second order systems of real differential equations.*

Dorothy McCoy, Iowa, July, phychology, *The complete existential theory of eight fundamental properties of topological spaces.*

N. H. McCoy, Iowa, June physics, *Commutation formulas in the algebra of quantum mechanics.*

Sister Marie Cecilia Mangold, Catholic, June, education, *On the loci of vertices of singly infinite systems of triangles circumscribed about a fixed conic.*

R. H. Marquis, Chicago, June, *A comparison of existing theories of algebraic numbers, and the isolation of an abstract theory.*

H. A. Meyer, Iowa, June, physics, *On certain inequalities with applications in actuarial science.*

E. L. Mickelson, Minnesota, July, physics, *On the approximate representation of a function of two variables.*

E. R. C. Miles, Rice, June, (1) *Boundary value problems for potentials of a single layer in the plane.* (2) *Potentials of general masses in single and double layers. The relative boundary value problems.*

H. J. Miles, California, May, astronomy, *On a generalization of Plücker's surface.*

A. K. Mitchell, Johns Hopkins, June, *The derivation of tensors from tensor functions.*

W. K. Morrill, Johns Hopkins, June, *A problem of ambiency.*

Max Morris, Chicago, June, *Nets of collineations in n-space.*

E. D. Mouzon, Illinois, January, economics, *Equi-modal frequency distributions.*

C. F. Muckenhaupt, Massachusetts Institute of Technology, June, physics, *Almost periodic functions and vibrating systems.*

C. O. Oakley, Illinois, May, physics, *Differential equations containing absolute values of derivatives.*

Gordon Pall, Chicago, June, *Problems in additive theory of numbers.*

G. P. Parkinson, Wisconsin, August, *Pairs of curves in a S_n. A theory of parallelism in sub-spaces.*

F. D. Perez, Chicago, August, *The Hilbert-Schmidt theory of linear integral equations in general analysis for quaternionic-valued functions.*

J. C. Polley, Cornell, June, physics, *Rational surfaces mappable by* $C_{3n}:8A^nB^{n-1}$.

Bryan Priestman, McGill, October, major in physics, minor in mathematics, *Propagation of Quanta.*

W. T. Reid, Texas, June, *Properties of solutions of an infinite system of ordinary linear differential equations of the first order with ordinary boundary conditions.*

J. H. Roberts, Texas, June, *Concerning non-dense plane continua.*

Robin Robinson, Harvard, February, *On the differential geometry of surfaces in non-euclidean space.*

E. B. Roessler, California, May, architecture, *A certain family of autopolar sextics.*

Samuel Silberfarb, Chicago, August, *Representation by indefinite ternary quadratic forms.*

C. A. Spicer, John Hopkins, June, *P-curves of Cremona transformations.*

L. H. Swinford, California, May, Greek, *On rational autopolar quintic curves.*

P. M. Swingle, Michigan, June, *A certain type of continuous curve and related point set.*

S. C. Wang, Columbia, major in physics, *The problem of the normal hydrogen molecule in the new quantum mechanics.*

J. H. Webb, Wisconsin, June, major in physics, minor in mathematics, *Potential distribution due to a buried sphere under the influence of a point electrode.*

Rose A. Whelan, Brown, June, *Approximate solutions of certain general types of boundary value problems from the standpoint of integral equations.*

F. G. Williams, Cornell, June, physics, *Families of plane involutions of genus 2 or 3.*

Kathryn Wyant, Missouri, August, astronomy, *The ideals in the algebra of generalized quaternions over the field of rational numbers.*

Leo Zippin, Pennsylvania, June, *A study of continuous curves and their relation to the Janiszewski-Mullikin theorem.*

Professor K. T. Compton, of the department of physics of Princeton University, has been appointed president of the Massachusetts Institute of Technology.

Dr. E. R. van Kampen, who is Professor Schouten's assistant at the University of Delft, has been appointed associate in mathematics at Johns Hopkins University.

Dr. L. W. Nordheim, of the University of Göttingen, has been appointed visiting professor of theoretical physics at the Ohio State University for the spring quarter.

Dr. T. J. J. See, who, as captain in the Navy, has been in charge of the naval chronometer and time station at Mare Island, Calif., for the last twenty-seven years, has retired from active service.

Professor C. A. Wheeler, of the Connecticut Agricultural College, has retired.

Dr. A. Wintner, who has been Professor Lichtenstein's assistant at the University of Leipzig, and is at present a Rockefeller Fellow at Rome, has been appointed associate in mathematics at the Johns Hopkins University.

Professor J. B. Johnson, head of the department of mathematics at Baylor University, died December 18, 1929.

Mr. R. E. Peterson, instructor in mathematics at Pennsylvania State College, died in the summer of 1929.

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teacher's agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candiates for a first appointment, are invited to put their names on record with the office and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to department or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

The Chauvenet Prize

In the year 1925, the ASSOCIATION established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the CARUS MONOGRAPHS are expository in character and on this score might be included. They carry their own reward in the form of a liberal cash honorarium to each author.

It is believed that clear expositions of mathematical subjects are greatly needed in this country and that the CHAUVENET PRIZE will tend to stimulate such production.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period 1929-1931.

Note that the prize is to be awarded only to a *member* of the ASSOCIATION—one more of the many good reasons for membership.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.
INDIANA, Earlham College, May 2-3.
IOWA, Ames, Iowa, May 2-3.
KANSAS, February 15.
KENTUCKY, Lexington, Ky., April 5.
LOUISIANA-MISSISSIPPI, Cleveland, Miss., March 7-8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA. May 10
MICHIGAN, Ann Arbor, Mich., March 22.
MINNESOTA, Carleton College, May 17.
MISSOURI.
NEBRASKA, Peru, Neb., May 9.
OHIO, Columbus, Ohio, April 3.
PHILADELPHIA, Philadelphia, Pa., November 29.
ROCKY MOUNTAIN, Denver, Colo., April 11-12.
SOUTHEASTERN, Atlanta, Ga., May 2-3.
SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, Calif., March 8.
TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE SIXTEENTH ANNUAL MEETING OF THE KANSAS SECTION

The sixteenth annual meeting of the Kansas Section of the Mathematical Association of America was held in the High School building, Topeka, on February 15, 1930. Professor C. H. Ashton, University of Kansas, chairman of the Kansas section, presided.

There were fifty-eight in attendance, among them the following twenty-seven members of the Association: W. H. Andrews, Maud Arnett, C. H. Ashton, Wealthy Babcock, Leon Battig, Florence Black, E. E. Colyer, A. S. Croom, Lucy T. Dougherty, W. H. Garrett, W. A. Harshbarger, A. J. Hoare, Emma Hyde, W. C. Janes, C. F. Lewis, O. B. Loewen, Anna Marm, Nina McLatchey, U. G. Mitchell, O. J. Peterson, A. W. Philips, C. A. Reagan, B. L. Remick, J. A. G. Shirk, G. W. Smith, J. J. Wheeler, A. E. White.

The following officers were elected: Chairman, Professor Emma Hyde, Kansas State Agricultural College; Vice-chairman, Professor J. J. Wheeler, University of Kansas; Secretary-treasurer, Lucy T. Dougherty, Junior College, Kansas City.

The forenoon session was a joint meeting with the Kansas Association of Mathematics Teachers, Professor C. H. Ashton presiding. Professor C. A. Reagan, of Friends University, spoke on "Creating enthusiasm for mathematics", especially by helping the student catch the spirit, and realize the power and beauty of the subject. Professor W. Irwin, of the economics department of Washburn College, spoke on "An outsider's appreciation of mathematics", pointing out that, according to current economic theory, there must be a certain volume of gold to maintain a certain volume of trade and that there must be some law which will determine this ratio; and he challenged the mathematicians to discover the solution, and devise a formula which will serve as a basis for preserving prosperity, and preventing panics.

The other papers presented before the joint session were:

1. "Some fundamental ideas in relativity theory", by Professor J. J. Wheeler University of Kansas.

2. "An unusual triple conjunction of Jupiter and Saturn," by Professor W. H. Garrett, Baker University.

Abstracts of these papers follow:

1. The paper by Professor Wheeler was prompted by the fact that a quarter century of relativity theory and a decade of experimentation designed to check its conclusions have now elapsed. Attention is called to work of earlier mathematicians that has made a concise mathematical theory of relativity possible. The study of surfaces begun by Gauss and greatly extended and generalized by Riemann has been of inestimable value. The vector concept of Gibbs and Heaviside may be said to have led, sometimes by devious paths, to the calculus of Ricci and Levi-Civita.

In his special theory Einstein has two particularly bold and outstanding postulates; the "principle of relativity" and the "principle of constant light velocity." In the general theory enunciated ten years later there is found one equally bold and important: the "principle of equivalence" of a gravitational field to a transformation of coordinates. As to the evidence justifying these physical assumptions and supporting the conclusions drawn, it is noted that data has been sought in the three lines pointed out by Einstein himself: (1) rotation of the line of apsides of a planetary orbit; (2) the bending of a ray of light in a gravitational field; (3) the shift of lines in the spectrum of certain types of stars. The evidence has perhaps convinced all who wished to believe and many others have passed from doubt to acceptance while many of the original skeptics are skeptics still.

2. Professor Garrett gave first a brief survey of the development of celestial mechanics, emphasizing particularly the contribution made in the theory of Jupiter and Saturn by G. W. Hill. By means of his sixty-eight tables of Jupiter and seventy-one tables of Saturn, the ephemerides of the two planets were computed from July, 1940 to March, 1941 in heliocentric coordinates and then transformed to geocentric coordinates. It was shown that during this period Jupiter and Saturn will be in conjunction three times with respect to the earth, an event of unusual occurrence. The presentation included diagrams, two of which showed the motions of the planets with respect to the sun and the earth at the time of conjunction. This problem was undertaken by Professor Garrett at the suggestion of Director E. B. Frost while a member of the 1929 summer staff at Yerkes Observatory.

The Associations met in separate session for the afternoon meetings, and the following papers were presented before the Kansas section of the Mathematical Association of America:

3. "Application of mathematics to the solution of physical problems," by Professor J. D. Stranathan, department of physics, University of Kansas, by invitation.

4. "Singular solutions of differential equations," by Professor B. L. Remick, Kansas State Agricultural College.

5. "On the rational plane quintic with three cusps," by Professor O. J. Peterson, Kansas State Teachers College, Emporia.

6. "On plane quartic curves obtained by quadratic transformation of curves of lower order," by Mr. Leon Battig, Kansas State Agricultural College.

Abstracts of these papers follow:

3. The purpose of the paper was to convey a fair idea of just how mathematics is used in the solution of physical problems. The physicist has discovered simple laws which describe the behavior of a system under certain well-defined conditions. Hence, in a complex problem with given conditions, he can usually recognize one or more relations existing among the physical quantities involved. These relations are written formally as mathematical equations. It is always the physicist's hope to discover as many in-

dependent relations as there are involved unknown physical quantities, and to solve these equations simultaneously for the desired physical quantity. While the group of relations may consist of algebraic, differential, or integral equations, it is true that most physical behavior can be phrased in terms of differential equations; hence the need for thorough training in this branch of mathematics. After solution of the group of equations, it is the physicist's hope to interpret these results in terms of physical behavior. As an illustration of the points emphasized, the problem of the simple pendulum was carried through, showing how the free or natural motion of the pendulum may be oscillatory or non-oscillatory, depending on the frictional resistance encountered in swinging. The frequency of oscillation was shown to be independent of the mass of pendulum, of the amplitude of swing (approximately), and of the frictional resistance (approximately). The problem was made more general by considering the motion of one pendulum hung from another. Interpretation of the mathematical solution predicted beats between the forced and the natural oscillations of the lower pendulum, the phenomenon of resonance, and the relative phase of forced vibration and excitation. All of these conclusions drawn from mathematical treatment were experimentally demonstrated to be correct.

4. This paper takes account of the connection between the envelope of a system of integral curves and the singular solution of the corresponding ordinary differential equation. In particular it contains a brief discussion of the extraneous loci which may be found when seeking the envelope and a statement as to their distribution; also a short historical sketch.

5. This paper limits itself to a study of the positions of the six singular points of a three-cusped rational plane quintic curve. If the plane be divided into three sets of four triangles by the six lines determined by four singular points of which either one or three are cusps, then the two remaining singular points are in triangles of the same set.

6. Mr. L. Battig's paper treated of the two classes of plane quartic curves, rational and elliptic, that may be obtained by quadratic transformation of conics and plane cubics. With the exception of quartics with a triple point all rational quartics may be thus obtained from conics. Quartics with a triple point result from the transformation of singular cubics. Elliptic quartics (of deficiency one) result from the transformation of properly placed non-singular cubics (of deficiency one), since deficiency is unchanged under quadratic transformation.

LUCY T. DOUGHERTY, *Secretary*

THE MARCH MEETING OF THE MICHIGAN SECTION

The seventh annual meeting of the Michigan Section of the Mathematical Association was held at the University of Michigan, Ann Arbor on March 22, 1930 in conjunction with the Michigan Academy of Science, Arts, and Letters.

Professor R. C. Shellenbarger, chairman, presided at the morning program and at the luncheon meeting. Professor Theodore Lindquist, chairman elect, presided at the afternoon program.

Fifty-seven persons registered, but there were others in attendance. Thirty-three members of the Association who attended the meeting are as follows: N. H. Anning, W. L. Ayres, J. W. Baldwin, W. D. Baten, Stanley Bolks, W. M. Borgman, Jr., J. W. Bradshaw, J. B. Brandeberry, Carl J. Coe, J. J. Corliss, S. E. Crowe, Albertus Darnell, Watson M. Davis, Lloyd C. Emmons, J. P. Everett, Peter Field, K. W. Folley, W. B. Ford, J. W. Glover, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, D. K. Kazarinoff, Theodore Lindquist, H. L. Olson, G. Y. Rainich, Louis J. Rouse, R. H. Schoonover, R. C. Shellenbarger, E. R. Sleight, G. G. Speeker, T. O. Walton, J. B. Winslow.

The following officers were elected for the ensuing year: Chairman, Theodore Lindquist, Michigan State Normal College; Secretary-treasurer, L. A. Hopkins, University of Michigan; Member of the Executive Committee, Clark L. Herron, Hillsdale College. Attention was called to the work of the two special committees which had been appointed in 1929 and whose reports were scheduled on the program:—first, the committee of which Professor L. C. Plant was chairman on the "Content of Mathematics Courses"; and second, the committee on "Legal Requirements of Mathematics Teachers" with Professor L. C. Emmons as chairman.

The following fifteen papers were read:

1. "On the motion of a top," by Mr. J. J. Corliss, University of Michigan.

2. "A note on cyclotomic equations," by Professor N. H. Anning, University of Michigan.

3. "Transformations with preservation of solid angles," by Professor G. Y. Rainich and Mr. E. H. Hildebrandt, University of Michigan.

4. "Content of mathematics courses," by Professor L. C. Plant, Michigan State College.

5. "On the normals to the parabola," by Mr. L. C. Mathewson, presented by Mr. Marvin Pahl, Albion College.

6. "Some generalizations of Pappus (Guldin's) theorems," by Mr. D. K. Kazarinoff, University of Michigan.

7. "On a certain system of differential equations," by Mr. G. C. Munro, University of Michigan.

8. "Combining certain probabilities which are constant," by Dr. W. D. Baten, University of Michigan.

9. "The evaluation of a certain generalized determinant," by Mr. W. O. Menge, University of Michigan,

10. "The Borel theorem and its generalizations," (The Chauvenet Prize Memoir) by Professor T. H. Hildebrandt, University of Michigan.

11. "A comparison of two expansions of the inverse sine function," by Professor A. L. Nelson, College of the City of Detroit.

12. "Constructional representation of numerical values involving complex

radicals, index 2, and its application in the inscribing or circumscribing of the general group of regular polygons," by Mr. C. E. Smith, Redford High School.

13. "Transformations of nets and surfaces," by Professor V. G. Grove, Michigan State College.

14. "Relation of mathematics to electrical engineering," by Professor J. H. Cannon, University of Michigan. (By special invitation).

15. "Legal requirements of mathematics teachers," by Professor L. C. Emmons, Michigan State College.

LOUIS A. HOPKINS, *Secretary*

THE NINETEENTH MEETING OF THE IOWA SECTION

The nineteenth meeting of the Iowa Section of the Mathematical Association of America was held with the Iowa Academy of Science at Iowa State College, Ames, Iowa, on May 2 and 3, 1930.

The attendance was about fifty, including the following twenty-eight members of the Association: F. A. Brandner, E. W. Chittenden, L. M. Coffin, Julia T. Colpitts, N. B. Conkwright, Marian E. Daniells, J. M. Earl, C. W. Emmons, C. H. Fischer, Annie W. Fleming, C. Gouwens, Gertrude A. Herr, E. C. Ingalls, F. M. McGaw, J. V. McKelvey, M. M. McKelvey, E. E. Moots, I. F. Neff, J. F. Reilly, H. L. Rietz, B. D. Roberts, Maria M. Roberts, E. R. Smith, G. W. Snedecor, J. S. Turner, L. E. Ward, C. W. Wester, Roscoe Woods.

The section chairman, Professor E. W. Chittenden, presided at both the Friday afternoon and Saturday morning sessions, relieved for a time by the Vice-chairman, Professor L. M. Coffin. Dinner was enjoyed together at the Memorial Union Friday evening and was followed by a short toast program.

At the business session officers for 1930–31 were elected as follows: Chairman, G. W. Snedecor, Iowa State College; Vice-chairman, E. C. Ingalls, Iowa Wesleyan College; Secretary-treasurer, J. F. Reilly, University of Iowa.

The necrology committee made the following report: "Since our last meeting, the Iowa Section of the Mathematical Association of America has lost through death two of its esteemed members—Professor E. A. Pattengill of Iowa State College, and Professor Fred Reusser of Buena Vista College. In recognition of their services as teachers of collegiate mathematics, their wholesome influence among students in their institutions, and their cooperation in furthering mathematical education, be it resolved that we express our deep sense of loss by a rising vote, and that this resolution be spread on the minutes and that the secretary send a copy to Mrs. Pattengill and to the nearest available relative of Mr. Reusser." H. L. Rietz, E. R. Smith, Committee.

A vote of thanks was tendered the department of mathematics at Iowa State College for the courtesies shown the section, and for the excellent arrangements made for our meeting.

The following recommendation was unanimously approved: "That a subject matter requirement be adopted for the certification of teachers of high school mathematics in Iowa, and that such subject matter requirement conform to that proposed by the North Central Association of Colleges and Secondary Schools."

The program consisted of eighteen papers, and an address by the retiring chairman, as follows:

1. "The distribution of the zeros for functions defined by a Heine series", by E. R. Smith, Iowa State College.

2. "Stencils in geometrical construction", by F. M. McGaw, Cornell College.

3. "Conformal and isohedral mapping of the earth on a plane", by Orlando C. Kreider, Iowa State College, by invitation.

4. "On the summation $\sum x^n$", by J. F. Reilly, University of Iowa.

5. "A lemniscate potential boundary value problem", by Archie Higdon, Iowa State College, by invitation.

6. "On certain appropriate scales in the graphic representation of frequency", by H. L. Rietz, University of Iowa.

7. "An algorithm for writing coefficients of a polynomial with known zeros," by C. W. Wester, Iowa State Teachers College.

8. "Progressive totals and summation as a substitute for accumulative multiplication", by A. E. Brandt, Iowa State College, by invitation.

9. "A note on the extraction of the square root on a calculating machine," by A. E. Brandt.

10. "Some theorems concerning parabolas having a common focus", by B. D. Roberts, Parsons College.

11. "Statistical control of a grading system," by G. W. Snedecor, Iowa State College.

12. "Some curves associated with two given curves", by C. H. Fischer, University of Iowa.

13. "The decomposition of $(x^p-1)/(x-1)$ for $p=41$, and $p=43$", by Cornelius Gouwens, Iowa State College.

14. "A representation of the law of mortality", by Arthur Ollivier, University of Iowa.

15. "The factors of a^n-1, $n\leqq 50$, $a=3, 5, 6, 7, 10, 11, 12$", by J. S. Turner, Iowa State College.

16. "Annuities certain for fractional periods", by J. F. Reilly, University of Iowa.

17. "The limaçon boundary value problem", by Daniel Hutton, Iowa State College, by invitation.

18. "An elementary proof of Descartes' rule of signs", by C. W. Wester, Iowa State Teachers College.

Abstracts of these papers follow:

1. Starting from results obtained by E. B. Van Vleck and Pringsheim, Professor Smith showed that the zeros of a function defined by a Heine series

are in any region finite in number. He also showed the possibility of defining a region about the origin which is entirely free of zeros.

2. Professor McGaw exhibited three stencils that he had made for the purpose of drawing accurate diagrams in his course in "Modern geometry". Stencil number one—the general triangle with about thirty points perforated, such as incenter, circumcenter, nine-point center, centroid, excenters, symmedians, Brocard points, etc. Stencil number two—the quadrilateral and quadrangle with all the necessary points for development of harmonic relations. Stencil number three—A general figure giving convenient cross-ratio forms for particular values.

3. In his discussion of Mercator's conformal projection Mr. Kreider found the ratio of the area of a portion of the sphere between two meridians to the corresponding area on the map in terms of the latitude. The paper was illustrated by several maps. Equal area projection was also discussed.

4. In his paper Professor Reilly emphasized the necessity of stating the final term in the usual formula given for the summation $\sum x^n$.

5. In his paper Mr. Higdon discussed a solution of the equation $\nabla^2\psi=0$ for the lemniscate by the use of definite integrals. The method involved the mapping of the region into a circle and obtaining the corresponding solution by means of a Poisson's integral which may be reduced to elliptic integrals of the first and second kinds.

6. In this paper Professor Rietz discussed the significance and usefulness of changes in scale in the graphic representation of certain frequency distributions. In particular special consideration was given to the general nature of distributions whose analysis is likely to be facilitated by the scales of arithmetic probability paper and of logarithmic probability paper.

7. Letting $s(r, k)$ represent the sum of the products r at a time of the first k numbers in the set $a_1, a_2, \cdots, a_n$, Professor Wester showed that $s(r, k) = s(r, k-1)+a_k s(r-1, k-1)$. This suggested arranging these sums, for purposes of calculation, in an array with k for the number of the column and r for the number of the row. The kth column then will contain the coefficients, after the first, of the polynomial whose zeros are the numbers $-a_1, -a_2, \cdots, -a_n$, and whose leading coefficient is unity.

8. A method was presented by Professor Brandt which enables one to eliminate all multiplication from problems involving the sums of squares or of products such as are met with in correlation studies and in inventories. This method has been developed for use with punched cards and sorting and tabulating machines. An explanation of the method step by step and an illustrative problem were given.

9. The fact that the sum of the first n odd numbers is n^2 and that any number can be expressed in the form $a^2+(2a+b)b+R$ enables one to use division in extracting square roots. Any number of places can be determined by subtracting successive odd numbers and an equal number of places further may be secured by subtraction. The true root will be equal to or less than the

one extracted but can never be less by more than one in the last place. Professor Brandt showed that this method is especially useful in connection with calculating machines having automatic division. The method was also extended to the extraction of cube roots.

10. By means of the orthogonality of confocal parabolas Professor Roberts showed that two parabolas having a common focus and principal axes inclined at an angle α intersect at an angle $\alpha/2$, that if tangents be drawn to such parabolas at the points where a line from the focus intersects them, these tangents intersect at an angle $\alpha/2$, and that the locus of the intersection of such pairs of tangents is the "radical axis" of the parabolas.

11. The methods in vogue for controlling grading systems statistically are inadequate in this particular: they fail to make allowance for differences in ability among the groups to which the control applies. Such differences arise from (1) sectioning according to ability, (2) random sampling, and (3) differences in standards among departments and colleges in the same institution. Professor Snedecor showed that differences in ability among groups can be evaluated by using standard tests and the methods of multiple correlation. The result of an adequate control is that a given grade represents the same standard of attainment in every group of students in the institution.

12. In his paper Mr. Fischer suggested a general method of deriving curves from two given rational curves illustrated by the derivation of an ellipse from two concentric circles. The method consists of forming various combinations of the coordinates of two given rational curves, expressed in parametric form.

13. In Kraitchik's "Recherches sur la Théorie des Nombres" (1924) are displayed the polynomials Y and Z of the well known decomposition of $4X = 4(x^p-1)/(x-1)$ into the form $4X = Y^2 - (-1)^{(p-1)/2}\, pZ^2$ for p an odd prime. The largest value of p there given is $p=37$. Legendre has given a method of obtaining these coefficients but it breaks down for $p=41$ and $p=43$. In this paper Professor Gouwens gave the polynomials Y and Z for these two values of p.

14. In his paper Mr. Ollivier reported on an attempt to represent a portion of the "American Experience Table of Mortality" by means of a parabola and a cosine series.

15. Professor Turner in his paper exhibited the factors of a^n-1, $n \leqq 50$, $a=3, 5, 6, 7, 10, 11, 12$, giving 48 results not found in Bickmore's paper (Messenger of Mathematics, vol. 25, p. 43).

16. In this paper Professor Reilly suggested a definition of the annuity symbols for fractional periods based upon interpolation with higher orders of differences.

17. Mr. Hutton gave a solution of the problem in potential theory where the contour is a limaçon.

18. Basing his proof on that given by L. E. Dickson in *First Course in Theory of Equations*, Professor Wester gave a shorter elementary proof of Des Cartes' rule of signs.

An abstract of Professor Chittenden's address on "General topology" follows:

Topology is concerned with the properties of sets of points which are invariant under bicontinuous one to one transformations. General topology applies the methods of the theory of abstract sets founded by Fréchet to the study of these questions. In this address the following topics were emphasized: the influence of the topological viewpoint on the development of the theory of abstract sets, the dimension theory of Urysohn and Menger, and the logical interrelations of the fundamental properties of abstract sets.

J. F. REILLY, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at Carleton College, Northfield, Minn., on Saturday, May 17, 1930. Sessions were held at 11:00 o'clock and at 2:30 o'clock, with luncheon in the dining hall of Carleton College. Professor Fredrick Wood, chairman of the section, presided at both sessions.

Eighty-one persons attended the meeting, including the following twenty-four members of the Association: R. W. Brink, A. Bogard, W. H. Bussey, Elizabeth Carlson, June Constantine, H. H. Dalaker, Margaret Eide, Gladys Gibbens, C. H. Gingrich, W. L. Hart, Dunham Jackson, C. M. Jensen, W. H. Kirchner, W. H. McEwen, Marie Ness, M. F. Rosskopf, Inez Rundstrom, Edward Saibel, A. K. Solum, F. J. Taylor, Ella Thorp, Marion B. White, Gilbert Winklemann, Fredrick Wood.

Following the afternoon session those attending the meeting witnessed the beautiful May Fete given by the students of Carleton College.

During the afternoon session a vote of thanks was adopted in appreciation of the cordial hospitality of Carleton College and of the efforts of its department of mathematics. Officers for the following year were elected as follows: Chairman: C. H. Gingrich, Carleton College; Secretary, A. L. Underhill, University of Minnesota; Members of the Executive Committee: H. H. Dalaker, College of Engineering, University of Minnesota; Mrs. Margaret Eide, State Teachers College, River Falls, Wis.; A. Bogard, College of St. Teresa, Winona, Minn.

The following eight papers were presented:

1. "Some remarks on the *Rhind Mathematical Papyrus* and the *Source Book in Mathematics*", by Professor W. H. Bussey, University of Minnesota.

2. "A derivation of the equation for the normal surface", by Miss Marian Wilder, University of Minnesota.

3. "Some methods for the classical problems of geometry", by Professor Gladys Gibbens, University of Minnesota.

4. "Some qualities that should be possessed by a teacher of mathematics", the Chairman's address, by Professor Fredrick Wood, Hamline University.

5. "Mathematics as applied to biological problems", by John Stanley, University of Minnesota.

6. "A lemma in the theory of Fourier series", by Myron Rosskopf, University of Minnesota.

7. "The combination of dependent observations", by Phillip J. Roulon, University of Minnesota.

8. "A proof of H lder's inequality", by W. H. McEwen, University of Minnesota.

Abstracts of the papers follow:

1. Mr. Bussey spoke briefly of two new books recently published: "The Arnold Buffum Chace Edition of the Rhind Mathematical Papyrus" and "A Source Book in Mathematics" by David Eugene Smith. The purpose of his talk was to stress the desirability of having the two books available in every department of mathematics library for use by students and faculty.

2. The derivation of the equation of the normal surface can be based on the assumption of a frequency function $z=Cf(xy)$, where $f(xy)=e^{-ax^2-2bxy-cy^2}$. From the definitions of the correlation coefficient and the standard deviations, equations in a, b, c are derived. The solutions of these equations, together with a determination of C, when substituted in the frequency function, give the standard form of the normal equation.

3. This paper will appear in an early number of the Monthly.

4. The paper suggested that a teacher should be a genuine student of the content and historic background of mathematics, and have a broad training in cultural subjects. He should love his subject and have imagination and intellectual curiosity. He should be interested in students, be gentle and kind, fair and square, and remain young enough to understand his students. Illustrations were given from various fields of mathematics to show the use of these qualities in stimulating the interest of the student and awaking him to the possibilities of doing original thinking.

5. The recent great advances made in physics by the use of mathematical reasoning lead to the question of whether or not living material lends itself to such treatment. It is possible to define several types of what may be called "biological entities". These are first, simple and compound, where the simple entity is a single organism, the compound, a group of organisms. Further division may be made into simple thinking, and simple non-thinking, compound with simple non-thinking elements, compound with simple thinking elements, thinking singly, and finally, compound with simple thinking elements thinking cooperatively and more or less in unison.

Examples of these entities are given and the feasibility of mathematical treatment is discussed in each case.

Certain analogies exist between these types. From a mechanistic viewpoint, the main characteristic of living matter is extreme heterogeneity of structure.

Under certain restrictions, the number of thoughts possible to a human be-

ing is finite, and small creatures are necessarily incapable of intricate thought.

Finally, the heterogeneous living system is oriented by means of a progressive extension, starting with the rigid motion, with all particles alike, passing en route through such systems as the non-rigid, compressible motion, with elements unlike, etc.

6. If $f(x)$ is a continuous function in the interval $(-\pi, \pi)$ with the period 2π and if $\int_{-\pi}^{\pi} f(x) \cos nx\, dx = 0$ and $\int_{-\pi}^{\pi} f(x) \sin nx\, dx = 0$ for all $n = 0, 1, 2, \cdots$, it follows that $f(x) \equiv 0$. The lemma allows a very concise proof of the δ, ϵ part of the proof of convergence in Fourier series.

7. The problem of this paper is suggested by the situation which arises when a series of tests are applied to an experimental group and to a control group of students in order to test the efficacy of some educative process. In case the tests are independent, one can sometimes infer a significant difference between the groups even when the difference for each test is not significant. The paper points out the danger of applying the method when the tests are correlated; and it suggests methods of getting the best value from a set of correlated observations, and of measuring its significance. These methods will be developed in a later paper.

8. This paper offers a proof of the relation

$$\int_a^b \phi dx \leqq (b-a)^{(r-1)/r}\left\{\int_a^b \phi^r dx\right\}^{1/r},$$

where $\phi > 0$ and r is any real number $\leqq 1$. It is first shown by means of Schwarz's inequality that

$$\int_a^b \phi^i dx < (b-a)^{1/(i+1)}\left\{\int_a^b \phi^{i+1} dx\right\}^{1/(i+1)}, \qquad i = 1, 2, \cdots, (n-1),$$

and by combining these $n-1$ relations, that the desired inequality holds for any positive integral value n. This fact is used to prove that

$$\int_a^b \psi^p dx \leqq (b-a)^{1-p}\left\{\int_a^b \psi dx\right\}^p$$

where p is any real number <1, and the proof is then completed by identifying ϕ^r with ψ and $1/r$ with p.

R. W. Brink, *Acting Secretary*

THE MAY MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-seventh regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at Goucher College, Baltimore, Md., on Saturday, May 10, 1930. Sessions were

held in the morning and in the afternoon; Professor W. F. Shenton, Chairman of the Section, presided at both sessions.

Fifty-four persons attended the meeting, including the following forty-five members of the Association: O. S. Adams, Beatrice Aitchison, G. F. Alrich, R. N. Ashmun, H. G. Avers, Clara L. Bacon, W. J. Berry, G. A. Bingley, C. C. Bramble, J. L. Clayton, Teresa Cohen, A. Cohen, G. R. Clements, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, J. B. Eppes, Mary Ewin, P. J. Federico, Michael Goldberg, W. M. Hamilton, F. E. Johnston, H. P. Kaufman, L. M. Kells, W. D. Lambert, C. L. Leiper, Florence P. Lewis, Florence M. Mears, F. D. Murnaghan, C. H. Rawlins, Jr., Lowell Reid, W. F. Reynolds, A. W. Richeson, H. M. Robert, Jr., H. A. Robinson, R. E. Root, W. F. Shenton, T. H. Taliaferro, J. H. Taylor, Marian M. Torrey, John Tyler, P. Wernicke, J. Williamson, R. C. Yates, O. Zariski.

The following officers were elected: *Chairman*, Clara L. Bacon, Goucher College; *Secretary*, Edgar W. Woolard, The George Washington University; additional members of the executive committee, J. Williamson, The Johns Hopkins University, and G. A. Bingley, St. Johns College.

During a discussion which followed the program, encouragement was given to the preparation of papers appropriate to the general membership of the Association—papers more in line with the teaching membership than the graduate membership. The executive committee was asked to make an early effort to arrange programs for the meetings, and to provide for the entertainment of the group when no formal invitation was received from some institution.

An invitation from the chairman to hold the next meeting of the Section at American University was accepted.

The following five papers were presented:

1. "On linear subspaces of a given space," by Dr. Zariski, Johns Hopkins University.

2. "Mathematical aspects of a theory of the frequency distribution of species," by Florence P. Lewis, Goucher College.

3. "The potential of a spherical zone," by W. D. Lambert, U. S. Coast and Geodetic Survey.

4. "Some episodes from the history of the infinite," by Tobias Dantzig, University of Maryland.

5. "On a problem of regions," by H. A. Robinson, Johns Hopkins University.

Mr. O. S. Adams also presented a brief statement of several of his recent problems in the American Mathematical Monthly.

An abstract of one of these papers follows:

3. Mr. Lambert's paper dealt with the Newtonian potential of a very thin layer of attracting matter on the surface of a sphere and bounded by a circle of any size, the attracted point being also on the surface. This problem has important geophysical applications. Woodward ("On the form and position of sea level," U. S. Geological Survey Bulletin No. 48, Washington, 1888) showed

that the problem could be reduced to the evaluation of an elliptic integral, but did not actually use this method. It turns out to be a complete elliptic integral of the third kind and in this paper three methods of evaluating it numerically are given. A method is also given for computing the potential of any layer of constant surface density bounded by arcs of great circles, the attracted point being on the surface. The calculation likewise involves complete elliptic integrals of the third kind. In both problems the key to the solution is the suitable choice of the variable of integration. It is expected that these developments will be incorporated in a publication of the U. S. Coast and Geodetic Survey.

EDGAR W. WOOLARD, *Secretary*

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SOME CONSTRUCTIONS FOR THE CLASSICAL PROBLEMS OF GEOMETRY

By GLADYS GIBBENS, University of Minnesota

The purpose of the following paper is to give a brief outline of some of the methods used by Mr. Gottfried Lenzer of St. Paul, Minnesota for handling the classical construction problems of geometry: the squaring of a circle, the duplication (or halving) of a cube, and the trisection of an angle. In March, 1928, the University of Minnesota was notified that Mr. Lenzer had bequeathed to the University a series of copyrighted drawings and explanatory notes concerning these problems, and it became my interesting task to go through the bequest, reorganize the work, and publish the results. While Mr. Lenzer was apparently not acquainted with the literature on the subject, he was not a circle-squarer in the sense of one who believed he had accomplished the impossible, but rather a man who found interest and companionship during many years of a solitary existence in the study of the subject and the execution of the constructions he had formulated. Although his results do not constitute any material addition to knowledge in the present state of things, they are of interest in themselves as well as for their revelation of the mind of a man who regarded them as his most cherished possession but who made no attempt to publish them during his lifetime.

Mr. Gottfried Lenzer was a native of Germany who lived in St. Paul for many years, and who was an engineer by profession. I do not know how his interest in these problems was originally awakened, but to judge by the dates on the drawings it was of long standing, and he continually reworked the constructions and developed his methods, trying to inter-relate the problems. His bequest consists of some sixty drawings, dating from 1911 to 1927, when the entire set was copyrighted, and a scant set of explanatory notes. The drawings are all on large paper, and are beautifully executed in colored inks. They are all very complicated, but are symmetric with respect to all convenient lines and points, an obvious esthetic pleasure to the engineer-draughtsman. Mr. Len-

zer's achievement seems quite remarkable for a man who was not mathematically trained, and who at no time attempted to formulate a general method for his constructions, but who handled each special case of the problem under consideration on its own merits; however he must have had some such general method in mind and in the following paper I have attempted to formulate these methods from the drawings and notes.

I. *The Trisection of an Angle*: *A Solution*.

Place the angle AOA' (Figure 1) that is to be bisected so that its internal bisector is the vertical line $Y'OY$ and its external bisector the horizontal line

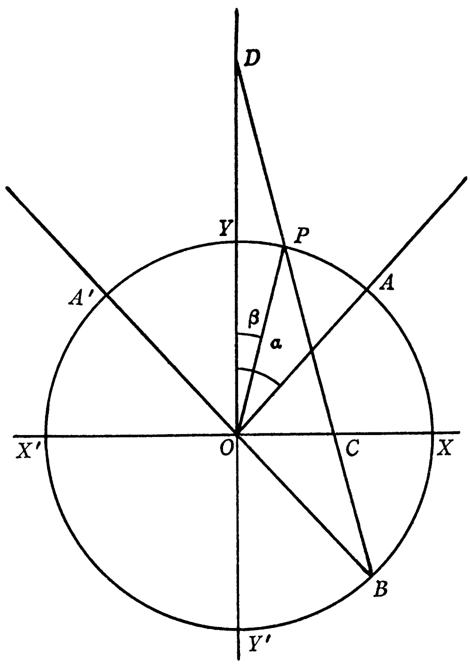

FIG. 1

$X'OX$. Draw a convenient circle having the vertex O as center and meeting the sides of the given angle at A' and A. Extend the side $A'O$ of the given angle to meet this circle again at B. Now mark off on a straight edge two radii, CP, PD of the circle, and shift the straight edge through B until C lies on OX, P on the circle, and D on OY. It is obvious that if $CP=OP$, then $PD=OP$, and the three conditions can be fulfilled. (It is at this point that the construction ceases to be one by ruler and compass in the classical sense; we are using a marked ruler, and what might be termed three-point contact with a given figure.) Draw OP; it is a trisecting radius and its symmetric OP' with respect to OY is the other. Proof. Let $\alpha = \angle AOD = \frac{1}{2}\angle AOA'$, $\beta = \angle POD = \angle PDO$;

then $\angle OPC=2\beta=\angle OBP$; $\angle OCP=90°-\beta$; $\angle BOC=90°-\alpha$ and $\angle BCO=90°+\beta$. But in the triangle OCB, the sum of the three angles is 180°, so that $(90°-\alpha)+(90°+\beta)+2\beta=180°$, or $\beta=\frac{1}{3}\alpha$. That is, OP is a trisecting radius for the half-angle AOD and hence for the original angle AOA'.

If we wish to trisect the remaining arc of the circle $AXY'X'A'$, we shift the marked rule in similar fashion through B so that C is on OX, P on the circle and D on OY'. The proof goes through in the same way.

2. *Squaring the Circle: An Approximation.*

Mr. Lenzer's drawings are symmetric with respect to the horizontal and vertical diameters of the circle, but in order to avoid confusion of notation I shall give only one of the constructions involved; the remaining ones are obvious. Let the circle with center O (Figure 2) be the circle to be squared; for

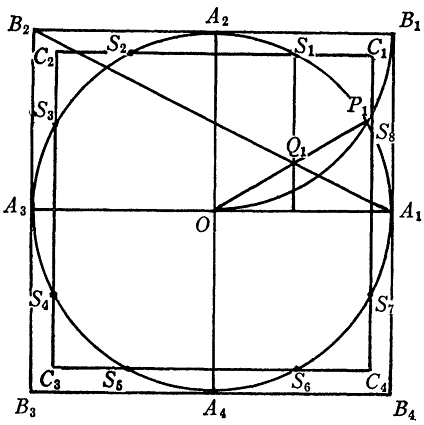

FIG. 2

convenience consider its radius as one unit. Circumscribe about it the square $B_1B_2B_3B_4$ with horizontal and vertical sides, labeling the points of tangency A_1, A_2, A_3, A_4 as indicated. With A_2 as center, draw a quarter unit-circle OB_1, meeting the original circle at P_1; draw OP_1 and B_2A_1 meeting at Q_1. At Q_1 erect a vertical line meeting the circle at S_1. This same construction can be made using A_2 for the center of the circle and B_1A_3 for the diagonal of the half square; then seven other points S symmetric to S_1 with respect to A_3A_1, A_4A_2 and the diagonals of the square will be obtained. The square $C_1C_2C_3C_4$ through these eight points has an area approximately equal to that of the given circle.

Closeness of the approximation. If we set up a rectangular coordinate system with A_3A_1 and A_4A_2 as x- and y-axes respectively, then the equation of OP_1 is $y\sqrt{3}=x$ and of B_2A_1 is $x+2y=1$; they meet at Q_1 whose coordinates are $(x=2\sqrt{3}-3,\ y=2-\sqrt{3})$. The vertical line Q_1S_1, $x=2\sqrt{3}-3$, meets the given circle $x^2+y^2=1$ at S_1 whose y-coordinate is given by $y^2=12\sqrt{3}-20$. The

area of the square $C_1C_2C_3C_4$ having $2y$ as its side is $4y^2=4[12\sqrt{3}-20]=3.1384$. The area of the original circle is 3.1416. The approximating square then is too small; its area being .999 times that of the circle. But this error is well within the limits of ordinary drawing; if the original circle has a radius of 10 inches, the side of the approximating square is too short by less than .004 inches, less than the ordinary pen line.

The square whose perimeter is approximately equal to the circumference of the given circle. Inscribe a circle in $C_1C_2C_3C_4$, meeting OS_k at T_k, $(k=1, 2, \cdots, 8)$ Then the square $D_1D_2D_3D_4$ through these points T_k has a perimeter approximately equal to the circumference of the original circle. The proof is analogous to the one just given, and the error is of the same order.

The drawings also include constructions for circling the square, but since it is essentially the inverse of the preceding, no details will be given.

3. *Halving the Cube: An approximation.*

Let $A_1A_2A_3A_4$ (Figure 3) represent one face of the given cube. Divide the half-diagonal OA_1 into four equal parts by the points C_1, D_1, E_1. Draw a circle with C_1 as center and C_1A_1 as radius, and a line through C_1 parallel to A_4A_1.

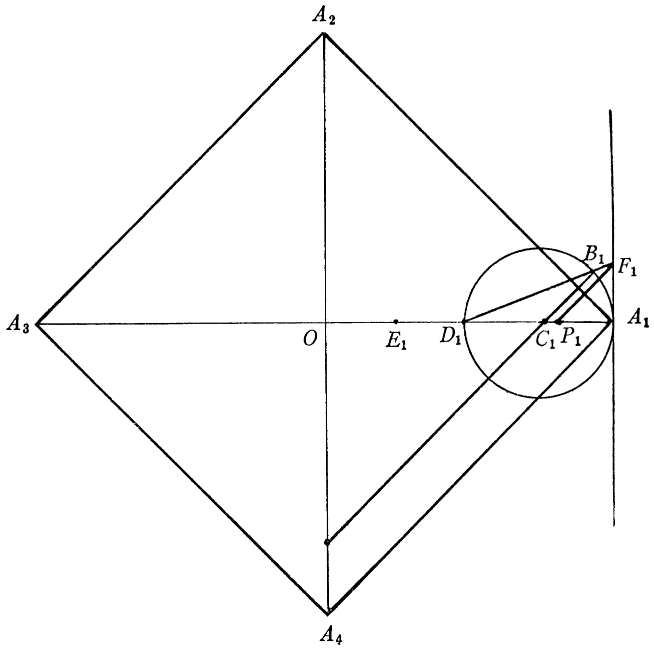

FIG. 3

Let B_1 be their point of intersection. Draw D_1B_1, extending it to meet a parallel to OA_2 through A_1 at F_1. Now through F_1 draw a parallel to C_1B_1, meeting OA_1 at P_1. Then OP_1 is a half-diagonal of one face of the required cube whose volume is approximately equal to half the volume of the original cube.

Analytic discussion of the degree of the approximation. Let us take the half-diagonal of the given cube as 4; then the length of an edge of the cube is $4\sqrt{2}$ and its volume is $128\sqrt{2}$. The equation of the circle C_1 is $(x-3)^2+y^2=1$; it meets the line C_1B_1 whose equation is $y=x-3$ at the point $B_1:(3+\frac{1}{2}\sqrt{2},\ \frac{1}{2}\sqrt{2})$. The equation of B_1D_1 is $x-y(\sqrt{2}+1)-2=0$, meeting A_1F_1 or $x=4$ at $(4, 2\sqrt{2}-2)$. Then the equation of F_1P_1 is $y-2\sqrt{2}+2=x-4$, so that the coordinates of P_1 are $(6-2\sqrt{2},\ 0)$. The edge of the cube having $P_1P_2P_3P_4$ as one face (where P_2,P_3,P_4 are the corresponding points to P_1 on the other diagonals) is $6\sqrt{2}-4$ and the volume of the cube then is $8(3\sqrt{2}-2)^3=16[45\sqrt{2}-58]$. But the volume of a cube just half the original is $64\sqrt{2}$, so the approximating cube is smaller than the desired one, with an error of .3%. If the edge of the original cube were 10 inches, the width of the strip between the sides of the approximating cube and the true half-cube would be less than .002 inches, well within the limits of drawing.

In a single sentence in the notes, Mr. Lenzer states that an analogous method could be used for duplicating a cube, but none of the drawings show such a construction.

4. *Construction of a Regular Pentagon.*

Draw a circle with OA_3 as radius (Figure 4) and a concentric circle whose radius is twice as large. Draw the horizontal and vertical diameters as indicated.

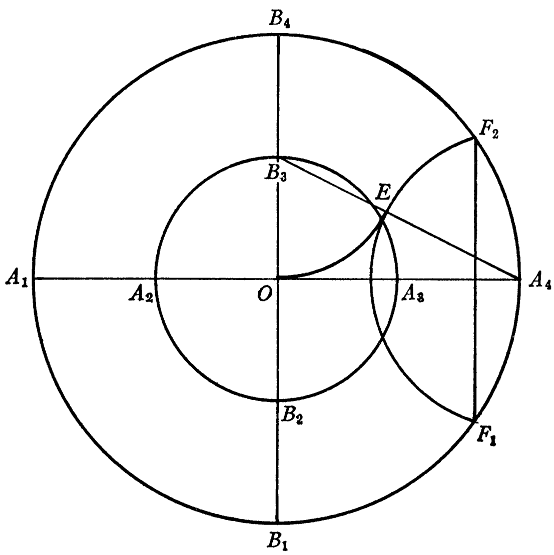

FIG. 4

Draw B_3A_4, meeting a circle with B_3 as center and radius OB_3 at E. Draw a circle with A_4 as center and radius A_4E, meeting the large circle at F_1 and F_2. Then F_1F_2 is a side of a regular pentagon inscribed in the circle $A_1B_4A_4B_1$.

Analytic Proof. Let us take A_1A_4 and B_1B_4 as x- and y-axes respectively, and OA_4 as a unit. Then the equation of B_3A_4 is $x+2y=1$; the equation of the circle with center at B_3 is $x^2+(y-\frac{1}{2})^2=\frac{1}{4}$; and the coordinates of E are $(\frac{1}{5}\sqrt{5}, \frac{1}{10}(5-\sqrt{5}))$. The length (EA_4) is $\frac{1}{2}(\sqrt{5}-1)$ and the equation of the circle with center at A_4 is $(x-1)^2+y^2=\frac{1}{2}(\sqrt{5}-1)$. It meets the original circle $x^2+y^2=1$ at F_1 and F_2 whose coordinates are

$$x = \tfrac{1}{4}(1 + \sqrt{5}) = \cos 36°, \quad y = \mp \tfrac{1}{4}\sqrt{(10 - 2\sqrt{5})}.$$

so that F_1F_2 subtends at 0 an angle of 72° and is a side of the inscribed pentagon.

ON OVALS

By N. HANSEN BALL, MASSACHUSETTS INSTITUTE OF TECHNOLOGY

1. *Introduction.* During recent years many geometers have shown increased interest in those properties of plane curves which do not depend upon their analytical expression. It is surprising how many such properties have already been discovered, although the field has been barely touched. It is the purpose of this note to give a brief review of the most important publications on this subject. While some of these properties hold for any type of curve whatever most of them are valid only for curves which are closed and convex, i.e., which have at most two intersections with any line in the plane. Theorems of the latter type only shall be dealt with here.

2. *The four-vertex and allied theorems.* The four-vertex theorem states that on every closed, convex curve, or oval, there are at least four points where the radius of curvature has an extreme value. In this and other theorems of the same type, the further assumption is made that the radius of curvature is continuous for each point and is never zero. Since an ellipse has exactly four such points, if it can be established that no oval can have less than four, the minimum number is shown to exist and is definitely determined. In 1912, A. Kneser[1] gave a purely geometrical proof of this theorem. Shortly afterward W. Blaschke[2] gave three different proofs for it. One was based on the application of Fourier series and another, due to G. Herglotz, made use only of elementary differential geometry. The third, and most elegant, proof depends only on the equations, valid for any closed curve,

$$(1) \qquad \oint \rho(\phi) \sin \phi d\phi = 0 \text{ and } \oint \rho(\phi) \cos \phi d\phi = 0,$$

where ϕ is the angle between the tangent to the curve and a fixed line in the

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[1] A. Kneser, *Bemerkungen über die Anzahl der Extremen der Krümmung auf geschlossenen Kurven*, Festschrift, Heinrich Weber, 1912.

[2] W. Blaschke, *Die Minimalzahl der Scheitel einer geschlossenen konvexen Kurve*, Rendiconti del Circolo Matematico di Palermo, vol. 36 (1913), pp. 220–222; and *Vorlesungen über Differentialgeometrie*, Vol. I, p. 15.

plane, and $\rho(\phi)$ is the radius of curvature of the oval. This equation can be interpreted as saying that the center of gravity of a circle, loaded with mass of density, ρ, is at the center of the circle. Then, assuming that ρ is continuous and has the same sign at all points on the curve, Blaschke shows that this is impossible if ρ has only two extrema, and furthermore he shows that if the number of extrema of ρ is finite, then they can occur only in an even number. H. Mohrmann[3] has also given a proof of this theorem.

G. Szegö[4] and W. Süss[5] have given proofs of a similar theorem which states that on any oval with a continuous radius of curvature, there are at least three pairs of points such that the tangents at the points of a pair are parallel, and the curvatures are there equal. Szegö proves it by the use of Fourier series, while Süss puts equation (1) in the form

$$\int_0^{\pi/2}\left[s(\phi)\cos\phi - s\left(\frac{\pi}{2}+\phi\right)\sin\phi\right]d\phi = 0,$$

where $s(\phi)=\rho(\phi)-\rho(\pi+\phi)$, and then makes use of the continuity of $s(\phi)$ to show that $s(\phi)$ has at least three zeros in the interval, $0\leqq\phi\leqq\pi$.

Of a similar nature is the theorem[6] in affine geometry that on every oval there are at least six sextactic points, a sextactic point being one where the osculating conic to the oval is stationary. We also have the theorem that there are six points on an oval where the affine normals pass through the center of gravity of the area enclosed by the oval.

Also of interest is the theorem of A. Emch[7] that in every oval at least one square can be inscribed. This is proved by the use of the lemma that two distinct rhombs with corresponding parallel sides or parallel axes can never be inscribed in the same oval. This theorem has been generalized by S. Kakeya[8] to read: In every oval there can be inscribed at least two rectangles similar to a given rectangle, except if the given rectangle is a square.

One of the most striking relations between geometry "im Grossen" and geometry "im Kleinen" is the theorem that if an oval is elliptically curved throughout (i.e., the osculating conic section at every point is an ellipse), any five points of the oval always lie on an ellipse.

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[3] H. Mohrmann, *Die Minimalzahl der Scheitel einer geschlossenen konvexen Kurve*, Rendicont del Circolo Matematico di Palermo, vol. 37 (1914), p. 267.

[4] G. Szegö, *Lösungen zu Aufgaben*, Archiv der Mathematik und Physik, vol. 28 (1920), p. 183.

[5] W. Süss, *Kurzer Beweis eines Satzes von W. Blaschke über Eilinien*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 33 (1924), 2 Abt. p. 32, or Tôhoku Mathematical Journal, vol. 24 (1925), pp. 66–67.

[6] Blaschke, *Differentialgeometrie*, vol. II, pp. 43–46.

[7] A. Emch, *Some properties of closed convex curves in a plane*, American Journal of Mathematics vol. 35 (1913), pp. 407–412.

[8] S. Kakeya, *On the inscribed rectangles of a closed convex curve*, Tôhoku Mathematical Journal vol. 9 (1916), pp. 163–166.

3. *The curvature-centroid and curvature-axis.* In 1838, J. Steiner[9] introduced the curvature-centroid as the point with respect to which the pedal curve of an oval had a minimum area. He showed geometrically that if the oval be considered as loaded with a mass density proportional to the curvature, its center of gravity is the curvature-centroid. N. M. Ferrers[10] and T. Hayashi[11] have given analytical proofs of this theorem. One of these depends on the following fact. If (a, b) are the coordinates of the point under consideration, the area of the pedal curve is

$$A = \frac{1}{2}\oint [(x - a)\sin\phi - (y - b)\cos\phi]^2 d\phi.$$

It is easily shown that the values of a and b determined by the equations $\partial A/\partial a = \partial A/\partial b = 0$ are the coordinates of the center of gravity for the curve when loaded as above. Hayashi also showed that the curvature-centroids of an oval and its evolute are identical. By a method of proof similar to that outlined above for the four-vertex theorem, he showed that the radii vectors of an oval and its pedal curve have each at least four extrema.

B. Su[12] introduced a generalisation of the curvature-centroid by determining that line, which he calls the curvature-axis, about which the oval has a minimum moment of inertia, if it be loaded with a mass of density proportional to its curvature. He shows that the curvature axes of a parallel series of ovals form a pencil of lines, through the common curvature-centroid of the series, which reduce to a single line when and only when

$$\oint \rho(\phi)\sin 2\phi d\phi = \oint \rho(\phi)\cos 2\phi d\phi = 0.$$

In particular the curvature-axes of a series of parallel curves of constant breadth are one and the same line, since the curves of constant breadth satisfy the above conditions.

4. *The extremal chords of an oval.* T. Hayashi[13] gave the first solution of this problem. He divided it into two parts, first, the maximum chord in a given direction, and second, the extremals of these maximum chords. Then by very simple analytical considerations he was able to show that the tangents to the

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[9] J. Steiner, *Von dem Krümmungs-Schwerpuncte ebener Curven*, Crelle's Journal, vol. 21 (1838), pp. 33–63 and 101–133, or *Werke*, vol. 2, pp. 97–159.

[10] N. M. Ferrers, *Note on a geometrical theorem of Mr. Steiner*, Quarterly Journal of Pure and Applied Mathematics, vol. 4 (1861), pp. 92–94.

[11] T. Hayashi, *On Steiner's curvature-centroid*, Science Reports of the Tôhoku Imperial University, vol. 13 (1924), p. 109.

[12] B. Su, *On the curvature-axis of the convex closed curve*, Science Reports of the Tôhoku Imperial University, vol. 17 (1928), pp. 35–42.

[13] T. Hayashi, *The extremal chords of an oval*, Tôhoku Mathematical Journal, vol. 22 (1923), pp. 387–393.

oval at the extremities of the maximum chord in a given direction are parallel, and that the tangents at the extremities of an extremal chord are perpendicular to the chord, i.e., the extremal chords are double normals of the oval.

5. *Various Inequalities.* The majority of the theorems on ovals are relations giving the upper or lower limits to ratios between the area and perimeter, the area and breadth, etc. Fundamental among these is the following inequality,[14] valid for the affine as well as for the Euclidean plane:

$$2/F_{22}\sqrt{(F_{12}^2 - F_{11}F_{22})} \geqq \max \sqrt{(D_1/D_2)} - \min \sqrt{(D_1/D_2)},$$

where F_{11}, F_{22} are the areas of two ovals K_1 and K_2 and D_1, D_2 are the areas of similar and similarly oriented triangles circumscribed about the ovals K_1 and K_2, respectively. F_{12} is the Minkowski mixed area of the two ovals defined by the equation,

$$F_{12} = \oint (x_1 dy_2 - y_1 dx_2).$$

If we let K_2 be a unit circle, we get as a special case of this inequality the theorem, valid only in the Euclidean plane, that

$$L^2 - 4\pi F \geqq \pi^2(\max r - \min r)^2, \tag{2}$$

where L, F are the perimeter and area of an oval and r is the radius of a circle inscribed in a triangle which is circumscribed about the oval.

This is a sharper relation than $L^2 - 4\pi F \geqq 0$ which expresses the well-known isoperimetric property of the circle, a property recognized by the Greek geometers, but first rigorously proved by Minkowski.[15]

If we represent by Δ the breadth, or minimum separation of two parallel tangents, and by D the diameter, or maximum separation of two parallel tangents, of an oval, we have, among others, the following expressions for the lower limit to the area of an oval:

$$\frac{L^2 - 4D^2}{12}; \quad \frac{(L - 2D)\sqrt{(4LD - L^2)}}{4}; \quad \frac{D(L - 2D)}{3}.$$

The first two were proved by T. Kubota[16] from the following two lemmas:

I. If we are given any polygon $B_1B_2 \cdots B_k$ of perimeter L and such that $B_1B_2 = B_1B_3 = \cdots = B_1B_k = D$, otherwise $B_iB_{i+1} < D$ then it is possible to construct a quadrilateral $C_1C_2C_3C_4$ of perimeter L such that

$$C_1C_2 = C_1C_3 = C_2C_3 = C_1C_4 = D \text{ and hence } C_3C_4 = L - 3D$$

with an area less than that of $B_1B_2 \cdots B_k$.

[14] W. Blaschke, *Eine Verschärfung von Minkowskis Ungleichheit für den gemischten Flächeninhalt*, Hamburger Abhandlungen, vol. 1 (1922), pp. 206–209.

[15] H. Minkowski, *Gesammelte Abhandlungen*, vol. 2, pp. 242–249.

[16] T. Kubota, *Eine Ungleichheitsbeziehung über die Eilinien*, Tôhoku Mathematical Journal, vol. 24 (1925), pp. 60–63. T. Kubota, *Einige Ungleichheitsbeziehungen über Eilinien und Eiflächen*, Science Reports of the Tôhoku Imperial University, vol. 12 (1923), pp. 45–65.

II. If we have a convex polygon $A_1A_2 \cdots A_n$ with fixed values of D and L, then we can construct a polygon $B_1B_2 \cdots B_k$, of area equal to or less than that of $A_1A_2 \cdots A_n$, and having the properties assummed in lemma I.

The area reaches the lower limit given by the first expression only when the oval consists of a straight line counted doubly.

The third relation can be derived from the first by making use of the fact that $L \geqq 2D$.

For upper limits to the area of an oval we have the following, in addition to (2) above:

$$\tfrac{1}{4}\pi D^2;\ \tfrac{1}{4}(2\Delta L - \pi\Delta^2).$$

These were also established by Kubota.[17]

Rosenthal and Szâsz have shown[18] that $L \leqq \pi D$, where the equality holds only for curves of constant breadth.

If we let $p(\phi) = -x \sin\phi + y \cos\phi$ be the Minkowski "Stützgerade Funktion" we have

$$L = \oint \rho d\phi \text{ and } F = \tfrac{1}{2} \oint p\rho d\phi$$

from which can easily be derived the theorem,[19] due to Hurwitz and Blaschke, that the perimeter or area of an oval lies between the perimeters or areas, respectively, of the largest and smallest osculating circles.

From the same expressions we also find that

$$R \geqq \tfrac{1}{2}D(1 - 2DL^{-1}) \text{ and } r \leqq \tfrac{1}{2}D,$$

where R, r are the radii of the largest and smallest osculating circles of the oval.

By making use of a Fourier expansion of ρ and $1/\rho$ it can be shown that the average value of *the radius of curvature* of an oval, averaged with respect to s, is equal to or greater than the *radius* of a circle with the same perimeter; and the *curvature* of an oval, averaged with respect to ϕ, is equal to or greater than the *curvature* of a circle with the same perimeter.

In the affine plane we have the two following relations:[20]

$$8\pi^2 F - S^3 \geqq 0\colon 4\pi A - (3\sqrt{3})F \geqq 0,$$

where F is the area, S the affine perimeter, and A the area of the largest inscribed triangle, of an oval.

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[17] T. Kubota, *Einige Ungleichheitsbeziehungen über Eilinien und Eiflächen*, Science Reports of the Tôhoku Imperial University, vol. 12 (1923), pp. 45–65.

[18] Rosenthal and Szâsz, *Eine Extremaleigenschaft der Kurven konstanter Breite*, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 25 (1917), pp. 278–282.

[19] T. Hayashi, *The extremal chords of an oval*, Tôhoku Mathematical Journal, vol. 22 (1923), pp. 387–393.

[20] W. Blaschke, *Differentialgeometrie*, vol. 2, pp. 50, 61.

We also have the theorem[21] that for an oval which is elliptically curved throughout, the area or affine perimeter of the largest osculating ellipse is equal to or greater than the area or affine perimeter, respectively, of the oval; and the equality holds only when the oval is an ellipse. This corresponds to one half of the theorem concerning osculating circles mentioned above.

6. *Conclusion.* There are still a large number of unsolved problems in this field. For instance Blaschke has proposed the following theorems:

I. If a point lies inside all osculating circles of an oval, it also lies inside every circle which has more than two points in common with the oval.

II. If a line lies outside all osculating circles of an oval, it does not touch any circle having more than two intersections with the oval.

Intuitively these theorems seem almost self-evident, but apparently no rigorous proofs have yet been given for them.

It would be very interesting to find out whether the affine analogon of the curvature-centroid has any interesting properties.

It would also be desirable to have a more geometrical discussion of the extremal chords of an oval.

ON THE EXPRESSION OF AN INTEGER AS THE SUM OF AN ARITHMETIC SERIES

By LAURENS EARLE BUSH, University of North Carolina

T. E. Mason in the American Mathematical Monthly, vol. 19 (1912), p. 46, found the number of ways in which an integer can be expressed as the sum of consecutive integers or of consecutive positive integers. In this paper we shall consider the more general question of expressing an integer as the sum of an arithmetic series of integers or of positive integers.

Without loss of generality we may consider the expression of positive integers only, since corresponding theorems for negative integers follow immediately from those for positive integers. We shall consider an integer itself as an arithmetic series of one term, and such a series we shall call *trivial.* In the theorems which follow trivial series are counted unless specifically excluded. We also admit zero as an integer and consider a series beginning or ending with zero as different from the same series with the zero suppressed.

Let $K=2^{e_0}p_1^{e_1}p_2^{e_2}\cdots p_r^{e_r}$ be any positive integer, where $p_1, p_2, \cdots, p_r$ are distinct odd primes:

Theorem 1. *The number of different ways in which K can be expressed as the sum of an arithmetic series of integers with a specified even common difference, d,*

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[21] T. Kubota, *Eine Bemerkung zur affinen Geometrie,* Science Reports of the Tôhoku Imperial University, vol. 12 (1923), pp. 1–5.

is the same as the number of distinct positive divisors of K, i.e., in $\Pi^r{}_{i=0}(e_i+1)$ *ways.*

For, if K is the sum of such a series beginning with the integer a and having n terms, then

$$K = \tfrac{1}{2}n[2a + (n - 1)d],$$

which, solved for a, gives

$$a = \frac{K}{n} - (n - 1)\frac{d}{2} \cdot \tag{1}$$

Since d is even, n must be a divisor of K. Conversely, every distinct positive divisor, n, of K leads to a series of the required type containing n terms.

Corollary: *Every composite integer can be expressed as the sum of a non-trivial arithmetic series of integers with a specified even common difference, d, in at least two different ways; every prime in one, and only one, way; and unity can not be so expressed.*

Theorem 2: *The number of different ways in which K can be expressed as the sum of an arithmetic series of integers with a specified odd common difference, d, is twice the number of distinct positive odd divisors of K, i.e., in* $2\Pi^r{}_{i=1}(e_i+1)$ *ways.*

For, if K is the sum of such a series beginning with the integer a and having n terms, a has the value given by (1). We have two cases, according as n is odd or even.

If n is odd, the last term in (1) is an integer and n must be a divisor of K. Conversely, every positive odd divisor, δ, of K leads to a series of the required type containing δ terms.

If n is even, the last term in (1) is half of an odd integer, therefore K/n must also be half of an odd integer, or $\delta = 2K/n = K/\tfrac{1}{2}n$ must be an odd divisor of K. Conversely, every positive odd divisor, δ, of K leads to a series of the required type containing $2K/\delta$ terms.

Hence, each distinct positive odd divisor of K leads to two different expressions of K of the required type, one having an odd number of terms, the other an even number. This leads to the

Corollary: *Exactly half of the different expressions of K as the sum of an arithmetic series of integers with a specified odd common difference, d, have an even number of terms.*

If $d=1$ we have one of the results obtained by Mason:

Corollary: *The number of different ways in which K can be expressed as the sum of consecutive integers is* $2\Pi^r{}_{i=1}(e_i+1)$.

We may also state the

Corollary: *Every composite integer can be expressed as the sum of a non-trivial arithmetic series of integers with a specified odd common difference, d, in at least one way; a power of 2 in only one way; an odd prime in three, and only three, different ways; and unity in one, and only one, way.*

It is to be noted that in the above corollary and in the corollary to theorem 1, if $d=\pm K$, we are admitting as non-trivial the series of two terms consisting of zero and K. For example, unity can be expressed as the sum of two or more consecutive integers only in this way, but, for $d=3$, we have $1=-1+2$.

In the theorems which follow we shall for convenience in notation consider only positive and zero common differences. That in doing this we lose no generality is evident, since any expression for K as the sum of a series with common difference d becomes a similar expression with common difference $-d$ if written in reverse order. In fact, the theorems stated below are equally true for negative values of d if we replace d by its absolute value in (2) and (4).

Theorem 3: *The number of different ways in which K can be expressed as the sum of an arithmetic series of positive integers with a specified positive even or zero common difference, d, is the number of distinct positive divisors, δ, of K which satisfy the inequality.*

$$\delta(\delta - 1)d < 2K. \tag{2}$$

For, if the series is to have only positive terms it is both necessary and sufficient that the smallest term be positive. Since d is positive the first term is the smallest, and from (1) we obtain the condition

$$n(n - 1)d < 2K. \tag{3}$$

Since the permissible values of n are those, and only those, which are divisors of K, the theorem follows.

Corollary: *A prime number can not be expressed as the sum of a non-trivial arithmetic series of positive integers with a positive even common difference, d.*

For, if K is prime, its only positive divisors are unity and itself. Unity leads to the trivial expression, while K does not satisfy (2), for, since $d\geqq 2$ and $K\geqq 2$,

$$K(K - 1)d \geqq 2K(K - 1) \geqq 2K.$$

Theorem 4. *The number of different ways in which K can be expressed as the sum of an arithmetic series of positive integers with a specified positive odd common difference, d, is equal to the number of distinct positive odd divisors, δ, of K which satisfy inequality (2) plus the number of distinct positive odd divisors, δ, of K which satisfy the inequality*

$$\delta(\delta + d) > 2\,Kd. \tag{4}$$

For, as in the last theorem, (3) is a necessary and sufficient condition that the series contain only positive terms. However, when d is odd each odd divisor, δ, leads to two values of n, namely, $n=\delta$ and $n=2K/\delta$. Hence, those, and only those, values ∂ which satisfy (2) will lead to a series of an odd number of positive terms; while those, and only those, which satisfy

$$\frac{2\,K}{\delta}\left(\frac{2\,K}{\delta} - 1\right)d < 2K,$$

or its equivalent (4), will lead to a series of an even number of positive terms.

Corollary: *A power of two is not expressible as the sum of a non-trivial arithmetic series of positive integers with a positive odd common difference.*

For a power of two has only the odd positive divisor unity, which does not satisfy (4), since $K \geqq 1$ and $d \geqq 1$, and therefore

$$\delta(\delta+d)=1+d \leqq (2K-1)\ d+d=2Kd.$$

Hence, the trivial series is the only one.

Corollary: *An odd prime, K, can be expressed as the sum of a non-trivial arithmetic series of positive integers with a specified positive odd common difference, d, such that $d<K$, in one, and only one, way.*

For, the only odd divisors of K are unity and K. Unity does not satisfy (4) as shown above. K does not satisfy (2), for $K \geqq 3$, $d \geqq 1$, and $\delta(\delta-1)d = K(K-1)d \geqq 2K$. But the prime K always satisfies (4), for since $d<K$,

$$K(K+d) = K^2 + Kd > Kd + Kd = 2Kd.$$

Therefore, there is one, and only one, expression for K other than the trivial one.

Another of Mason's results is contained in the following

Corollary: *The number of ways in which K can be expressed as the sum of consecutive positive integers is equal to the number of positive odd divisors of K, i.e., in $\Pi^r_{i=1}(e_i+1)$ ways.*

For, we have $d=1$ and (2) and (4) reduce to $\delta(\delta-1)<2K$ and $\delta(\delta+1)>2K$. Any odd divisor, δ, of K which does not satisfy one of these relations will evidently satisfy the other. Furthermore, the same positive odd divisor, δ, can not satisfy both, for if it did, we should have

$$\delta^2 < 2K + \delta \quad \text{and} \quad \delta^2 > 2K - \delta,$$

i. e.

$$\delta < 2\frac{K}{\delta} + 1 \quad \text{and} \quad \delta > 2\frac{K}{\delta} - 1.$$

Since K/δ is an integer, we have an odd integer, δ, lying between the two consecutive odd integers $2(K/\delta)-1$ and $2(K/\delta)+1$, which is impossible.

Corollary: *A necessary and sufficient condition that K be expressible as the sum of two or more consecutive positive integers is that it be not a power of 2.*

In order to illustrate the theorems stated above, let us take $K=12=2^2 \cdot 3$. The divisors of 12 are six in number, namely, 1, 2, 3, 4, 6, and 12, of which two, namely, 1 and 3, are odd. If $d=2$, by theorem 1, we have six expressions for 12:

$$12 = 12$$
$$12 = 5 + 7,$$
$$12 = 2 + 4 + 6,$$

$$12 = 0 + 2 + 4 + 6,$$
$$12 = -3 - 1 + 1 + 3 + 5 + 7,$$
$$12 = -10 - 8 - 6 - 4 - 2 + 0 + 2 + 4 + 6 + 8 + 10 + 12.$$

By theorem 3, since for $d=2$, (2) is satisfied by the divisors 1, 2, and 3 only, we have the first three of these expressions alone consisting of positive integers. If $d=3$, by theorem 2, we have four expressions for 12:

$$12 = 12,$$
$$12 = 1 + 4 + 7,$$
$$12 = -34 - 31 - 28 - 25 - 22 - 19 - 16 - 13 - 10 - 7 - 4 - 1 + 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 + 26 + 29 + 32 + 35,$$
$$12 = -9 - 6 - 3 + 0 + 3 + 6 + 9 + 12.$$

By theorem 4, since for $d=3$, (2) is satisfied by the odd divisors 1 and 3 only, while no odd divisor satisfies (4), we have the first two of these expressions alone consisting of positive integers. The only expressions of 12 as the sum of consecutive integers are

$$12 = 12,$$
$$12 = 3 + 4 + 5,$$
$$12 = -11 - 10 - 9 - 8 - 7 - 6 - 5 - 4 - 3 - 2 - 1 + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12,$$
$$12 = -2 - 1 + 0 + 1 + 2 + 3 + 4 + 5.$$

Of these, only the first two consist of positive integers.

A SHORT ACCOUNT OF THE HISTORY OF SYMMETRIC FUNCTIONS OF ROOTS OF EQUATIONS

By H. GRAY FUNKHOUSER, Washington and Lee University

A function is symmetric with respect to any number of letters if an interchange of any two of the letters does not change the function. Thus $\alpha\beta\gamma$ and $\alpha+\beta+\gamma+\alpha^2\beta^2\gamma^2$ are symmetric functions. The simplest ones are those involved in the relations between the coefficients and roots of a quadratic equation. If α and β are the roots of the equation $x^2+bx+c=0$, then $\alpha+\beta=-b$ and $\alpha\beta=c$, b and c being symmetric functions of the roots α and β.

It shall be the purpose of this paper to seek out the first evidence of the knowledge of the relationship between the coefficients and roots of an equation;

to discover, if possible, the faint beginnings or shadowy forerunners of the subject of symmetric functions, to trace their historical development up to the nineteenth century, and to discuss briefly the mathematicians who contributed to the progress of the theory.

The study is not exhaustive and the aim has been to present in concise form an account of the early factors which have made this phase of algebra what it is today, and to supply the student of the theory of equations with some historical material in this specific part of his subject. It has been my particular experience to find that new interest and vitality come to a subject when there is linked up with it a knowledge of its origin and the vicissitudes of its progress, and an acquaintance with those renowned and worthy scholars of previous generations who have provided such a noble heritage for the student of today. One approaches his work with a somewhat different conception of it when he realizes that he is sampling a stream that has been flowing steadily for centuries and will continue to flow in like manner after he is gone.

Original sources in the early history of the subject have naturally been difficult to find and interpret. In preparing this paper I have been fortunate in having at my disposal the library of Professor David Eugene Smith of Teachers College, Columbia University, in addition to the library of the American Mathematical Society and the New York Public Library. These libraries, especially the first named, have provided such a wealth of material as to make the study profitable and interesting.

Original sources have been consulted in all instances mentioned with the exception of Peletier, Vieta and Vandermonde. The authority for the statements in regard to the first two men is Charles Hutton—*Mathematical and Philosophical Dictionary* (London, 1796). For a reproduction of Vieta's work on roots and coefficients, I have consulted Francis Maseres—*Tracts on the Resolution of Cubic and Biquadratick Equations* (London, 1803), page 531. The basis for the paragraph on Vandermonde is a statement on page 51 of Faà de Bruno's *Théorie des Formes Binaires* (1876).

As is the case with the development of algebra as a whole, the subject of symmetric functions to a perhaps greater extent waited upon the introduction and improvement of symbolism. It was hard to perceive with any clarity relations between roots and coefficients when the equation was wrapped up in a paragraph of words representing it. Prior to the time of Franciscus Vieta (1540–1603), who was among the first to employ letters to represent numbers, we find very little trace of the thing we are seeking. We might say that "algebraic shorthand" was almost essential to the development of the subject. It was only when the eye was able to aid the mind that progress began to be noticeable.

Luca Pacioli (c. 1445–1509) in his *Sūma* (1494), a general summary of mathematics up to that time, devotes a part to algebra under the name—L'Arte Magiore; ditta dal vulgo la Regola de la Cosa, over Alghebra e Amucabala.[1] In

[1] *Sūma de Arithmetica, Geometria, Proportioni e Proportionalita*, Venice, 1494, folio 67.

this he solves equations by completing the square and seems generally to have a clear notion of the quadratic. Although I can find no mention of it, he very probably recognized some relation between the roots and the numerical parts of the equation.

The starting point of our subject seems to be with that versatile and eccentric Italian mathematician of the middle of the sixteenth century, Girolamo Cardano (1501–1576), whose name has come down in connection with the first solution of the cubic equation. His work on algebra, *Ars Magna*, was published in 1545. In this he gives an equation,[2] $x^3+72=11x^2$ and the roots $\sqrt{40}-4$, $\sqrt{40}+4$, 3. It should be noticed that the correct roots are $-\sqrt{40}+4$, $\sqrt{40}+4$, 3; but this fault does not affect his knowledge of the relations involved. He says that the difference between the positive and the negative roots is always equal to the coefficient of the second term—"differentia aequationum verarum et fictarū semper est numerus quadratorū." He adds $\sqrt{40}+4$ and 3. From this sum he subtracts $\sqrt{40}-4$ obtaining 11, the coefficient of x^2.

He also shows, in effect, that when the second term is missing the sum of the negative roots equals the sum of the positive ones.

Jacques Peletier (1517–1582), writing on algebra thirteen years later, gives a method of finding the roots of an equation among the divisors of the absolute term when the root is rational, whether integral or fractional, and he observes that the root always "lies hid" in the term and is some one of its divisors.[3]

Rafael Bombelli (born c. 1530) in his algebra[4] published in 1572 shows about the same knowledge as that set forth by Cardan.

Vieta's work on algebra, which gave a great impetus to all branches of algebra and especially to that which we are considering, did not become known until after his death. It was published by Alexander Anderson in 1615. Chapter 14 of this work contains in four theorems the general relation between the roots of an equation and the coefficients of its terms when all the roots are positive. The first theorem in Vieta's original style is as follows:

Si B+D in A−A quadratum aequatur
B in D, A explicabilis est de qualibet illarum duarum B vel D.
3N−1Q aequetur 2. Fit 1N 1 vel 2.

This in modern notation is

$$\text{If } (a+b)x-x^2=ab;\text{ then } x=a \text{ or } b.$$
$$3x-x^2=2,\ x=1 \text{ or } 2.$$

The cubic, biquadratic and quintic are given in similar fashion.

This seems to show that Vieta had a very clear idea of the relation he expresses but such is not quite the case. I quote from Cajori's *History of Elementary Mathematics:*

"Vieta arrived at a partial knowledge of the relation existing between the coefficients and the roots of an equation. Unfortunately he rejected all except

[2] *Artis Magnae, sive de regulis algebraicis*, 2nd edition, Basel, 1570, p. 10.

[3] *L'algèbre départie en deux livres*, Lyons, 1554.

[4] *L'algebra parte maggiore dell' arimetica divisa in tre libri*, Bologna, 1572.

positive roots and could not therefore fully perceive the relations in question. His nearest approach to complete recognition of the facts is contained in the statement that the equation $x^3-(u+v+w)x^2+(uv+vw+uw)x-uvw=0$ has three roots u, v, w. For cubics this statement is perfect if u, v, w, are allowed to represent any numbers. But Vieta is in the habit of assigning to letters only positive values so that the passage means less than at first sight it appears to do."[5]

However we can appreciate how great an advance Vieta made when there is considered the few fragmentary statements that have come from his predecessors. He paves the way and provides an introduction to the first man who really has a place in the history of symmetric functions of roots of equations, a man, who for clearness and grasp of material at hand in not only this topic but also in other phases of algebra could well hold his place a century later.

The man was Albert Girard (1595–1632) and his work on algebra is a little 34-leaf pamphlet called *Invention Nouvelle en l'Algèbre*, published in 1629.

Girard gives the triangle later known as Pascal's triangle and uses it as the basis for developing a theorem on symmetric functions, although he had no idea of them as such. It is interesting to note how he makes and interprets the triangle. I shall translate his words as literally as possible:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

"When many units are put as on the side and other numbers in the middle, you find such a figure by means of addition. It may be called the triangle of extraction. The unity above signifies simple arithmetic and the others, algebra, i.e. 1, 1 is the degree of (1) and 1,2,1 the degree of (2), then 1,3,3,1 is called the degree of (3) and thus always to infinity."[6]

The symbol (1) is a substitute for Girard's* notation for the first power of the unknown; (2) and (3) are respectively symbols for the second and third powers.

Designating by "première faction" the sum of a group of numbers; "deuxième faction" the sum taken two at a time, etc., he states the theorem—"If a multitude of numbers is proposed, the multitude of the products of each faction can be expressed by the triangle of extraction and by the same rank as the multitude of numbers." Thus, if there are four numbers, the row 1,4,6,4,1 signifying the fourth power, is taken. The 1 denotes the unit of the greatest, the 4 the first faction, the 6 the second faction, etc.

In an illustrative equation, $1\ (4)=4\ (3)+7\ (2)-34\ (1)+24$, he states that the exponent of the highest quantity is 4 which signifies that there are four solutions, neither more nor less. But he says "in order to see the thing in its

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[5] Revised edition, New York, 1917, page 230.

[6] Loc. cit., page 42.

* Girard's symbol was a circle enclosing the number. Parentheses instead of circles are used here for convenience in printing.

perfection" it is necessary to take the signs arranged in a different order, thus

$$1\ (4)-7\ (2)-24\ (0)=4\ (3)-34\ (1);$$

then the coefficients with their proper signs are 4, -7, -34, -24, which are the four factions of the four solutions.[7]

He gives an equation, $x^4-4x+3=0$, which has two imaginary roots and shows that the rule for forming the coefficients from the roots holds true in all cases.

Girard arrives rather casually and uniquely at the development of the sums of the powers of the roots in terms of the coefficients. He is discussing the above mentioned relationship between coefficients and factions when he assumes that someone for the sake of simplicity, instead of saying the sum, the product two by two, three by three, etc., may like to say the sum, the sum of the squares, the sum of the cubes, etc. In order to show that these two ideas do not represent the same thing, he gives the following example:

"If the coefficients of the second, third, fourth terms, etc. are A, B, C, etc., then in any degree of equation,

A	will	solutions
A^2-2B	be	squares
	the	
$A^3-3AB+3C$	sum	cubes
$A^4-4A^2B+4AC+2B^2-4D$	of	biquadrates"[8]

This is the well known theorem which bears the name of Newton, but we find it given almost as completely a century before by Girard.

Charles Hutton (1737–1823), in his *Mathematical and Philosophical Dictionary* (1795–6), puts his stamp upon Girard in these words:

"He was the first person who understood the general doctrine of the formation of the coefficients of the powers from the sum of the roots and their products. He was the first who discovered the rules for summing the powers of the roots of any equation."[9]

Thomas Harriot (1560–1621) at about the same time as Girard was also discovering relations of the roots to coefficients, but he lacked the insight and knowledge of his contemporary. His work on algebra was published ten years after his death by his friend, Walter Warner. It was probably written[10] about 1610. He studied equations from the standpoint of their formation from binomial factors or roots and thus recognized the relation between the roots and coefficients when the roots are positive. The following is an example of his method:

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[7] Loc. cit., p. 43.

[8] Loc. cit., p. 46.

[9] Vol. 1, p. 63.

[10] D. E. Smith, *History of Mathematics* (1923), Vol. 1, p. 388.

$$\begin{array}{ll} a=+b & a-b=0 \\ a=+c & a-c=0 \\ a=+d & a-d=0 \\ \hline & aaa-baa+bca \\ & \quad -caa+bda \\ & \quad -daa+cda-bcd=0 \end{array}$$[11]

Cajori says, "He was the first to decompose equations into their simple factors, but he failed to recognize imaginary or even negative roots."[12]

William Oughtred (1574–1660) in the *Clavis Mathematicae* published in 1631 follows Vieta to a large extent but does not extend the relation we are studying.

For the next seventy-five years, up to the time of Newton, there is little worth noting in the development of symmetric functions. René Descartes' geometry, which first appeared in 1637, begins with a discussion of the nature and roots of equations. Descartes generates equations in a manner similar to Harriot by the multiplication of binomial factors. He shows how an equation can be depressed by a binomial composed of the unknown plus or minus a root. Being interested in equations only from the geometrical viewpoint, he proceeds to their transformation with no further regard for roots and coefficients as such.

In 1673 appeared the unique work of John Wallis (1616–1703), *De Tractatus Algebra, Historicus et Practicus*. Wallis devotes a short chapter to the subject, "The composition of coefficients." This relates to the manifest number of roots that any equation can have and to the ease of perceiving that "the coefficient of the second term, reckoning downward from the highest, is the aggregate of all the roots, the coefficient of the third term is the aggregate of the rectangles," etc.[13] This is the sum total of his contribution and he either did not know of Girard's work or considered it of not sufficient importance to mention. The former seems more likely. Also, this pioneer historian was so wrapped up in Harriot's algebra that he had eyes for little else.

As is often the case, the man who makes a rediscovery of some important truth, or gives his results to the world under more favorable circumstances than the original discoverer, gathers to himself the honor of having his name perpetuated while that of the former goes unsung. Such is the fact that we note in this connection with reference to Girard and Newton. Almost a hundred years after the time of Girard the *Arithmetica Universalis* of Isaac Newton (1642–1727) appeared, and in it, under the heading of transmutation of equations, is the theorem on the sums of powers of the roots. From this time on, every work on symmetric functions begins with a presentation of "Newton's theorem."

For the sake of comparison it may be worth while to give it as Newton did in 1707:

[11] *Artis Analyticae Praxis ad Aequationes Algebraicas .... Resoluendas*, London, 1631, p. 18.

[12] *History of Mathematics* (1919), page 157.

[13] English edition, London, 1685, p. 142.

"Let us suppose now, that the known quantities of the terms of any equation under the signs changed are p, q, r, s, t, v, viz., that of the second p, that of the third q, of the fourth r, of the fifth s, and so on. And the signs of the terms being rightly observed, make $p=a$, $pa+2q=b$, $pb+qa+3r=c$, $pc+qb+ra+4s=d$, $pd+qc+rb+sa+5t=e$, $pe+qd+rc+sb+ta+6v=f$, and so ad infinitum, observing the series of the progressions.

"And a will be the sum of the roots, b the sum of the squares of each of the roots, c the sum of the cubes, d the sum of the biquadrates, e the sum of the quadrato-cubes, f the sum of the cubo-cubes, and so on."[14]

Newton gives no proof or demonstration of how he obtained this theorem, but passes on immediately to the use of the sums of powers in determining the limits of the roots of equations. In an English edition of *Arithmetica Universalis* in 1769, a commentary by Rev. Theaker Wilder shows the derivation of the theorem, stating that it "follows easily from the algebraical expression of the quantities and the binomial theorem."[15]

The middle of the eighteenth century finds the relation between .oots and coefficients fairly well understood. Crousaz's algebra in 1726, Saunderson's in 1740, Clairaut's in 1749, and Maclaurin's in 1748, all contain chapters on roots and coefficients.

In the years just following Newton a number of mathematicians were interested in demonstrating and proving the Newtonian formula. Among the most important proofs may be mentioned Maclaurin's[16] in 1748 and Euler's[17] in 1750. The latter is called a double demonstration of Newton's theorem, the first of which is by the use of logarithmic differentiation and development in series, the second being algebraic and elementary in type. The second, however, is an excellent proof, the demonstration of which has been characterized by Leopold Kronecker (1823–1891) as very exact and elegant. It is similar to that of Maclaurin's in finding the sum of the rth powers in an equation of the nth degree when $r \geqq n$, but differs for the case of $r<n$.

Jean Castillon (1709–1791), who edited an edition of the *Arithmetica Universalis* in 1761, includes a demonstration in an extensive commentary on the theorem.[18] In an appendix to the above edition there is also a proof by Georg F. Baermanns (1717–1769). He writes that he early found out the usefulness of the theorem in the theory of limits of equations and that he deplored the lack of a proper demonstration among the mathematicians of his time. After searching in vain he at last himself developed a "praiseworthy" one. He reverses

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[14] *Arithmetica universalis sive de compositione et resolutione arithmetica liber*, Cambridge, 1707, p. 251.

[15] Page 393.

[16] *A Treatise of Algebra*, London, 1748, pp. 286 ff.

[17] *Gemina theormatis Neutonioni, quo traditur relatio inter coefficientes cuius vis aequationis algebraicae et summas potestatum radicum eiusdem*, Opuscula Varii Argumenti, 1750, vol. 2, p. 108.

[18] Page 70,

common practice in representing coefficients by Greek letters and he makes use of increments and a sort of differential notation.

As we scan the progress of the subject after the impress of Newton upon it, we find that the next man whose name is known to modern students of symmetric functions is Dr. Edward Waring, Lucasian Professor at Cambridge from 1760 until 1798. He was one of the first to write extensively on the theory of equations and he developed three important theorems on symmetric functions. These appear in Chapter I of his *Meditationes Algebraicae*, published in 1770.

The first theorem is a formula for finding immediately, as a function of the coefficients, the sum of the nth powers of the roots. Waring does not make known the method which leads to this theorem. The second theorem, the least useful of the three, appears as a formula for finding the sums of all the powers of the roots. The third contains a method for finding any rational and entire symmetric function of the roots in terms of the coefficients. This theorem is the beginning of the fundamental theorem of modern symmetric functions.

To Alexandre Théophile Vandermonde (1735–1796), chemist, musician, and political economist as well as mathematician, belongs the distinction of having first constructed tables of symmetric functions. Coming into possession of Waring's formulas a year after the publication of *Meditationes Algebraicae*, he worked out a set of these functions.[19] However, he gave no method for their calculation, and his work, being little known, was overshadowed by the results of Hirsch twenty-five years later.

At almost exactly the same time that Waring was working out his first theorem, Leonard Euler (1707–1783), in Russia, was directing his thoughts along the same lines with the result that he also obtained a like theorem. His conclusions were embodied in a memoir read before the Saint Petersburg Academy in January, 1771, under the title *Observationes Circa Radices Aequationum*, which was published the same year. One of the most important applications of the theory of symmetric functions, the method of elimination, is due to Euler.[20] This method which finds the final equation by means of symmetric functions was first presented by him before the Berlin Academy of Sciences in 1748. Two years later, Gabriel Cramer (1704–1752) in his treatment on algebraic curves[21] followed the same method in a somewhat more general way.

Among the great mathematicians of the eighteenth century who were interested in symmetric functions was Joseph Louis Lagrange (1736–1813) whose work, however, must be regarded not as an end but rather as a means to an end. He studied symmetric functions for the purpose of developing and improving methods for the solution of higher equations.

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[19] Mémoires de l'Académie des Sciences (1771).

[20] Cajori, *History of Mathematics*, page 235. Meyer Hirsch, *Literal Calculus and Algebra*, translated by J. A. Ross (1827), page 143.

[21] *Introduction à l'Analyses des Lignes Courbes* (Geneva, 1750), page 660.

Lagrange's first edition of the *Traité de la Résolution des Équations Numériques de tous les Degrés* appeared in 1798. In a note on the method of approximation by recurring series, he develops the algebraic division method of proving Newton's theorem.[22] This is perhaps the simplest of all methods and is one of several generally given in modern textbooks.

In this work of Lagrange's there is one of the first uses of the modern symbol $\sum$ for summation in connection with symmetric functions. The symbol appears before in Wallis's algebra but is used there for "cosine." Its introduction as the symbol for summation is due to Euler[23] in 1755. Lagrange is also the first to state and fully recognize the theorem that every rational and entire function of the roots can be expressed in terms of the coefficients. He says of this theorem "C'est la un des principes les plus féconds de la théorie des équations."[24]

Chief among Lagrange's contributions is a noteworthy formula[25] which Louis Poinsot (1777–1859) praised "aussi remarquable par son élégance que par sa généralité." It gives an expression for finding directly the sum of like negative powers of the roots of an equation. This theorem has been the foundation of a number of researches by Lagrange's successors, among whom the most important have been Cauchy, who has given a number of demonstrations of it,[26] and Eugène Rouche[27] who gave a rigorous and yet simple proof.[28] Lagrange also developed Newton's theorem by means of logarithms.

The first presentation of symmetric functions in a modern way is due to Meyer Hirsch (1765–1851). In the preface to his algebra (1809) he says, "I begin with symmetric functions; they are the foundation of all others."[29] Then follow several proofs of Newton's theorem and a general demonstration that every rational symmetric function of the roots can be expressed as a function of the coefficients. The first tables of symmetric functions that have become generally known and used were constructed by Hirsch. These contain all values up to equations of the tenth degree. He constructed these by means of the functions of the like powers of the roots, giving demonstrations of his method of work. This method requires excessive calculation when the functions are extended beyond the simpler forms, and hence was supplanted by the later formulas of Cayley and Sylvester.

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[22] *Traité de la Resolution des Équations Numériques* (Second edition, Paris, 1808), p. 133.

[23] Johannes Tropfke, *Geschichte der Elementar-Mathematik* (Berlin, 1921), Band II, page 33.

[24] Op. cit., p. 193.

[25] Memoirs of Berlin Academy, 1768, *Traité de la Résolution des Équations Numériques,* page 211.

[26] *Mémoire sur Quelques Séries Analogues à la Série de Lagrange,* Mémoires de l'Académie des Sciences (1830), Tome IX, page 104. *Oeuvres complètes d'Augustin Cauchy* (Paris 1882–1911), Séries I, Tome II, page 73.

[27] Journal de l'École Polytechnique, vol. 22, p. 193 ff.

[28] J. A. Serret, *Cours d'Algèbre Supérieure* (3rd edition, 1866), page 462.

[29] *Sammlung von Aufgaben aus de Theorie der Algebraischen Gleichungen,* Berlin, 1809, page ix.

BOND YIELDS AND BOND PRICES

By W. D. A. WESTFALL, University of Missouri

1. *Introduction.* Let P be the purchase price of a bond of unit redemption price, dividend rate r, investment yield i, and with n interest periods to maturity Then

$$(1) \qquad P(n) = (1 + i)^{-n} + r(1 + i)^{-1} + r(1 + i)^{-2} + \cdots + r(1 + i)^{-n}$$

or

$$P(n) = 1 + \frac{1 - (1 + i)^{-n}}{i}(r - i)$$

and

$$(2) \qquad \frac{dP}{dn} = \frac{(1 + i)^{-n} \log (1 + i)}{i}(r - i).$$

Hence dP/dn has the sign of $r-i$. It follows that the premium $P-1$, paid when $r>i$, and the discount $1-P$, for $r<i$, increase with n. Similarly it may be shown that the absolute value of $(1-P)/P$ increases with n. These well known facts may be stated as follows:

The change in purchase price and the proportional changes in purchase price of a bond, to alter the investment rate from r to i or from i to r, increase with increasing length of time to maturity.

It is the purpose of this note to determine whether this holds when the change in investment yield is from i_1 to i_2, where $i_1 \neq r \neq i_2$.

2. Let P_1 be the purchase price of a bond to yield i_1, and P_2 to yield i_2. We shall take $i_1 > i_2$ and $r > 0$. Then from (1) it is seen that $P_2 - P_1$ is positive for positive n and an analytic function of n for all finite values of n. We shall investigate the sign of its derivative,

$$\frac{d}{dn}\left(P_2 - P_1\right) = (1 + i_2)^{-n} A_{(n)},$$

where

$$(3) \qquad A(n) = \frac{r - i_2}{i_2} \log (1 + i_2) - \frac{r - i_1}{i_1}\left(\frac{1 + i_2}{1 + i_1}\right)^n \log (1 + i_1).$$

For $n=0$ this has the positive value[1]

$$r[i_2^{-1} \log (1 + i_2) - i_1^{-1} \log (1 + i_1)] + \log (1 + i_1) - \log (1 + i_2).$$

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[1] Note that $x^{-1} \log (1+x)$ decreases for $x>0$ with increasing x, since

$$\frac{d}{dx} \frac{\log (1+x)}{x} = \frac{x-(1+x) \log (1+x)}{x^2+x^3} = -\frac{\log (1+\xi)}{2\xi+3\xi^2}$$

for some ξ, $0<\xi<x$, by Cauchy's formula, and this is negative.

From (3) it is seen that $A(n)$ is a univariant function of n which takes the sign of $r-i_2$ as n approaches infinity. The derivative of P_2-P_1 is therefore positive for all values of n if $r>i_2$. For $r<i_2$ the derivative is positive for n less than the root n_0 of $A(n)=0$ and is negative for all higher values of n. Combining these results with the statements in the introduction we have the theorem:

Theorem: *The change in the purchase price of a bond, to alter the investment rate from i_1 to i_2, increases with increasing n if either investment rate is less than or equal to the dividend rate. If both investment yields are greater than the dividend rate it will increase to a maximum at n_0 and thereafter decrease.*

3. To discover whether the two proportional changes in purchase price, $(P_2-P_1)/P_1$ and $(P_2-P_1)/P_2$, increase or decrease with increasing n it will be sufficient to consider the first alone, since their derivatives have the same sign.

$$\frac{d}{dn}\frac{P_2 - P_1}{P_1} = \frac{P_1P_2' - P_2P_1'}{P_1^2} = \frac{P_2^2}{P_1^2}\frac{d}{dn}\frac{P_2 - P_1}{P_2}.$$

From (2) it follows that this derivative is positive if $i_1>r_2>i_2$. To discuss the other cases we write

$$\frac{d}{dn}\frac{P_2 - P_1}{P_1} = \frac{(1 + i_2)^{-n}A(n)}{P_1^2},$$

where

$$\begin{aligned} i_1i_2A(n) = {} & \left\{r - (r - i_1)(1 + i_1)^{-n}\right\}(r - i_2)\log(1 + i_2) \\ & - \left\{r - (r - i_2)(1 + i_2)^{-n}\right\}(r - i_1)(1 + i_2)^n(1 + i_1)^{-n}\log(1 + i_1). \end{aligned}$$

$A(n)$ has the positive value

$$r\left\{ i_2^{-1}\log(1 + i_2) - i_1^{-1}\log(1 + i_1)\right\} + \log(1 + i_1) - \log(1 + i_2)$$

for $n=0$, and the sign of $r-i_2$ as n approaches infinity. Hence the derivative of $(P_2-P_1)/P_1$ is positive for $n=0$. It is also positive as n approaches infinity if $r>i_2$ but negative if $r<i_2$. To discover its sign for values in between we write it in the form

$$\frac{d}{dn}\frac{P_2 - P_1}{P_1} = \frac{(1 + i_1)^{-n}(1 + i_2)^{-n}}{i_1i_2P_1^2}B(n),$$

where

$$\begin{aligned} B(n) = {} & \left\{r(1 + i_1)^n - (r - i_1)\right\}(r - i_2)\log(1 + i_2) \\ & - \left\{r(1 + i_2)^n - (r - i_2)\right\}(r - i_1)\log(1 + i_1). \end{aligned}$$

$B(0)$ is positive and its derivative,

$$dB/dn = r(1 + i_2)^n\log(1 + i_1)\log(1 + i_2)C(n),$$

where

$$C(n) = \left\{\left(\frac{1 + i_1}{1 + i_2}\right)^n - \frac{r - i_1}{r - i_2}\right\}(r - i_2),$$

has the sign of $C(n)$. The latter is a univariant function positive for $n=0$ and has the sign of $r-i_2$ as n approaches infinity. For $r>i_1>i_2$, $C(n)$ is positive for all n. For $r<i_2<i_1$ it is positive for $n<n_1$, the root of

$$(r-i_2)\ (1+i_1)^n = (r-i_1)\ (1+i_2)^n,$$

and is negative for $n>n_1$. In the first case $B(n)$ is positive for all positive n. In the second case it is positive and rises to a maximum at $n=n_1$, thereafter decreasing to zero at n_2, the root of $B(n)=0$. For higher values of n it is negative. In the first case the derivative of $(P_2-P_1)/P_1$ is positive for all positive values of n. In the second case it is positive for all values such that $n>n_2$ and is negative for all higher values of n. We have as before the theorem:

Theorem: *The proportional changes, $(P_2-P_1)/P_1$ and $(P_2-P_1)/P_2$, in the purchase price of a bond, to alter the investment yield from i_1 to i_2, increase with increasing n if either yield is less than or equal to the dividend rate. If both investment yields are greater than the dividend rate these proportional changes increase to a maximum at $n=n_2$ and thereafter decrease.*

ON THE GEOMETRY OF CLOCKS

By J. M. FELD, Columbia University

In a paper on *Homographic circles or clocks* by L. Hoffman and E. Kasner,[1] was introduced the term *homographic clocks* for circles whose points are in homographic correspondence with the points of the unit circle. If $t=e^{i\theta}$ (θ real) represents the unit circle on the Gaussian plane, then any homographic clock Z is given by the equation

$$z = (A + Be^{i\theta})/(C + De^{i\theta}).$$

The clock is termed positive or negative according as z moves on the circle Z in the same sense that t moves on the unit circle or not. Thus the clocks $Ae^{i\theta}$ and $Be^{-i\theta}$ are positive and negative, respectively.

The object of this paper is to study the geometry associated with a sum of two clocks $Ae^{i\theta}$ and $Be^{-i\theta}$. Let

$$z = Ae^{i\theta} + Be^{-i\theta}, \tag{1}$$

where $z=x+iy$ and A and B are constants. By separating the real and imaginary parts in (1) we get

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[1] Bulletin of the American Mathematical Society, vol. 34 (1928), p. 495.

$$x = (a_1 + b_1)\cos\theta - (a_2 - b_2)\sin\theta,\ \ y = (a_2 + b_2)\cos\theta + (a_1 - b_1)\sin\theta,$$

where $A = a_1 + ia_2$ and $B = b_1 + ib_2$. Letting $a_1 + b_1 = p$, $a_2 - b_2 = q$, $a_2 + b_2 = r$, $a_1 - b_1 = s$, the Cartesian equation of the curve (1) becomes

$$(r^2 + s^2)x^2 - 2(pr - qs)xy + (p^2 + q^2)y^2 = (ps + qr)^2.$$

Thus (1) represents an ellipse having its center at the origin.

From the form of equation (1) we obtain a method for constructing an ellipse. Let A and B be any two vectors, OA and OB, respectively. We form the linkage OAB by adding vector B to A. As OA rotates counterclockwise about O, the link AC rotates clockwise, and at the same rate as OA, about A. The point C generates the ellipse.

Let $|z| = m$ and $\bar{z} = x - iy$. From (1) we get

$$m^2 = (Ae^{i\theta} + Be^{-i\theta})(\overline{A}e^{-i\theta} + \overline{B}e^{i\theta})$$

or

$$m^2 = A\overline{A} + B\overline{B} + A\overline{B}e^{2i\theta} + \overline{A}Be^{-2i\theta}. \tag{2}$$

For maximum and minimum m we must have

$$e^{4i\theta + 2\pi ik} = \overline{A}B/A\overline{B}\ (k \text{ an integer})$$

or

$$\theta = \tfrac{1}{2}\pi k + \tfrac{1}{2}(\arg B - \arg A). \tag{3}$$

When k is zero or even, m is a maximum; and when k is odd, m is a minimum. Replacing θ in (2) by $\frac{1}{2}(\arg B - \arg A)$ and by $\frac{1}{2}\pi + \frac{1}{2}(\arg B - \arg A)$ we get the lengths of the semi-major and semi-minor axes of the ellipse, respectively. Letting a and b represent the semi-major and semi-minor axes we find

$$a = |A| + |B|,\ \ b = |A| - |B|.$$

If ϕ is the angle that the major axis makes with the axis of reals, we get from (3)

$$\phi = \arg A + \tfrac{1}{2}(\arg B - \arg A) = \tfrac{1}{2}(\arg A + \arg B).$$

The major axis therefore bisects the angle formed by the vectors OA and OB.

The distance of the foci from the center O is given by

$$c = (a^2 - b^2)^{1/2} = 2\,|AB|^{1/2}.$$

If in (1) A and B are regarded as parameters and if we make $AB = C$, where C is a constant, (1) represents a family of confocal ellipses since

$$\tfrac{1}{2}(\arg A + \arg B) = \tfrac{1}{2}\arg C \quad \text{and} \quad 2|AB|^{1/2} = 2|C|^{1/2}.$$

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

ON SOME LOCI CONNECTED WITH THE ORTHOPOLE-GEOMETRY

By R. GOORMAGHTIGH, Bruges, Belgium

In the March, 1930 issue of this Monthly[1] was published a paper by O. J. Ramler on *The orthopole loci of some one-parameter systems of lines referred to a fixed triangle.*

Several properties obtained in that article have already been mentioned[2] and it may be interesting to recall, with some notes on the literature of the subject, a few other theorems about the loci considered.

1. *Locus of the orthopoles of the Simson lines.* In *Mathesis* (1914), p. 152, we have proved the following theorem:

If TT' is a diameter of the circumcircle of a triangle A, B, C, the orthopole of the Simson line of T is the projection of the orthocenter H on the Simson line of T'.

Hence, the locus of the orthopoles of the Simson lines of a triangle is the *pedal curve of the Steiner deltoid for the orthocenter H* (See our question No. 17927 in the *Educational Times*, 1915, p. 39, and solutions by C. E. Youngman and R. F. Davis, *ibid*, 1915, p. 164).

2. *Locus of the orthopoles of the tangents to the circumcircle.* Neuberg has proved in his *Mémoire sur les Projections et Contre-projections d'un triangle fixe* 1890 p. 76, that the locus is the Steiner deltoid.

More generally, when a straight line L remains tangent to a circle concentric to the circumcircle, the locus of the orthopole of L is a trochoidal curve (L. Poli; Question 2490 in *Nouvelles Annales de Mathématiques*, 1920, p. 160; our proof is given in a paper presented by V. Thébault to the *Société Scientifique de Bruxelles*, 1926).

3. *Locus of the orthopoles of the straight lines passing through a fixed point P.* The fact that the conic, locus of these orthopoles, is always an ellipse is proved by A. A. Krishnaswami Ayyangar, in his paper *A simple exposition of the allied properties of the Simson line, the Steiner envelope, and the orthopole* (*Educational Review*, Madras, October, 1924).

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[1] Vol. 37 (1930), pp. 130–136.

[2] R. Goormaghtigh, *On the properties of the orthopole*, Tohoku Mathematical Journal, vol. 30 (1926), pp. 77–125.

The barycentric equation of the orthopolar conic of P, in the pedal triangle of P, is

$$a^2\alpha^{-1} + b^2\beta^{-1} + c^2\gamma^{-1} = 0,$$

a, b, c being the length of the sides A, B, C.
This equation is given in Neuberg's article *Sur les cercles polaires relatifs à un triangle fixe* (*Bulletin de l'Académie royale de Belgique, classe des Sciences*, July-August, 1910).

The equation shows that, *if the pedal triangle of P is projected on a plane into a triangle similar to A B C, the orthopolar conic is projected into a circle* (see article in the *Tohoku Mathematical Journal, 1925*.

The properties about the length of the axes of the orthopolar conic are also to be found in Ayyangar's paper; it contains the following determination of the directions of the axes: *they are perpendicular to the Simson lines of the exiremities of the circumdiameter passing through P.*

We have further shown (*Nouvelles Annales de Mathématiques*, 1914, p. 121) *that the orthopolar conics of a triangle are all inscribed in the Steiner deltoid* and (*Mathesis*, 1914, p. 121) that *the orthopolar conic of a point P is inscribed in the triangle when P is the image of H through the circumcenter.*

4. Finally it may be interesting to note another locus connected with the subject:

When a straight line L passes through a fixed point, the locus of the image of the orthopole of L, through L, is a limaçon.

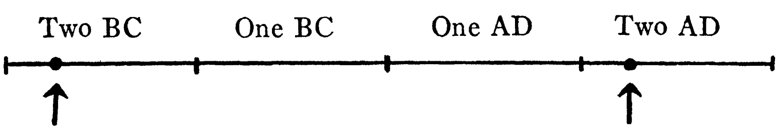
For a proof see our article *Sur l'orthopôle et certains limaçons de Pascal associés au triangle*, in *Nouvelles Annales de Mathématiques*, 1918, p. 242.

A Letter to the Editor

By Eward S. Smith, Denver, Colorado

In the Monthly for March (1930) I find on page 150 what seems to me an error, in the selection of the dates 570, 470 and 370 B.C. as dates for which the present year is a multicentennial.

When Dionysius Exiguus devised an improved system of chronology, he could have saved future historians and computers some annoyance if he had designated the year following B.C. one as the year zero. He was possibly not acquainted with zero and its helpful properties; anyway the year after B.C. one was A.D. one. It is easy to see, where the two years concerned are near together, one being B.C. and the other A.D., that in order to compute the interval, the two numbers must be added together and the sum diminished by one.



This diagram shows that the interval between say, the Ides of March, B.C. 2, and the Ides of March A.D. 2, is not $2+2$ but $2+2-1=3$ years.

Unless there is some pertinent consideration that I have failed to notice, we should take the year 1931 instead of the present year as the correct time to celebrate the multi-centennial of events that occurred in B.C. 570, 470, 370, 270.

Thus, if Anaximenes were still living on Earth (he is probably in a better place now) he would in 1931 be $570+1931-1=2500$ years old, and would undoubtedly still be studying (and perhaps teaching) mathematics.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of the New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y. and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Advanced Mathematics for Students of Physics and Engineering. By D. Humphrey. Oxford University Press, 1929. Part I, viii+120 pages. Part II, viii+176 pages. $4.25.

Plane and Spherical Trigonometry. By R. D. Carmichael and E. R. Smith. Boston, Ginn and Co., 1930. x+200 pages. $1.60.

Analytic Geometry. By D. R. Curtiss and E. J. Moulton. Boston, D. C. Heath and Co., 1930. xiv+338+18 pages.

The Size of the Universe. By Ludwik Silberstein. Oxford University Press, 1930. vii+216 pages. $3.50.

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Statistical Methods for Research Workers. By R. A. Fisher. Third edition, Revised and Enlarged. London, Oliver and Boyd, 1930. xvi+284 pages, 6 tables 15s.

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Statistical Tables and Graphs. By Bruce D. Mudgett. Boston, Houghton Mifflin Co., 1930. viii+194 pages. $1.75.

Grundlagen der Analysis. By E. Landau. Leipzig, Akademische Verlagsgesellschaft M. B. H., 1930, xiv+134 pages.

Elementargeometrie der Ebene und des Raumes. By Max Zacharias. Berlin, de Gruyter, 1930. 252 pages. Rm 13.50.

Darstellende Geometrie. ii. *Perspektive Ebene Gebilde, Kegelschnitte.* By Robert Haussner. Sammlung Göschen 143. Berlin, de Gruyter, 1929. 168 pages.

Élèments de Mathèmatiques financières. Opèrations à long terme. By R. Thery. Paris, Vuibert, 1930. viii+92 pages.

Calculus. By E. J. Miles and J. S. Mikesh. New York, The McGraw-Hill Book Company, 1930. xiv+638 pages. $3.75.

Projective Geometry. By J. W. Young. The Carus Mathematical Monographs, No. 4. Chicago, The Open Court Publishing Co., 1930. ix+185 pages. $2.00.

The Adjustment of Errors in Practical Science. By R. W. M. Gibbs. Oxford University Press, 1930.

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Differential Geometry of Three Dimensions. By C. E. Weatherburn. Volume II, Cambridge University Press, 1930. xii+240 pages. $4.25.

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Differential Equations. By F. R. Moulton. New York. The Macmillan Company, 1930. xvi+396 pages. $5.50.

Gewöhnliche Differentialgleichungen. By G. Hoheisel. Second edition, enlarged. Sammlung Göschen 920. Berlin, de Gruyter, 1930. 159 pages. Rm. 1.80.

REVIEWS

Special curves. By R. C. Archibald, Encyclopaedia Britannica, vol. 6, 1929, pp. 887–899.

This remarkable article contains the description of some 120 curves. The reader is greatly aided by an alphabetical list, followed by a systematic discussion arranged in sections in this order: plane algebraic curves according to their degrees, plane transcendental curves, general classes of plane curves, and curves of double curvature. In addition to short historical notes and a wealth of bibliographical information concerning these curves, there is a concise but very clear account of the fundamental geometric properties of the curves in question (with figures), together with their equations in rectilinear, polar, and other coordinate systems. An attentive reader will find here many interesting details never before mentioned in the literature. For instance, it appears that the curves resulting from addition of two simple harmonic motions along two orthogonal axes, which usually are known under the name of Lissajous curves, were actually studied first by Nathaniel Bowditch, author of the well known book on navigation, some 40 years before Lissajous. Although no one would expect completeness in a brief monograph for the Encyclopaedia, there are still some interesting classes of curves which, perhaps, should have been mentioned here. Such are, for instance, the class of curves of constant width

studied since Euler, and certain curves introduced by Tchebyscheff. There is no doubt, however, that R. C. Archibald has contributed a reference source that will prove in many instances more useful and more reliable than some of the large treatises on the subject mentioned at the end of the article.

J. TAMARKIN

Le théorème de Picard-Borel et la théorie des fonctions méromorphes. By Rolf Nevanlinna. Borel Series of Monographs. Gauthier-Villars et Cie, Paris, 1929. vii+174 pages.

The main object of this book is the study of the distribution in the x-plane of the roots of the equation $f(x)=z$ for all (finite or infinite) values of z, where $f(x)$ is a transcendental function of x meromorphic for all (finite) values of x. A number of the results obtained are extensions of classical results of Hadamard, Borel, and others for entire functions. Some of the theorems of the book have not previously appeared in the literature; others have been proved quite recently by such writers as Lindelöf, Valiron, F. Nevanlinna, and of course the author himself.

If $f(x)$ is such a function as is considered above, and if the equation $f(x)=z$ has no roots x for a particular z, that value z is called an *exceptional value*, and the classical theorem of Picard asserts that there are at most two exceptional values. Nevanlinna's first fundamental theorem (1925) studies not merely the distribution of possible roots x of the equation $f(x)=z$ but involves likewise the average convergence, for $|x|$ becoming infinite, of $f(x)$ to the value z, and it asserts that if suitable measures of these two quantities are considered together, *there are no exceptional values*. Or, as Nevanlinna expresses it, the *total affinity* of the function $f(x)$ for the value z is independent of z.

We must refrain for lack of space from going into detail concerning the results established. The notion of *order* of a meromorphic function is introduced and the analogues of the results of Hadamard for entire functions proved. This leads to the representation of such a function by canonical products. The second fundamental theorem gives more detailed information about the distribution of the roots x of the equation $f(x)=z$, and has varied and beautiful applications. Finally there are studied systems of meromorphic functions connected by a linear relation and functions assumed meromorphic merely in a finite circle.

The treatment throughout the book is clear and elegant, and many of the theorems are of great interest and beauty. The results are in the main established by the systematic use of the formula of Jensen in the form:

$$\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\,|f(\rho e^{i\theta})|\,d\theta+\int_0^{\rho}\frac{n(r,\infty)-n(0,\infty)}{r}dr+n(0,\infty)\log\rho$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\left|\frac{1}{f(\rho e^{i\theta})}\right|d\theta+\int_0^{\rho}\frac{n(r,0)-n(0,0)}{r}dr$$
$$+\,n(0,0)\log\rho+\log|c_\lambda|$$

where $\overset{+}{\log}\ \alpha$ is defined as $\log \alpha$ or zero according as $\alpha \geqq 1$ or $0 \leqq \alpha < 1$, where $n(r, 0)$ and $n(r, \infty)$ are the numbers of zeros and of poles of $f(x)$ for $|x| \leqq r$, and where in the neighborhood of the origin we have

$$f(x) = c_\lambda x^\lambda + c_{\lambda+1} x^{\lambda+1} + \cdots, \quad c_\lambda \neq 0.$$

We can recommend to anyone who is interested in the advances now being made in the theory of functions of a single complex variable, or who wants an indication of some of the work done in the past decade, or who would have presented to him some of the important unsolved problems in the theory, this book by R. Nevanlinna.

J. L. Walsh

The Calculus. By Hans H. Dalaker and Henry E. Hartig. McGraw-Hill Book Company, New York, 1930. viii+254 pages. $2.25

This textbook is designed for a first course in calculus and follows to a large extent the tradition of Granville and others. The authors say: "Especial effort has been made to guide the student to an understanding of basic principles and inner meanings." The first two chapters show a serious attempt to justify this statement. In them are discussed very carefully matters of notation and the subjects of limits and continuity. One feels that the average student can master this material without undue difficulty and thus lay a firm foundation for the study of the calculus.

In Chapter III the derivative is defined. Here, following the Granville tradition, the symbol dy/dx is used. Then the two usual interpretations of the derivative are given, viz, *the derivative as a rate of change*, and *the derivative as the slope of a curve.* The former interpretation usually causes the student a great deal of trouble; and one is inclined to think that the explanation given on page 12 will carry little meaning to him. As to the latter interpretation one must remember that most students studying calculus have not studied tangents except in the case of the circle. It would seem that the explanation of the slope would be more convincing if the definition of the tangent to any curve were given. The students will probably require a great deal of help from the teacher before they will have mastered these two interpretations of the derivative.

One wonders why the symbol dy/dx is used for the derivative at the beginning of the course when it causes so much uncertainty in the minds of so many students. They fail to realize that the derivative is the limit of a ratio, and think that in some unknown way Δy has become dy and Δx has become dx. Certainly there are many teachers who feel that the symbols $D_x y$, y', and $f'(x)$ produce better results.

The first ten chapters of the book are devoted largely to a mastery of the technique of differentiation. Chapters XI–XV deal with applications of differentiation to geometry and physics. There is a wealth of well-graded exercises

in these chapters. However, one feels that the problems in the applications to geometry can be handled to a large extent mechanically by the student. Chapter XVI deals with the *law of the mean and indeterminate forms* in the manner that is usually found in textbooks in calculus.

After a detailed study of differentiation, integration is studied in great detail. A student who works through the lists of exercises on indefinite integration will have a good knowledge of the technique of the subject. Following the drill on the indefinite integral, the authors define the definite integral. The treatment of this important subject is excellent. The basic principles of definite integration are brought out in such a way that the student should be able to apply them readily to a large variety of problems. One is impressed by the excellence of the figures in the chapter on multiple integrals. The treatment of this subject has been kept to an irreducible minimum; and one is inclined to believe that it could have been profitably expanded at the cost of curtailing some of the earlier material on technique.

In addition there are a brief chapter on partial differentiation and an excellent one on infinite series. The book closes with chapters on *Expansion of Functions*, *Hyperbolic Functions*, *Change of Variables*, and *Approximate Integration*.

One is inclined to think that the book will be widely used. It has the virtues of the old Granville text, and also many that the earlier book did not have. But in spite of the efforts that the authors have made to stress fundamental principles, a large percentage of the students who use this book will probably acquire little more than a knowledge of the technique of differentiation and integration.

One wonders if the writers of textbooks of calculus are losing an opportunity to serve the natural sciences. Men working in chemistry and biology are crying for help in mathematics. A dean of a medical school says that biologists should understand the fundamental principles of calculus, but that they never need the mastery of the technique of differentiation and integration which is required by most teachers of mathematics. Is a knowledge of so much technique necessary even for engineers and physicists? The long lists of exercises on differentiation remind one of those on factoring in textbooks in elementary algebra of thirty years ago. One ventures to hope that in the near future a textbook in calculus will appear which will stress fundamental principles and in which technique is reduced to a minimum. Isn't it possible that such a book will meet the needs of biologists, chemists, engineers, physicists, and mathematicians better than the books which give so much attention to technique?

W. G. SIMON

Calculus. By Egbert J. Miles and James S. Mikesh. The McGraw-Hill Book, New York, 1930. xii+638 pages. $3.75

The main thesis of the authors is that the concept of rate of change should be given the major rôle in a first course in the calculus, and that everything can

be and should be developed around rate of change as a unifying principle. They believe that many of the things that are usually done are bad to do; they know exactly what they want to do, and they proceed to do it throughout 638 pages. No change from tradition is too great to maintain rate of change in its central position. Thus the differential is practically eliminated, integration is anti-differentiation, and no use is made of the integral as the limit of a sum. Whether or not it is advisable to emphasize rate of change is debatable—and although it is of no moment, the reviewer believes it is not desirable— but surely such far-reaching consequences as these require very careful consideration.

Limits, rate of change, differentiation and applications, integration and applications occupy the first thirteen chapters (263 pages). In these chapters differentiation is confined to algebraic functions, and the scope of the integration is thus grestly restricted. Chapter XIV on "Appoximations" contains among other things Maclaurin's and Taylor's series. An unexpected chapter (XV) on functions of several variables contains not only a brief treatment of partial differentiation, but also exact derivative (differential) equations, and even line integrals. In Chapter XVI the exponential and logarithmic functions are introduced, the discussion is detailed, and the many physical and chemical applications lead rather naturally to the consideration of first order linear differential equations. The trigonometric functions are introduced in Chapter XVII and treated at such length that the chapter runs to 114 pages. Section 104, "Solution of Triangles," is exactly that; the reason for its insertion in a text on the calculus is quite unknown to the reviewer.

The maintenance of rate of change as a unifying principle leads the authors to make substantial changes in notation. No d's nor deltas are introduced, the derivative of $f(x)$ when $x=x_1$ is the constant $f'(x_1)$, the derivative function is $f'(x)$, the integral of $f(x)$ is $\int f(x)$. In differentiation the "delta-process" has been completely eliminated. In this connection the authors say: " . . . . the student feels he is continuing work already begun in algebra, and his attention is focused on the principle involved rather than upon the manipulatory processes." We do not feel that this claim is justified. Specifically we refer the reader to pages 38, 39, and 81 where the derivative of $u^n(x)$ is obtained. Integration without the differential notation is by no means impossible, but it does seem to involve added difficulties for both student and instructor. The excessive use of the solidus is displeasing, typographically, and this and other usages make for errors which are all too numerous throughout the book.

We have noted the inclusion of functions of several variables and of differential equations. If we leave these topics aside, the scope of the book, despite the six hundred odd pages, is distinctly less than that of the average American text. Curvature, evolutes, singular points, centers of gravity, moments of inertia, and mechanics are entirely omitted. One might ask how the authors employ so many pages. Nearly 40% of the space is occupied by illustrative examples, worked out in complete detail and with full explanations. The problems, nominally 1100 in number, actually total about 2100 if distinct parts

be considered separate problems. The uniform simplicity of these disturbs us. With the exception of two theoretical problems (#3 on page 245, and #13 on page 475) there is not a problem in the entire book which constitutes a real challenge to a good sophomore. The text is full, accurate, and thorough, there is careful motivation, and the explanations are unusually detailed. We appreciate fully that there is honest workmanship, that this text adheres to definite aims, and that it endeavors to set up standards of accurate thought. Yet we must add our personal conviction that, pedagogically, this is not a good book. There is too much detailed explanation, too much illustration; and the student who studies faithfully an assignment of seven pages will have little energy and no inclination to tackle problems for himself. There is little opportunity here for the student to develop initiative and mathematical robustness. In our judgment, the authors fail to understand either the weaknesses or the strengths of the American sophomore.

B. H. BROWN

Enciclopedia delle Matematiche Elementari. L. Berzolari, G. Vivanti, D. Gigli (general editors). Volume 1, Part 1. Milan, Ulrico Hoepli, 1930, 9+450 pages.

The present "Part 1", is to be followed at once by "Part 2," announced as already under press. Volume 1 is devoted to analysis, and is to be one of three volumes constituting this encyclopedia. The second volume is to deal with geometry and the third with applications of mathematics, history of mathematics and didactic questions. The present part comprises seven independent articles together with indices etc., namely: I. Logic, by Alessandro Padoa; II. General arithmetic, by Duilio Gigli; III. Practical arithmetic, by Ettore Bortolotti and Duilio Gigli; IV. Theory of numbers and indeterminate analysis, by Michele Cipolla; V. Progressions, by Aldo Finzi; VI. Logarithms, by Aldo Finzi; VII. Mechanical calculation, by Giuseppe Tacchella.

This is the first work of this sort in the Italian language and the editorial commission has approached its task with an enthusiasm that suggests not merely service in a needed field, but also patriotic fervor, as seen throughout the preface dated "1929 (A. VII)." The form and subject matter may well be contrasted with that of Pascal's "Repertorium" (not mentioned in this work) whose expanded German editions are proving to be of the highest service for ready reference. The treatment here used is intended to meet the demand of teachers and younger students who do not have ready access to basic sources and the material to be covered is intended to be roughly that touched by the end of the first two years of university courses.

Any elaborately detailed and critical analysis of each of the seven condensed treatises which are here offered to the reader will be inappropriate. The American teacher will find most of the material elsewhere in a language more familiar to him. Despite excellent historical references, completeness in this book is not promised. The value of the work cannot consist in its completeness, nor

in its bringing to light unsuspected historical sources. It must be judged by the degree to which it renders available to college and secondary school teachers the important elements touching upon the basic mathematical courses, and the good judgment with which topics from so vast a possible field are selected and woven into smoothly flowing expositions. The authors have refrained from that hint of superiority that might be suggested by the familiar phrase, "vom höheren Standpunkte aus" and incidentally have missed much of the lively interest for the graduate student that characterizes Klein's work of a somewhat similar purpose (*Elementarmathematik vom höheren Standpunkte aus*, Third Edition, 1924–1928). One ought rather to compare this work with the H. Weber and J. Wellstein, *Encyklopädie der elementar-Mathematik* (Vol. 1, 4th Edition 1922, Vol. 2, 2nd Edition 1907. Vol. 3, 1st Edition 1907).

The first section (of 79 pages) is a serious study of the formal elementary aspects of modern logic as used by Peano and later by Russell and others. This section is not referred to in later parts and would seem to be somewhat out of keeping with the general tenor of the work, despite the fact that mathematics uses logic and to a considerable extent is logic. Controversial questions and all references to paradoxes are suppressed, but the reader may agree with the reviewer in feeling that the theories have not yet been subjected to that intensive critical analysis by many independent workers which is the only practicable test of scientific stability. The astonishing number of 143 independent (?) axioms are listed consecutively.

The section on general arithmetic (covering 130 pages) deals with what is often called the number-concept, starting with finite classes and natural numbers and concluding with the transcendence of e and π. The treatment is excellent although necessarily far from novel. It may be too abstract to suit many college teachers in this country, but would be worth any effort required to understand it.

The third section devoted to "practical arithmetic" is what many persons might call the abstract theory of arithmetic. Apart from some interesting historical topics the section deals in a logical but thoroughly elementary way with the fundamental operations of arithmetic, and the proofs incident thereto (assuming much prior logical foundation). Questions concerning mensuration, percentage and other topics of commercial arithmetic find no place.

The fourth section treats of certain topics in elementary number theory. Apart from a very brief preliminary discussion, two main divisions deal respectively with linear congruences (Fermat-Euler theorem, the Gaussian, Wilson's theorem, Euclid's algorithm, etc.), and with higher congruences (roots of unity, quadratic congruences, etc.). In no other section does one find so many familiar topics omitted, but what has been retained shows an unusual attempt at continuity. The treatment is simple, interesting and in connection with topics treated reasonably comprehensive.

The fifth section, dealing with progressions, seems in strange company. It discusses only the traditional arithmetic, geometric, and harmonic series and

some of the most elementary extensions of the first two to figurate numbers. The use of $x/\!\!\wedge y$ to denote x^y is the most noticeable single item of interest. The section does, of course, form a natural bridge from linear congruences to the introduction of logarithms.

The sixth section, on Logarithms, is more than usually effective and covers material ordinarily assumed to be too detailed for elementary texts and too special for advanced mathematical courses.

The seventh section, on computing machines, suffers only from the common difficulty incidental to merely talking about mechanical devices whose virtues lie in their concrete utility. While one machine actually used would outweigh in significance any number of mere descriptions the reader is at least prepared to admit that there are many machines and for divers purposes.

The paper is poor, the binding flimsy, the figures cheaply drawn, and typographical mistakes not unknown (The table of contents lists "Errata, pag. 415." It should read "P. 451."). But throughout the presentation is scholarly, the emphasis well-placed, and the language simple, connected and interesting. Whether the completed work will prove eventually more attractive to American readers than the somewhat more conventional "Weber-Wellstein," will remain to be seen.

ALBERT A. BENNETT

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

PROBLEM FOR SOLUTION

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3441. *Proposed by Solomon Kullback, Brooklyn, New York.*

In a given triangle, inscribe an equilateral triangle having a given point, P, on one of its sides. Suggested by problem No. 3405.

3442. *Proposed by the late F. P. Matz*

Solve the equation $d/dw(dw/dr+2w/r)=0$.

3443. *Proposed by Warren A. Rees, Houston Junior College.*

To construct a quadrilateral, given the four feet of the perpendiculars from the point of intersection of the diagonals upon the four sides.

3444. *Proposed by Frank Morley, Johns Hopkins University.*

In an inversive plane, the general self-conjugate equation, $f(x,\bar{x})=0$, of degree three in x and $\bar{x}$ defines a bi-cubic curve, c. Since any circle has with such a curve six common points (intersections or common unique pairs) there are contact circles, touching thrice. There are, it is known, 120 contact circles. If we take three of these, the 9 points of contact either lie on a biquadratic b, or they do not. When they do, the circles are tied (or syzygetic); and the curve, b, meets c in the points of contact of a fourth circle, so that the circles are tied in sets of four. Prove that the four circles of a set touch a circle.

3445. *Proposed by Mannis Charosh, New Utrecht High School.*

If p is odd and greater than 1, prove that

(a) $$1^2 \cdot 3^2 \cdot 5^2 \cdots (p-2)^2 \equiv (-1)^{(p+1)/2} \bmod p;$$

(b) $$2^2 \cdot 4^2 \cdot 6^2 \cdots (p-1)^2 \equiv (-1)^{(p+1)/2} \bmod p.$$

3446. *Proposed by J. Rosenbaum, Milford, Conn.*

Given n points, $A_1, A_2, \ldots, A_n$, and n constants $a_1, a_2, \ldots, a_n$, show how to locate a point X such that the vectors

$$a_1 \cdot XA_1,\ a_2 \cdot XA_2, \cdots, a_n \cdot XA_n$$

shall form a closed polygon.

This is a generalization of problem 3395 (November, 1929).

Here the point X is the center of mass of the n masses $m_1, m_2, \ldots, m_n$ placed at $A_1, A_2, \ldots, A_n$ and proportional to $a_1, a_2, \ldots, a_n$.

3447. *Proposed by Vladimir F. Ivanoff, San Francisco, California.*

Prove that

$$2R = R_x \cos\alpha + R_y \cos\beta + R_z \cos\gamma,$$

where R is the radius of curvature of a given curve at the point $(x,y,z,)$; R_x, R_y, R_z are the radii of curvature of the projections of this curve on the co-ordinate planes, YOZ, XOZ, and XOY at the points (y_1, z_1), (x_1, z_1) and (x_1, y_1), respectively; and α, β, γ are the angles between R and R_x, R_y, R_z, respectively.

SOLUTIONS

3366. [1929, 168]. *Proposed by Otto Dunkel, Washington University.*

Given the two equations

$$x^{r+2} - bx^{r+1} \pm l(x-a) = 0, \qquad 0 < a < b,\ l > 0,$$

where r is any positive number, let r_1 and r_2 be the smallest and largest positive roots of the second equation (with the lower sign). Prove that all of the roots of the two equations except r_1 and r_2 lie within the circular ring with radii r_1 and r_2 about the origin as center.

Let $OA=a$, $OB=b$, $OR_1=r_1$, $OR_2=r_2$, and let R_1' R_2' be the harmonic conjugates of R_1, R_2 with respect to A and B. Show that neither equation has positive roots on the segments R_1R_1', R_2R_2' except the two roots of the second equation r_1, r_2.

Solution by the Proposer.

Setting $f(x)=x^{r+2}-bx^{r+1}-lx+la$, we see that $f''(x)=(r+1)x^{r-1}[(r+2)x-br]$ vanishes only once for positive values of x. Hence $f'(x)$ decreases from the negative value $-l$ at $x=0$ to a minimum at $x=br/(r+2)$; it then increases from that point on. Hence $f'(x)$ vanishes once and only once for positive values of x, and therefore $f(x)$ cannot have more than two positive roots. Since $f(0)=la$, $f(a)=-a^{r+1}(b-a)$, $f(b)=-l(b-a)$, $f(\infty)=\infty$, it follows that $f(x)$ has precisely two positive roots r_1 and r_2 such that $r_1<a<b<r_2$.

In the plane of the complex variable z designate by A and B the points determined by $z=a$ and $z=b$ on the positive real axis; and consider the family of circles with centers on the real axis and orthogonal to the circle with the diameter AB. Any point P on one of the circles (C_λ) of the family with the center C_λ determines the ratio $AP/BP=\lambda$ which is constant as the point moves on this circle. For $\lambda=0$ we have the null circle at A; and, as λ increases, C_λ moves to the left of A, the circles increase in size enclosing each of the preceding circles without contact or intersection. When the center is at $-\infty$, $\lambda=1$ and the circle has an infinite radius; it is the straight line perpendicular to AB at its mid-point. As λ increases from 1, C_λ moves from $+\infty$ towards B, the circles decrease in size, and each lies within each of the preceding circles. This continues until $\lambda=\infty$ and we then have the null circle at B.

Designate by R_1 and R_2 the points determined by the two roots above, r_1 and r_2. Then R_1 and R_2 determine two circles of the family with the ratios λ_1 and λ_2, the first with the diameter R_1R_1' containing A and the second with the diameter $R_2'R_2$ containing B. Let z be a root of either one of the two equations of the problem and suppose that $|z|<r_1$. Let Z be the point determined by z. Then

$$\frac{|z|^{r+1}}{l}=\left|\frac{z-a}{z-b}\right|=\frac{AZ}{BZ}.$$

Consider the circle of the family through Z; for this circle $\lambda=|z|^{r+1}/l<r_1^{r+1}/l$. But since r_1 is a root of $f(x)$,

$$r_1^{r+1}/l=(r_1-a)/(r_1-b)=AR_1/BR_1=\lambda_1.$$

Hence $\lambda<\lambda_1<1$ and the λ circle lies completely within the λ_1 circle. From this follows that $|z|>r_1$, and this contradicts the hypothesis. Hence no root of either equation lies within the circle with center at the origin and radius r_1. If $|z|=r_1$, then $\lambda=\lambda_1$, and z must lie on the λ_1 circle; it must therefore be at R_1. Hence it must be a root of the second equation only.

Now suppose that z is a root of either equation such that $|z|>r_2$. Then for the circle of the family through Z we have as before $\lambda=|z|^{r+1}/l>r_2^{r+1}/l=\lambda_2>1$. The λ_2 circle encloses the λ circle, and hence $|z|<r_2$, a contradiction. If $|z|=r_2$, then $\lambda=\lambda_2$, and Z must be at R_2. It has now been proved that all of the roots of either equation lie within the circular ring with the origin as cen-

ter and radii r_1 and r_2, except the two real roots of the second equation, which lie on the boundaries of the ring.

It has been shown that for any root z of either equation, excepting r_1 and r_2 we must have $r_1 < |z| < r_2$. Hence if λ is the ratio determined by z then

$$r_1^{r+1}/l < |z|^{r+1}/l < r_2^{r+1}/l, \text{ or } \lambda_1 < \lambda < \lambda_2.$$

It then follows that z cannot lie within either the λ_1 or the λ_2 circle. It cannot lie on the boundary of either circle, for if it lies, say on the λ_1 circle, we would have $|z|^{r+1}/l = r_1^{r+1}/l$, or $|z| = r_1$. This is impossible unless $z = r_1$.

3396. [1929, 492]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Let D, E, F be the feet of the altitudes, and A', B', C', the mid-points of the sides of a triangle ABC. Show that the double elements of the three involutions (BC, DA'), (CA, EB'). (AB, FC') are three pairs of opposite vertices of a complete quadrilateral.

I. Solution by Mabel M. Young, Wellesley College.

Let the sides of the triangle ABC meet the sides of the triangle DEF in O_a, O_b, O_c. Since the altitudes are concurrent, O_a, O_b, O_c are collinear. These points lie on the radical axis of the circumcircle and the nine-point circle of the triangle ABC, since each point is the radical centre of these circles and one of the circles having the sides of ABC as diameter. Hence O_a, O_b, O_c are the centers of the given involutions, which are determined on the sides of the triangle by these two circles. Of the three pairs of double points, either three or one will be real, according as the triangle ABC is acute or obtuse.

Let A_1A_2, B_1B_2, C_1C_2 be the double points of the involutions on a, b, c respectively, and let the complete quadrilateral which has A_1A_2, C_1C_2 as two pairs of opposite vertices have X_1X_2 as third pair. Two sides of the diagonal triangle of this quadrilateral are a and c. The third side, determined by X_1 and X_2, coincides with b, since it cuts a and c in points which are harmonic conjugates of B as to A_1A_2 and C_1C_2, respectively. The mid-point of segment X_1X_2 coincides with O_b, since it is collinear with O_a and O_c. Vertices X_1 and X_2 are themselves harmonic conjugates as to A and C. Hence $(O_bX_1)^2 = O_bA \cdot O_bC = (O_bB_1)^2$, and X_1, X_2 coincide with B_1, B_2.

II. Solution by A. Pelletier, Montreal, Canada.

Let ABC be an acute angled triangle. On BC as a diameter describe a circle cutting AD in G, and let the tangent at G cut BC in O. Then it follows that $A'O \cdot DO = BO \cdot CO = GO^2$, and hence O is the center of the involution determined by the pairs (BC, DA'). The circle with O as center and radius OG cuts BC in two points X, X' which are the double points of the involution. We shall suppose that X is the point within the segment BC. Similarly, we find Y, Y' on CA, and Z, Z' on AB, where again the unaccented letters designate the points

within the segments. Thus X, X' divide harmonically BC as well as $A'D$. Since XGX' is a right angle, GX bisects the angle BGC; and from this follows that $BX/CX = -BG/GC = -(BD/DC)^{\frac{1}{2}}$. Likewise we find that $CY'/AY' = (CE/EA)^{\frac{1}{2}}$ and $AZ/BZ = -(AF/FB)^{1/2}$. Hence

$$\frac{BX}{CX}\cdot\frac{CY'}{AY'}\cdot\frac{AZ}{BZ} = \left(\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}\right)^{1/2} = 1,$$

since the altitudes meet in a point. Hence X, Y', Z lie on a straight line. In a similar manner we show that Z, Y, X'; X, Y, Z'; Y', X', Z' lie in a straight line. Hence the quadrilateral X, Y', X', Y has ZZ' for its third diagonal.

Note by Otto Dunkel. The above solutions suggest a solution which may be based on the solution of 3387 [1930, 262]. In the latter solution it is shown that the conjugate circle of a triangle is coaxial with the nine-point circle and the circumcircle in all cases. If the triangle ABC has only acute angles, as will be assumed in what follows, then the conjugate circle is imaginary, and it follows that the nine-point circle does not cut the circumcircle in real points. Let (S) denote the circumcircle with the center S, and (N) the nine-point circle with the center N. The coaxial system of circles determined by these two non-intersecting circles has two limit points I and J; or, we may say that I and J are two points on the straight line SN which are conjugate to both (S) and (N). For definiteness, suppose that I is within the two circles. The perpendicular to IJ at its mid-point is the radical axis of the two circles. Let this perpendicular cut BC in O_x, and let the circle with center O_x and radius O_xI cut BC in X and X'. This circle passes also through J, and it is orthogonal to (S) and to (N). Hence X and X' are conjugate to both (S) and (N), and they are in consequence the double points of the involution on BC. We shall suppose that X is within the segment BC. Then XIX' is a right angle and XI bisects the angle BIC. Hence $BX/CX = -BI/CI$. In a similar manner we have

$$CY/AY = -CI/AI,\ AZ/BZ = -AI/BI.$$

From the product of these three equations we infer that AX, BY, CZ meet in a point. Hence X', Y', Z' lie in a straight line, since these three points are harmonic conjugates of X, Y, Z, respectively, with respect to the corresponding vertices of the triangle. By the use of suitable harmonic pencils it may be shown that Y', X, Z; X', Y, Z; Z', Y, X lie in straight lines, and the rest easily follows.

Also solved by Paul Wernicke.

3398. [1929, 542]. *Proposed by V. Ivanoff, San Francisco, California.*

Prove that

$$\frac{1}{R^2} = \frac{\sin^6 M}{R_x^2} + \frac{\sin^6 N}{R_y^2} + \frac{\sin^6 P}{R_z^2},$$

where R is the radius of curvature of a given curve in the point $(x_1y_1z_1)$; R_x, R_y, R_z are the radii of curvature of the projections of this curve in the co-ordinate

planes, YOZ, XOZ, and XOY in the points (y_1, z_1), (x_1, z_1), and (x_1, y_1), respectively; M, N, P are the angles between the tangent to the curve in the point (x_1, y_1, z_1) and OX, OY, and OZ, respectively.

Solution by Robert E. Moritz, University of Washington.

If we denote the infinitesimal arc of the curve at the point (x_1, y_1, z_1) by ds, then obviously $\cos M = dx/ds$, $\cos N = dy/ds$, $\cos P = dz/ds$ and therefore,

$$\sin^2 M = (dy^2 + dz^2)/ds^2, \ \sin^2 N = (dz^2 + dx^2)/ds^2, \ \sin^2 P = (dx^2 + dy^2)/ds^2.$$

Now

$$1/R_x^2 = (dy \cdot d^2z - dz \cdot d^2y)^2/(dy^2 + dz^2)^3;$$

hence

$$(\sin^6 M)/R_x^2 = (dy \cdot d^2z - dz \cdot d^2y)^2/ds^6,$$

and

$$\frac{\sin^6 M}{R_x^2} + \frac{\sin^6 N}{R_x^2} + \frac{\sin^6 P}{R_x^2}$$
$$= \frac{(dz \cdot d^2y - dy \cdot d^2z)^2 + (dx \cdot d^2z - dz \cdot d^2x)^2 + (dy \cdot d^2x - dx \cdot dy^2)^2}{ds^6},$$

which is the well known expression for $1/R^2$.

Also solved by M. S. Knebelman, J. D. Leith, T. L. Smith, Paul Wernicke, and the Proposer.

3400 [1929, 543]. *Proposed by W. O. Pennell, St. Louis, Mo.*

Find a function of x, $f(x)$, such that

$$f(x) = xf(x - r) = x(x - r)f(x - 2r) = x(x - r)(x - 2r)f(x - 3r), \text{ etc.}$$

and $f(r) = r$, where r is a given real quantity > 0.

Solution by H. T. Davis, Indiana University.

The problem is reduced to the solution of the difference equation,

$$u(t + 1) = r(t + 1)u(t),$$

by means of the transformation: $x = r(t+1)$, $u(x) = f(rx)$. Assuming a solution of the form $u(t) = \phi(t)\Gamma(t+1)$, we are led for the definition of $\phi(t)$ to the simple difference equation, $\phi(t+1) = r\phi(t)$. The solution of this equation by ordinary methods is $\phi(t) = \psi(t)r^{t+1}$, where $\psi(t)$ is any periodic function of unit period. Hence the general solution of the original equation will be,

$$f(x) = \psi(xr^{-1})r^{\rho}\Gamma(\rho) \quad \text{where} \quad \rho = xr^{-1} + 1.$$

In order to satisfy the condition $f(r) = r$, it is merely necessary that $\psi(1) = r^{-1}$. A particular solution is given by setting $\psi(xr^{-1}) = r^{-1}$.

The solution just given follows ordinary methods. It might be interesting, therefore, to show how the solution can be attained by a less known method involving the use of differential operators of infinite order.[1]

Using the symbolic operator $f(x-r)=e^{-rz}\to f(x)$, where $z=d/dx$, we may write the original equation in the form,

$$(1 - xe^{-rz}) \to f(x) = 0.$$

Let us seek a solution of the form $f(x)=\int_L e^{xt}v(t)dt$, where $v(t)$ is a function to be determined and L is some path of integration in the complex plane. Introducing this expression into the differential equation we obtain,

$$(1 - xe^{-rz}) \to f(x) = \int_L e^{xt}(1 - xe^{-rt})v(t)dt.$$

Integrating by parts the term multiplied by x, we get

$$(1 - xe^{-rz}) \to f(x) = - e^{xt-rt}v(t)\Big|_L + \int_L e^{xt}[v(t) + d\{e^{-rt}v(t)\}/dt]dt.$$

Let us now seek a value of $v(t)$ which satisfies the equation,

$$v(t) + d\{e^{-rt}v(t)\}/dt = 0,$$

and a path L at the extremities of which $e^{t(x-r)}v(t)$ is zero. Since we find $v(t)=Ce^{\mu}$, where $\mu=rt-e^{rt}r^{-1}$, we can choose L to be the path along the real axis from $-\infty$ to $+\infty$. Changing variables from t to s by means of the equation $s=e^{rt}r^{-1}$, L becomes a path from 0 to infinity. We thus obtain,

$$f(x) = C\int_0^\infty e^{-s}(rs)^{x/r}ds = Cr^{x/r}\Gamma(xr^{-1} + 1).$$

The path $L=\infty$ to 0, n loops about the origin, 0 to ∞ would give:

$$C(e^{2\pi inx/r} - 1)r^{x/r}\Gamma(xr^{-1} + 1).$$

The desired function which reduces to r for $x=r$ is obtained by letting $C=1$ in the less general preceding solution.

Also solved by G. D. Leith, Byron T. Roberts, T. L. Smith, and the Proposer.

3401 [1929, 543]. *Proposed by Paul Wernicke, Washington, D. C.*

Let a, b, c, d, be four lines in a plane no three of which are concurrent. Let

[1] See for example: T. Lalesco, *Sur l'équation de Volterra*, Journal de Mathématiques, (6), vol. 4, (1908), p. 194, and E. Hilb, *Lineare Differentialgleichungen unendlicher Ordnung*, Mathematische Annalen, vol. 82 (1921), p. 20.

f be the join of the intersections ac and bd and g the join of the intersections of ad and bc. Prove that

$$(\sin af/\sin fc)/(\sin ag/\sin gd) = (\sin bg/\sin gc)/\ (\sin bf/\sin fd).$$

I. Solution by Robert E. Moritz, University of Washington.

Let us first assume that no two of the given lines are parallel. Then any three of the six lines, a, b, c, d, f, g, which are not concurrent, will form a triangle. Let us denote the sides of any such triangle, say the triangle formed by the lines b, c, d, by $\bar{b}$, $\bar{c}$, $\bar{d}$, and the area of the triangle by $T(b, c, d)$. With this notation we have

$$(1) \qquad 2\cdot T(a, f, d) = \bar{a}\cdot\bar{f}\cdot\sin af = \bar{f}\cdot\bar{d}\cdot\sin fd,$$

$$(2) \qquad 2\cdot T(b, f, c) = \bar{f}\cdot\bar{c}\cdot\sin fc = \bar{b}\cdot\bar{f}\cdot\sin bf,$$

$$(3) \qquad 2\cdot T(a, g, c) = \bar{a}\cdot\bar{g}\cdot\sin ag = \bar{g}\cdot\bar{c}\cdot\sin gc,$$

$$(4) \qquad 2\cdot T(b, g, d) = \bar{g}\cdot\bar{d}\cdot\sin gd = \bar{b}\cdot\bar{g}\cdot\sin bg.$$

If now we divide the quotient of (1) divided by (2) by the quotient of (3) divided by (4), and divide out the common factor $\bar{d}/\bar{c}$, we obtain the relation in question.

The relation just established for the cases when no two of the given lines are parallel will continue to hold true when two or more of the given lines are parallel, for every such case may be considered the limiting case of a quadrilateral for which the relation is invariant. It must then hold true in the limit.

II. Solution by S. F. Bibb, Armour Institute of Technology.

The following is a general statement of the problem and its solution: Suppose that the n sides of a polygon are $a_1, a_2, a_3, \cdots a_n$. Let b_1 be a diagonal drawn from the intersection of a_na_1 to the intersection of a_2a_3, and b_2 a diagonal drawn from the intersection of a_1a_2 to the intersection of a_3a_4, and b_3 a diagonal drawn from the intersection of a_2a_3 to the intersection of a_4a_5, and similarly for $b_4, b_5, \cdots b_n$. Prove that

$$\frac{\sin a_1b_1 \sin a_2b_2 \sin a_3b_3 \cdots \sin a_nb_n}{\sin a_2b_1 \sin a_3b_2 \sin a_4b_3 \cdots \sin a_nb_{n-1} \sin a_1b_n} = (-1)^n.$$

Proof: By law of sines one obtains:

$$(\sin a_1b_1)/(\sin a_2b_1) = -(a_2)/(a_1),$$

$$(\sin a_2b_2)/(\sin a_3b_2) = -(a_3)/(a_2),$$

$$(\sin a_3b_3)/(\sin a_4b_3) = -(a_4)/(a_3)$$

$$\cdots\cdots\cdots\cdots\cdots,$$

$$\cdots\cdots\cdots\cdots\cdots,$$

$$(\sin a_nb_n)/(\sin a_1b_n) = -(a_1)/(a_n).$$

Multiplying these equations together one gets the result as stated. For $n=4$, this gives the solution of the proposed problem.

Also solved by E. M. Berry, W. E. Buker, Solomon Kullback, J. D. Leith. A. Pelletier, T. L. Smith, Abraham Sinkov, and the Proposer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Professor Enrico Bompiani, of the University of Rome, has recently delivered the following lectures as Visiting Professor of the American Mathematical Society: "Differential Equations and Projective Geometry," at Columbia University on May 16; "A Contribution to the Geometry of Paths," at Princeton University on May 19; "Projective Differential Geometry and the Italian School" (two lectures), at Cornell University on May 21, 22; "What is Geometry," at the Ohio State University on May 28; "Italian Contributions to Modern Mathematics," at the University of Iowa on June 30; "What is Geometry" and "Italian Contributions to Modern Mathematics," at the University of Chicago on July 1, 8. During the present Summer Quarter, Professor Bompiani is lecturing at the University of Chicago on "Analytic Projective Geometry" and on "Projective Differential Geometry of Hyperspaces." From October 5 to November 12 he will lecture at Johns Hopkins University on "The Differential Equations of Projective Geometry."

Professors P. W. Bridgman, of Harvard University, and Stephen Timoshenko, of the University of Michigan, are among the members of the newly formed scientific advisory board of the Westinghouse Research Laboratories.

Dr. Karl Menger, of the University of Vienna, has been appointed lecturer in mathematics at Harvard University for the first half of the year 1930–31.

Dr. J. A. Schouten wıll deliver a series of informal lectures at Harvard Unisity in the fall of 1930.

Dr. J. Frenkel, professor of theoretical physics at the Polytechnic Institute of Leningrad, will lecture on wave mechanics and conduct a seminar on problems in modern physics at the University of Minnesota during the academic year 1930–31.

Dr. Joseph Eugene Rowe, President of Clarkson Memorial College of Technology, was granted the honorary degree of LL.D. at the annual commencement of Gettysburg College held on June 9, 1930.

The following mathematicians have been awarded Guggenheim fellowships for 1930: Professor A. N. Gándara, of the National Preparatory School of

Mexico, for studies in differential geometry and harmonic analysis at the Massachusetts Institute of Technology; Professor H. S. Vandiver, for research in Europe on Fermat's last theorem, laws of reciprocity, and related topics in the theory of algebraic numbers.

Professor C. L. Arnold, of the Ohio State University, has retired from active teaching. He was given the rank of professor emeritus.

Dr. J. Hobert Bushey has been appointed to an assistant professorship in mathematics at Hunter College of the City of New York.

Assistant Professor W. F. Cheney, of Tufts College, has been appointed head of the department of mathematics at the Connecticut Agricultural College.

Dr. Alonzo Church has been appointed assistant professor of mathematics at Princeton University.

Dr. T. F. Cope has been appointed assistant professor and head of the department of mathematics at Marietta College.

Associate Professor Louise D. Cummings, of Vassar College, has been promoted to a professorship of mathematics.

Dr. H. B. Curry has been appointed assistant professor of mathematics at Pennsylvania State College.

Dr. Julia Dale, head of the department of mathematics at Delta State College in Mississippi, has been appointed assistant professor of mathematics at Duke University.

Dr. Marguerite D. Darkow has been promoted to an assistant professorship in mathematics at Hunter College of the City of New York.

Dr. Jesse Douglas has been appointed assistant professor of mathematics at the Massachusetts Institute of Technology.

Mr. R. D. Douglass, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematcs.

Professor W. E. Edington, of Purdue University, has been appointed professor and head of the department of mathematics at DePauw University.

Assistant Professor Philip Franklin, of the Massachusetts Institute of Technology, has been promoted to an associate professorship of mathematics.

Dr. B. P. Gill has been promoted to an assistant professorship of mathematics at the College of the City of New York.

Dr. E. L. Hill has been appointed assistant professor of theoretical physics at the University of Minnesota.

Assistant Professor B. P. Hoover, of the Carnegie Institute of Technology, has been promoted to an associate professorship of mathematics.

Dr. Jewell C. Hughes has been appointed to an assistant professorship in mathematics at Hunter College of the City of New York.

Assistant Professor R. P. Johnson, of the Carnegie Institute of Technology, has been promoted to an associate professorship of mathematics.

Assistant Professor R. B. Lindsay, of Yale University, has been appointed associate professor of theoretical physics at Brown University.

Associate Professor P. H. Linehan, of the College of the City of New York, has been promoted to a professorship of mathematics.

Dr. W. H. McEwen, of the University of Minnesota, has been appointed assistant professor of mathematics in Mount Allison University, Sackville, N. B.

Associate Professor M. L. MacQueen, of Southwestern University, has been promoted to a professorship of mathematics.

Assistant Professor Florence M. Mears, of Pennsylvania State College, has been appointed assistant professor of mathematics at George Washington University.

Dr. T. W. Moore has been appointed assistant professor of mathematics at Indiana University.

Associate Professor Marston Morse has been promoted to a professorship at Harvard University.

Associate Professor E. J. Oglesby, of Washington Square College, New York University, has been promoted to a professorship of mathematics.

Dr. G. A. Parkinson has been promoted to an associate professorship of mathematics in the extension division of the University of Wisconsin at Milwaukee.

Professor H. B. Phillips, of the Massachusetts Institute of Technology, has been granted leave of absence for the academic year 1930–31.

Assistant Professor Emory Potterstarke has been promoted to an associate professorship of mathematics at Rutgers University.

Dr. I. I. Rabi has been promoted to an assistant professorship of theoretical physics at Columbia University.

Mr. L. H. Rice, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

Dr. J. H. Roberts has been appointed adjunct professor of mathematics at the University of Texas.

Assistant Professor George Rutledge, of the Massachusetts Institute of Technology, has been promoted to an associate professorship of mathematics.

Mr. George Sauté has been appointed assistant professor of mathematics at Cleveland College.

Mr. C. N. Shuster has been promoted to an assistant professorship of mathematics at the State Teachers College at Trenton, N. J.

Dr. C. H. Smiley has been appointed assistant professor of mathematics at Brown University.

Mr. I. S. Sokolnikoff has been promoted to an assistant professorship of mathematics at the University of Wisconsin.

Dr. E. H. Taylor has been promoted to a professorship of mathematics at the Eastern Illinois State Normal College.

Assistant Professor R. W. Veatch, of Ursinus College, has been appointed assistant professor of mathematics at the University of Tulsa.

Assistant Professor J. L. Walsh has been promoted to an associate professorship of mathematics at Harvard University.

Dr. Dorothy W. Weeks has been appointed professor of physics at Wilson College.

Assistant Professor E. A. Whitman, of the Carnegie Institute of Technology, has been promoted to an associate professorship of mathematics.

Associate Professor D. V. Widder, of Bryn Mawr College, has been promoted to a professorship of mathematics.

Professor A. R. Wilson, of Haverford College, delivered a lecture on March 27 for the mathematics clubs of Rutgers University on the subject, "Space Filling Polyhedra."

Mr. Carleton R. Worth, of Rutgers University, has been granted leave of absence for the year 1930–31. He will study at the California Institute of Technology.

Associate Professor Mabel M. Young, of Wellesley College, has been promoted to a professorship of mathematics.

At the University of Chicago, Mr. Lawrence M. Graves and Mrs. Mayme I. Logsdon have been promoted to associate professorships in mathematics. Mr. Ralph G. Sanger and Mr. Clifford W. Mendel are appointed to instructorships, Mr. Ralph E. Huston to an assistantship, and Mr. Max Coral and Mr. Arnold E. Ross to research assistantships for 1930–31.

The following appointments to instructorships are announced:

University of British Columbia, Mr. F. J. Brand.

Duke University, Mr. F. G. Dressel of the University of Michigan and Mr. J. A. Greenwood of the University of Missouri.

University of Iowa, Dr. J. M. Earl.

Massachusetts Institute of Technology, Part time: Mr. H. A. Giddings of the versity of Vermont, Mr. E. M. Pease of Purdue University, Mr. J. G. Estes of the Texas Christian University. Full time: Mr. S. B. Littauer.

North Carolina State College, Mr. E. R. Elliott.

Princeton University, Dr. K. E. Rosinger (in the department of Philosophy.)

Rutgers University, Mr. H. B. Huntley, and Mr. H. S. Grant of the University of Buffalo.

Dr. T. E. McKinney, for twenty years professor of mathematics at the University of South Dakota, died on April 14, 1930, at the age of sixty-six.

Dr. J. L. Markley, professor emeritus of mathematics at the University of Michigan, died on April 20, 1930, in his seventy-first year.

Dr. W. E. Story, professor emeritus of mathematics at Clark University, died on April 11, 1930, at the age of seventy-nine.

Professor Florian Cajori died suddenly of pneumonia on August 14, 1930, at his home in Berkeley, California. He was a charter member of the Mathematical Association of America and was one of an original group of four (later enlarged to twelve) representatives of mid-western universities and colleges who made possible the re-establishment of the American Mathematical Monthly on a sound financial basis. A detailed account of his historical researches will be published in the *Monthly* in due course.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of .... Dollars, to be known as the .... Fund, and to be used

for[2] { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witness: .... Signature ....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

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BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.

INDIANA, Earlham College, May 2-3.

IOWA, Ames, Iowa, May 2-3.

KANSAS, February 15.

KENTUCKY, Lexington, Ky., April 5.

LOUISIANA-MISSISSIPPI, Cleveland, Miss., March 7-8.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA. May 10

MICHIGAN, Ann Arbor, Mich., March 22.

MINNESOTA, Carleton College, May 17.

MISSOURI.

NEBRASKA, Peru, Neb., May 9.

OHIO, Columbus, Ohio, April 3.

PHILADELPHIA, Philadelphia, Pa., November 29.

ROCKY MOUNTAIN, Denver, Colo., April 11-12.

SOUTHEASTERN, Atlanta, Ga., May 2-3.

SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, Calif., March 8.

TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The fourteenth annual meeting of the Rocky Mountain Section of the Mathematical Association was held at the University of Denver, Denver, Colorado, on April 11–12, 1930. There were three sessions, Professor G. W. Gorrell acting as chairman at each.

The attendance was forty-two including the following twenty-seven members of the association: C. F. Barr, J. Britton, A. G. Clark, J. R. Everett, J. C. Fitterer, G. W. Gorrell, S. G. Hacker, C. A. Hutchinson, D. Jackson, H. Karnow, A. J. Kempner, Miss Claribel Kendall, A. J. Lewis, G. H. Light, A. S. McMaster, J. Q. McNatt, W. K. Nelson, Miss Greta Neubauer, Miss L. R. Odell, E. J. Purcell, E. D. Rainville, A. W. Recht, W. J. Risley, L. J. Rote, Miss Mary Sabin, C. H. Sisam, Miss Adela M. Thom.

The following officers were elected for the coming year: Professor Claribel Kendall, University of Colorado; Vice-Chairman Professor C. F. Barr, University of Wyoming.

The following papers were read:

1. "Fregier's theorem" by Professor Francis Regan, Colorado Agricultural College, by invitation.

2. "Predicting occultations" by Professor A.W.Recht, University of Denver.

3. "Foci of algebraic curves" by Professor Claribel Kendall, University of Colorado.

4. "On the invariance of certain types of areas" by Professor A. G. Clark, Colorado Agricultural College.

5. "A problem in partial correlation" by Professor G. H. Light, University of Colorado.

6. "Focal surface of a normal congruence of an ellipsoid" by Professor J. R. Everett, Colorado School of Mines.

7. "A formula in terms of greatest integers giving parcel post charges as a function of weight and distance" by Professor W. K. Nelson, University of Colorado.

8. "The elliptic modular group and applications to the theory of functions" by Mr. Earl Rainville, University of Colorado.

9. "Theory of numbers and the multiplication table" by Professor A .J. Kempner, University of Colorado.

10. "Formulas of correlation in several variables" by Professor Dunham Jackson, University of Minnesota.

Abstracts of these papers follow:

1. Mr. Regan presented Fregier's theorem: if a variable chord PQ of a conic subtends a right angle at any fixed point V on the conic it passes through a fixed point F which lies on the normal to the conic at V. The proofs for the parabola, ellipse, and hyperbola were given. The theorems dealing with the locus of the Fregier points of each conic were developed, and several corollaries

growing out of the fundamental theorem were touched upon. All the work was developed from a purely analytic view point.

2. Professor Recht mentioned the value of occultations in determining the position of the moon and described an apparatus for making maps of the United States predicting within a minute the times of occultations.

3. In this expository paper Miss Kendall defined the foci of algebraic curves for the general case and for certain special cases. The four foci, two real and two imaginary, of the central conics were found. The finding of the real foci of a cubic which was the inverse of a hyperbola with respect to one of its vertices illustrated the theorem that foci always invert into foci. Mention was made of the locations of foci for cubics without singularities and for quartics with nodes at the circular points.

4. Professor Clark considered briefly the conditions under which the area cut from the curve $y=f(x)$, a polynomial, would be invariant. As an application of the conclusions, it was proved that the inflexion tangets of a general quartic cut equal areas from the curve.

5. This paper gives formulae for finding the grades that should be expected by a student who is taking mathematics, English, and history in his first year at college. The data were obtained from the actual grades for the first and second quarters.

6. Professor Everett outlined the general theory of linear congruences, and showed how linear congruences might be applied to normals of an ellipsoid. He also discussed the development and nature of the surfaces generated by normals of an ellipsoid.

7. The paper by Professor Nelson presented a formula using greatest integers which gives the parcel post charges in terms of weight and distance. The following refinements make the formula agree closely with the postal laws: (a) All packages of a given weight sent over 1800 miles have the same charges. (b) All packages of half a pound or less have postage charges dependent on the weight only. (c) When a package is sent to a point five miles either side of a zone boundary the charges are uncertain since the zones are not true circles. For such a distance the charges become indeterminate. (d) If the weight is over fifty pounds and the distance over three hundred miles, or the weight over seventy pounds regardless of the distance, the charges become infinite.

8. Mr. Rainville gave an expository account of some of the simpler outstanding properties of the elliptic modular group and the allied functions. Landau's proof of the restricted Picard theorem and some results from the work of Landau and Caratheodory were used as examples of the type of application to the theory of functions. Stress was laid on the fundamental importance of the modular functions in the general theory of analytic functions. The paper was based, in the main part, on Klein's *Theorie der Elliptischen Modulfunktionen* (1890); and L. R. Ford's *Automorphic Functions* (1929).

9. Professor Kempner explained how a large number of the elementary concepts of the theory of numbers (residues, Fermat's theorem, exponent to

which a number belongs, indices, primitive roots, etc.) can in a very simple and satisfactory manner be demonstrated by means of a square table which for a given fixed prime modulus gives both the residues of a^λ for λ fixed, a variable; and for a fixed, λ variable.

Some apparently new results will be presented on another occasion.

10. Professor Jackson's paper discussed applications of the geometrical interpretation of correlation coefficients less simple than those which are treated in papers published in recent volumes of the Monthly. In particular, it gave a geometrical derivation of the regression coefficients for a problem involving three statistical variables.

The members and friends of the association were guests of the University of Denver at a banquet on the evening of April 11. President Gorrell acted as toastmaster. The address of welcome was given by Chancellor Frederick Hunter of the University of Denver. The response was given by Professor A. J. Kempner of the University of Colorado.

Following this a very interesting and instructive address was given by the guest of honor, Professor Dunham Jackson, on "The significance of elementary mathematics in modern statistics."

A. J. LEWIS, *Secretary*

THE SEVENTH ANNUAL MEETING OF THE INDIANA SECTION

The seventh annual meeting of the Indiana section of the Mathematical Association of America was held on May 2–3, 1930 at Earlham College, Richmond, Indiana.

There were forty-five present at the meeting including the following twenty-three members of the Association: W. C. Arnold, R. W. Babcock, Gladys L. Banes, G. E. Carscallen, P. T. Copp, C. S. Doan, J. E. Dotterer, W. E. Edington, P. D. Edwards, E. D. Grant, G. H. Graves, H. E. H. Greenleaf, C. T. Hazard, D. F. Heath, Cora B. Hennel, Florence Long, Juna M. Lutz, T. E. Mason, J. A. Reising, C. K. Robbins, L. S. Shively, K. P. Williams, W. A. Zehring.

On Friday at 5:30 P.M. a reception was given to the visiting members and their guests. At 6:30 P.M. a complimentary banquet which was held in the dining room of the college was attended by sixty guests of the college. Professor E. D. Grant presided at the banquet and introduced President Denny of Earlham College, who made a brief address of welcome. Music was provided during the banquet by a trio of students of the college.

At eight o'clock a short pipe organ recital was presented in Stoddard Auditorium. The public lecture of the evening, under the auspices of Earlham College, was given by Professor Louis C. Karpinski of the University of Michigan

on the subject, "Mathematics and the Eternal Verities." Professor Karpinski pointed out that work and intensive intellectual application are the foundations of success not only in mathematics but in all the arts and sciences upon which the progress of civilization rests. In mathematics the student is inevitably confronted with these realities more than in any other subject of the school curriculum. Particularly for students who are likely ever to do creative work mathematics furnishes stimulus and inspiration. Even also for the great mass of students in our secondary schools mathematics is one of the few studies in which the fact of intellectual work or lack of work is made evident in every recitation. When this fact is combined with the fact that mathematics is the indispensable foundation for the study of engineering, economics, physics, and many other sciences, it becomes evident that mathematics must continue to hold a prominent place in our secondary school program.

At ten o'clock Saturday morning in Carpenter Hall the following officers were elected for the coming year: Chairman, P. D. Edwards, Ball State Teachers College; Vice-Chairman, G. E. Carscallen, Wabash College; Secretary-Treasurer, H. T. Davis, Indiana University.

A chairman's address in absentia by Professor Zinszer on "Sub-atomic Versus Interstellar Space," was read by Professor Grant. The first part of the paper consisted of a rapid review of the discoveries in the field of modern physics and their mathematical agencies. This included a brief description of Millikan's determination of the electronic radius, also Rutherford's determination of the approximate size of the nucleus of the gold atom. Following an exposition on the method of parallax and its application to astronomy, the paper took up a discussion of the galactic system and its dimensions.

The following seven papers were read:

1. "On non-composite plane curves of the form $C_6:8A^2B^2$," by Professor J. C. Polley, Wabash College, by invitation.

2. "Uses of vectors in geometry and trigonometry," by Professor R. W. Babcock, DePauw University.

3. "Number one and number naught," by Dr. A. F. Bentley, Paoli, Indiana, by invitation.

4. "Amount of training in mathematics required of high school teachers of mathematics in the various states," by Professor P. D. Edwards, Ball State Teachers College.

5. "Technique of instruction for large classes in mathematics," by Mr. C. E. Trueblood, Arsenal Technical Schools, Indianapolis, Indiana, by invitation.

6. "The predicted location of the 1930 center of population of the United States," by Professor L. S. Shively, Ball State Teachers College.

7. "The minimum essentials' place in mathematics courses," by Professor G. H. Graves, Purdue University.

Abstracts of these papers follow:

1. In this paper the author considers the web of sextic curves of the form $C_6:8A^2$, i.e., with eight double points. It is shown that: (1) on any non-composite

sextic C_6 of the web there are six points where a member of the pencil of cubics $C_3 - kC_3^1 = 0$ and C_6 have a common tangent and each point is a ninth double point on a non-composite web; (2) the locus of the ninth double point is a curve of order nine of the form $C_9:8A^3$.

2. Professor Babcock thinks that the concept of vector is within the grasp of the average pupil of geometry. Various problems involving intersections of lines of projections of lines may easily be solved by vector algebra. Several of the formulae of elementary trigonometry are easily derived by means of the scalar product. This work may be used for special projects for students who are possessed of intellectual curiosity.

3. Under Hilbertian technique, though not Hilbertian minimal presupposition, we may define a pure mathematics as any system of full consistency. Examining inductive numbers with the approach of Kronecker or Poincaré, rather than with the scaffoldings of Russell or of the Mengenlehre, we obtain a pure theory of number to which the distinctions of cardinal and ordinal are irrelavent. The cardinal is relegated to the "impure" or "foundation" regions of mathematics. Realistic and analytic (semantic) postulates for the investigation of Number One are constructed, and analysis is carried through by differentiation of operative zeros from realistic nulls, and by similar differentiation of infinities. A semantic number series and a semantic radix series are constructed. Under this construction proof is given that decimals are denumerable. The Cantorian proof of non-denumerability yields under analysis its realistic elements and fixations. Analysis of the ordinary form of proof that Null is a Zahl yields similar evidence of confusion between semantic symbol and realistic reference.

4. Practically all the states issue from two to a dozen different grades of high school certificates. The author discussed only the requirements for the certificates of highest grade. In most of the states a certificate of general validity for all subjects is granted. Consequently the teacher of high school mathematics may have studied no mathematics of college grade. In addition to a discussion of these requirements for the various states the author suggested some action on the part of college instructors of the state might improve the secondary teaching in Indiana.

5. During the past five or six years Mr. Trueblood has been experimenting with classes of a hundred students and finds that the results are satisfactory both from the standpoint of the teacher and the students. In this paper he outlined the method of conducting large classes and pointed out the special technique necessary.

6. Professor Shively predicted the location of the center of population of the United States basing his calculation upon estimates of population increase and of the distribution of population during the preceding decade. The results of the calculation, which was made as of January 1, 1930, are that the center has moved westward 13.2 miles and northward 2.8 miles with probable errors

of 1.25 and .84 miles respectively. This places it in N. Lat. 39°13′ and W. Long. 86°58′, a point a little to the southwest of Arney in Owen County, Indiana.

7. In this paper the author sketched a plan in use at Purdue University. The course is divided into a few heads and after the class periods allotted to each of these have been devoted to discussion, recitation, and illustrative examples as usual, a test is given which determines, with the class work, whether the student has "cleared his record" on that head. No grades are given during the course and only those who clear their record under all the heads by the end of the term receive credit for the course.

At the close of the meeting a resolution was adopted by the members expressing their appreciation to Earlham College and to the mathematics department for the splendid banquet and their efforts in making the meeting a success. Also the section expressed its appreciation to Professor Karpinski and Dr. Bentley.

V. V. LATSHAW, *Acting Secretary*

THE FOURTEENTH MEETING OF THE KENTUCKY SECTION

The fourteenth regular meeting of the Kentucky Section of the Mathematical Association of America was held at Transylvania College, Lexington, Kentucky, on Saturday, April 15, 1930. The Section was fortunate in having as its guest Professor W. D. Cairns, Secretary-Treasurer of the Association.

There were forty-three present, including the following twenty-one members of the Association: P. P. Boyd, W. D. Cairns, C. E. Caldwell, M. G. Carman, M. C. Dame, J. M. Davis, D. S. Dearman, A. R. Fehn, W. W. Garnett, Charles Hatfield, W. R. Hutcherson, C. G. Latimer, Elizabeth LeStourgeon, Mrs. A. R. Lyon, C. A. Maney, W. L. Moore, Smith Park, Sallie Pence, D. W. Pugsley, J. H. Simester, Guy Stevenson.

The chairman, Professor C. A. Maney, presided at both the morning and afternoon sessions. All present were guests of Transylvania College at luncheon.

The officers elected for the coming year were: Chairman, Professor J. M. Davis, University of Kentucky; Secretary, Professor A. R. Fehn, Centre College.

The program of the meeting was as follows:

Morning Session, 10:00 A.M.

1. "A certain identity in theta functions" by Mr. Smith Park, University of Kentucky and Eastern State Teachers' College.

2. "Trigonometric formulae by vector analysis" by Professor W. R. Hutcherson, Berea College.

3. "Mathematics—What's the use?" by Professor J. M. Davis, University of Kentucky.

4. "Difference equations" by Professor M. G. Carman, Murray State Teachers' College.

5. "A graphical solution of the equation $x^n+ax^2+bx+c=0$" by Professor Walter L. Moore, University of Louisville.

6. "Some simple methods and problems" by Professor J. M. Maxey, Asbury College, by invitation.

7. "Queen Dido's problem" by Professor Elizabeth LeStourgeon, University of Kentucky.

Luncheon 1:00 P.M.

8. "Current mathematical activities" by Professor W. D. Cairns, Oberlin College.

Afternoon Session, 2:30 P.M.

9. Mathematics for students of chemistry" by Professor A. R. Fehn, Centre College.

10. "The fundamental mathematical requirements of biology" by Professor D. S. Dearman, Kentucky Wesleyan College.

11. "The lure of mathematics" by Professor W. D. Cairns, Oberlin College.

12. Business Meeting.

Brief abstracts of some of these papers follow:

1. In this paper Mr. Park considered two sets S and S_1 of elements of algebraic numbers having a one-to-one correspondence. Two functions containing theta functions were constructed, the terms of the first being in one-to-one correspondence with the elements of S and the second with the elements of S_1. The terms of the two functions were thus in one-to-one correspondence, and by properly defining the variables of the first as linear functions of the variables of the second, making corresponding terms equal, he obtained the identity.

2. By use of two unit vectors A and B which make angles a and b with the i-axis, the dot and cross products produced the formulae for cos $(b-a)$ and sin $(b-a)$. The fundamental law for oblique spherical triangles, $\cos b=\cos c \cos a+\sin c \sin a \cos B$, was obtained by expansion of the expression $[A\times B]\cdot[C\times D]$, setting vector $B\equiv$vector D.

3. This paper discussed the importance of teachers of mathematics being prepared to answer this question whenever and however asked, and in the answer to seek to bring the questioner to realize that, aside from its all-pervading influence in all arts and science—indeed in all life, mathematics has a beauty, a harmony, a charm all its own; that it is in fact the most perfect means man has devised for discovering and realizing truth.

5. The nomographic method of D'Ocagne for the graphical solution of the equation $x^n+ax+b=0$ is extended to apply to the equation $x^n+ax^2+bx+c=0$ by the use of the parabola as the variable curve instead of a straight line. A mechanism based on Pascal's theorem is used to trace the arcs of the parabola.

7. This paper was a brief exposition of the method of determining first necessary conditions for a minimum or maximum in the case of the isoperimetric problem of the calculus of variations when the end points move along a fixed curve, with application to "Queen Dido's Problem," as related by Virgil.

9. This was a discussion of a paper by Professor F. Daniels of the University of Wisconsin reported in the January, 1928 issue of the American Mathematical Monthly.

10. This paper was a review of the article entitled "The fundamental mathematical requirements of biology" by J. Arthur Harris that appeared in the April, 1929, issue of the American Mathematical Monthly.

ELIZABETH LESTOURGEON, *Secretary*

UNIQUE DECOMPOSITION

By E. T. BELL, California Institute of Technology

I. AN ARITHMETICAL PROBLEM

1. *Introduction.* Before proceeding to a precise formulation of the problem of unique decompesition, it will be well for clearness to give a general description of its nature and of the situation in which it arises. What follows is merely descriptive; the exact statement can be given only after the postulational definitions of certain algebraic varieties—ova, commutative semigroups, modules, rays, rings, fields, irregular fields—in Section II. A fair image of the exact problem can be seen by having in mind as a background either the multiplicative or the additive part, but not both, of ordinary rational arithmetic, or the theory of the ideals only of an algebraic number field, the integers of the field being replaced by their corresponding principal ideals, or the set of all symmetric polynomials in n indeterminates ($n>1$), with rational integer coefficients, and their unique representations as polynomials in the relevant elementary symmetric functions. The brief descriptions of the algebraic varieties concerned are strictly provisional until the exact postulational statements.

2. *Algebraic Varieties.* A field is a set of elements closed under the four rational operations, division by zero excluded, and having unique zero and unit elements. A product is equal to zero only if one of its factors is zero. This is in distinction to the next.

An irregular field differs from a field only in the exclusion of each of m elements, $m>1$, as a divisor; m may be finite or infinity. The excluded elements are called irregular; the rest, regular. With respect to addition, multiplication, subtraction, irregular elements are indistinguishable from regular. The product of two irregular elements, neither the zero element, may be equal to the zero element, in which case the factors are called divisors of zero.

A set of elements closed under addition, multiplication and subtraction is called a ring.

A set of elements closed under addition and subtraction is called a module.

A set of elements closed under multiplication and division, the divisor being regular, is called a ray.

It may be remarked that the definitions of rings, modules and rays in the literature lack uniformity. For example, a module is sometimes defined as a set closed under addition, or under subtraction, without the explicit statement of the postulate that these operations shall have unique inverses. The sets closed under addition alone or multiplication alone, when only the associative and commutative laws are assumed, are identical abstractly, and they appear to be more fundamental for arithmetic than any of the foregoing. From such sets, we construct the others.

A set closed under a single operation R obeying the associative and commutative laws, and further subject to the postulate that $xRy=xRz$ implies $y=z$, is called a commutative semigroup. If a commutative semigroup contains only a finite number of distinct elements, it is an abelian (commutative) group. It is to be noted particularly that a commutative semigroup does not necessarily contain a modulus (element having with respect to R and elements of the set the properties of an identity); if the set does have a modulus, the modulus is necessarily unique. Inverses do not exist in a commutative semigroup without restriction by the adjunction of additional postulates. If such adjunction be made, we descend to modules or other less general varieties.

If from the definition of a commutative semigroup the postulate of cancellation, $xRy=xRz$ implies $y=z$, be deleted (as there exist sets for which it is false, for example in the algebra of logic), we obtain a still more rudimentary variety which does not seem to have been named. All that follows can be constructed from this variety. Lacking a better name, it may be called an ovum.

A variety which contains only either a finite number or a denumerably infinite number of distinct elements will be said to be countable. The elements of a countable variety can be ordered in 1, 1 correspondence with the positive rational integers if the variety is infinite, or with the first n integers if the variety contains precisely n (n finite) distinct elements. When such a correspondence is established, we shall say that the variety is ordered.

3. *The Problem.* Call the operation R under which the ovum Ω is closed composition, and the element of Ω obtained by composition from two or more elements of Ω their composite. An example of Ω occurs in the classical theory of composition of binary quadratic (arithmetical) forms.

Suppose now that Ω contains a countable set I of elements, and further that I contains a set P of elements such that:

3.1 Any element of I not in P is, except for permutations of the elements in the composite, uniquely a composite of elements in P.

3.2 No element in P is a composite of elements in P.

3.3 The sets I, P are maximal sets in Ω having the properties (3.1), (3.2). Then the elements of I may be called (by an obvious isomorphism) the rational integers of Ω, and those of P the rational primes; (3.1)–(3.3) express the law

of unique decomposition of integers in Ω into primes. More than one maximal set of integers or of primes in a set of integers may exist. Examples have been indicated in what precedes in which composition is multiplication, or addition, both being interpreted as in rational arithmetic, and in which it is neither. In view of this, we shall call $3.1-3.3$ and their implications the composite arithmetic Λ of Ω. It is evident that so long as we remain in Ω, no further development of its arithmetic—properties of its rational integers—is possible. Thus, for example, if composition be identified with the associative and commutative properties of either multiplication or addition, it is identified with those of both. This leads to the following problem, of which we shall present a denumerable infinity of solutions, as the process of construction of one solution is iterable. The complete arithmetic of the ordinary rational integers is presupposed.

Given the composite arithmetic Λ of Ω to obtain from it the following:

3.4 An ordering of its rational integers.

3.5 A set of functions (to be called elements of V) of integers of such that, with respect to suitably defined rational operations the elements of V form an irregular field V. If at any stage of the new processes necessary, elements undefined in the data are introduced, the new elements are to be adjoined to the data in such a manner that the enlarged set is self-consistent. The definition of the new processes will always be such that consistency is automatically provided. The adjoined elements, of course, demand in the interpretations of the original data extensions of given properties. Such extensions can be given only with reference to the specific nature of the original data. For example, the adjunction necessary when the original elements are integral ideals, is that of fractional ideals.

This is the first, or algebraic part of the problem. The second part is arithmetical. Three cases arise, according as the number of distinct elements in V is finite, or denumerably infinite, or neither. Many instances of the first and second exist, and at least one of the third. However, we shall attend only to the first and second, and particularly to the second, as the necessary modifications in what follows when V is finite are obvious, and the second is the case of greater interest. We shall assume then that the elements of V are denumerably infinite. The second part of the problem follows.

3.6 To select from V a countably infinite ring R of its elements such that, with a suitable definition of arithmetical divisibility for irregular elements, R has a complete arithmetic A, additive and multiplicative, isomorphic with that of the ordinary rational integers, order relations included.

3.7 To superimpose V, A upon Ω, Λ so that every proposition concerning V (or A) has a unique correspondent in Ω (or Λ) which is true or false according as its correspondent in V (or A) is true or false.

Finally, with the necessary restrictions imposed by irregularity, we may proceed to

3.8 Construct continua from A abstractly identical with those of the real or complex number systems.

A solution of 3.6 will automatically produce an ovum from which the whole abstract process may be generated afresh, and so on indefinitely.

The problem of constructing a complete arithmetic from the composite arithmetic alone of an ovum is here solved through an irregular field. I have also given a solution (or rather an infinity of solutions), in another paper, through a field. The solution by an irregular field is, however, intrinsically richer, as it necessarily demands a definition of arithmetical divisibility for divisors of zero and a unique decomposition of such divisors into primes. This, I believe, has not hitherto been done for divisors of zero in the arithmetic of any algebra. The reader who prefers may pass at once to IV, using II, III only for reference.

II. ALGEBRAIC VARIETIES

4. *Variety; Notation.* As each of the four rational operations in an irregular field is ultimately to receive an infinity of distinct interpretations, the customary $+$, $\cdot$, $-$, $\div$ will be replaced in the postulates by $*$, $\dagger$, $\S$, $\ddagger$, and each of these symbols has only the meaning assigned to it in the postulates of the variety concerned. For example, in an ovum, it is meaningless to say that $*$ is algebraic addition, as no inverse to $*$ is defined in the ovum. But, inversely, a true proposition in an ovum remains true in a module, a ring, a commutative semigroup, a field, etc., which contains the ovum.

4.1 *Definition.* A *variety* V is a system consisting of a relation $\sim$, called *equivalence*, a set Σ of marks $\alpha, \beta, \cdots, \gamma, \cdots$, called *elements* of Σ or of V, and one or more *operations* $*, \dagger, \S, \ddagger, \cdots$, which can be performed upon any pair α, β of elements of V to produce uniquely determined elements $\alpha * \beta$, $\alpha \dagger \beta$, $\alpha \S \beta$, $\alpha \ddagger \beta, \cdots$, of V such that the postulates 4.11–4.15, 4.17 for $\sim$, $*$, $\S$, $\ddagger, \cdots$, are satisfied.

4.11 If α, β are any elements of V, $\alpha \sim \beta$ is significant (well defined, and either true or false).

4.12 $\alpha \sim \alpha$.

4.13 If $\alpha \sim \beta$, then $\beta \sim \alpha$.

4.14 If $\alpha \sim \beta$ and $\beta \sim \gamma$, then $\alpha \sim \gamma$.

4.15 If $R(\alpha, \beta, \cdots, \gamma, \cdots)$, is any relation between elements $\alpha, \beta, \cdots, \gamma, \cdots$, of V, and $\alpha \sim \alpha'$, $\beta \sim \beta'$, $\cdots$, $\gamma \sim \gamma'$, $\cdots$, then $R(\alpha, \beta, \cdots, \gamma, \cdots)$, $R(\alpha', \beta', \cdots, \gamma', \cdots)$, are both true or both false.

4.16 *Definition.* If $\alpha \sim \beta$ is false, we write $\alpha \sim' \beta$.

4.17 If $\dagger\dagger$ and $\S\S$ are members of the set $*, \dagger, \S, \ddagger, \cdots$, and are not the same there exists in V at least one pair of elements α, β for which $\alpha \dagger\dagger \beta \sim' \alpha \S\S \beta$.

It is evident that equivalence, $\sim$, has with respect to elements of V all the properties of equality, $=$, with respect to real numbers, being reflexive (4.12), symmetrical (4.13), and transitive (4.14), while 4.15 permits the substitution of equivalent elements. Nevertheless, in the instances adduced later, $\sim$ bears but slight resemblance to $=$. For example, if (α, β, γ) is a one-rowed matrix, and $(\alpha, \beta, \gamma) = (\alpha', \beta', \gamma')$, the usual definition gives $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, and it does not apply to $(\alpha, \beta, \gamma) = (\alpha', \beta')$, whereas $\sim$ does.

4.18 *Definition.* If to the preceding postulates others be added, the resulting system will be called a *special variety*, provided the entire set of postulates is consistent.

Consistency can always be proved for a special variety by exhibiting an instance which satisfies the postulates. In what follows, all consistency proofs are deferred to IV, where all are implied simultaneously by the construction of the systems constructed.

5. *Commutative, Associative and Distributive Special Varieties.*

5.1 *Definition.* A special variety V is said to be *commutative* with respect to the operation $*$ of V if the postulate 5.11 is satisfied.

5.11. If α, β are any elements of V, then $\alpha * \beta \sim \beta * \alpha$.

5.2 *Definition.* A special variety V is said to be *associative* with respect to the operation $*$ of V if the postulate 5.21 is satisfied.

5.21 If α, β, γ are any elements of V, then $(\alpha * \beta) * \gamma \sim \alpha * (\beta * \gamma)$.

5.3 *Definition.* A special variety V is said to be *left-distributive* with respect to the ordered pair $(\dagger, *)$, where $\dagger$, $*$ are operations of V and are not the same, if the postulate 5.31 is satisfied.

5.31 If α, β, γ are any elements of V, then

$$\alpha \dagger (\beta * \gamma) \sim (\alpha \dagger \beta) * (\alpha \dagger \gamma).$$

5.4 *Definition.* If instead of 5.31 the postulate 5.41 is satisfied, V is said to be *right-distributive* with respect to the ordered pair $(\dagger, *)$.

(5.41) If α, β, γ are any elements of V, then

$$(\alpha * \beta) \dagger \gamma \sim (\alpha \dagger \gamma) * (\beta \dagger \gamma).$$

5.5 *Definition.* If the special variety V is both *left-distributive* and *right-distributive* with respect to the ordered pair $(\dagger, *)$, it is said to be *distributive* with respect to $(\dagger, *)$.

It is unnecessary here to develop the numerous consequences of these postulates, as only the simplest of them will be required.

6. *Modular Special Varieties.* A modulus is an identity element with respect to a given operation.

6.1 *Definition.* If in the special variety V there exists an element μ_* such that the postulate 6.11 is satisfied, V is said to be *modular* with respect to $*$.

6.11 If α is any element of V, then at least one of the following holds,

$$\alpha * \mu_* \sim \alpha \qquad \mu_* * \alpha \sim \alpha.$$

6.2 *Definition.* If only $\alpha * \mu_* \sim \alpha$ in 6.11 holds, μ_* is called a *right-modulus*; if only $\mu_* * \alpha \sim \alpha$ holds, μ_* is called a *left-modulus*; if both equivalences hold, μ_* is called a *modulus* with respect to $*$.

7. *Regular and Irregular Special Varieties.*

7.1 *Definition.* If V is a modular special variety having with respect to $*$ the modulus $\mu*$, and if α, β are elements of V such that $\alpha * \beta \sim \mu*$, α is called a *left-*

inverse of β with respect to $*$, and β a *right-inverse* of α with respect to $*$, and each of α, β is said simply to be an *inverse* of the other with respect to $*$.

7.2 *Definition.* A modular special variety is said to be right-regular if the postulate 7.21 holds, left-regular with respect to $*$ if the postulate 7.22 holds.

7.21 If α is any element of V, α has with respect to $*$ a unique right-inverse.

7.22 If α is any element of V, α has with respect to $*$ a unique left-inverse.

7.3 *Definition.* A modular special variety V is said to be *regular* with respect to $*$ if it is both right-regular with respect to $*$ and left-regular with respect to $*$.

7.4 *Definitions.* An element of a special modular variety V which does not have a unique right (left) inverse with respect to the operation $*$ of V, is said to be *right- (left) irregular* with respect to $*$; an element of V which is both right-irregular and left-irregular with respect to $*$ is said to be *irregular* with respect to $*$.

If V is right-irregular (or left-irregular, or irregular) and contains precisely m (m necessarily >0) right-irregular (or left-irregular, or irregular) elements with respect to $*$, we say that the *index of right-irregularity* (or *of left-irregularity* or *of irregularity*) of V with respect to $*$ is m.

As the definitions are similar for all three possibilities, we may state only those for irregularity.

If every element of V is irregular with respect to $*$, V is said to be *totally irregular* with respect to $*$. It is perhaps not obvious that totally irregular commutative and associative varieties exist; an infinity of such are given by the irregular elements, other than the unique zero element, of the matric varieties constructed in IV.

By referring to 7.4 and 7.2 we see that an element may be irregular in only one of two distinct respects: the element may have no inverse; the element may have more than one inverse. In the totally irregular varieties mentioned above each element has an infinity of inverses.

8. *Ordered Varieties.* The postulates for order fall into several cases, according to the degree of specialization of the variety concerned, each more restrictive than its predecessor. The most restricted set, which applies to rings and fields, regular or irregular, is abstractly identical with the postulates for inequality in the real number system.

8.1 *Definition.* If a variety V or a special variety V is *ordered* with respect to the relation $\rightarrow$, it is necessary that the postulates 8.11, 8.12 hold.

8.11 If α, β are any elements of V, one and only one of the following relations is true,

$$\alpha \sim \beta, \quad \alpha \rightarrow \beta, \quad \beta \rightarrow \alpha.$$

8.12 If α, β, γ are any elements of V such that $\alpha \rightarrow \beta$ and $\beta \rightarrow \gamma$, then $\alpha \rightarrow \gamma$.

8.2 *Definition.* The relation $\alpha \leftarrow \beta$ means $\beta \rightarrow \alpha$.

The preceding definitions apply to all varieties, and in particular to the V in 4.1, which is unspecialized. Specialization enters through the imposition of further postulates concerning an operation $*$ of V. *If V is unspecialized*, we say

that 8.11, 8.12 *are the necessary and sufficient conditions that V be ordered with respect to* $\rightarrow$.

There is an extensive theory possible for ordered special varieties. Of this, for brevity, we shall state only the postulates for ordered modular varieties, as these are the minimum necessary for the subsequent theory of unique decomposition. For the necessary definitions, see Section 6. Again for simplicity we shall attend only to the case in which the special variety has a modulus (as defined in 6.2).

8.3 *Definition.* The modular special variety V having with respect to $*$ the modulus μ_* is said to be *ordered* with respect to the relation $\rightarrow$ when and only when, in addition to 8.11, 8.12, the postulate 8.31 is satisfied.

8.31 If α, β are any elements of V such that $\alpha \rightarrow \beta$, and if γ is any element of V, such that $\gamma \rightarrow \mu_*$, then $\alpha * \gamma \rightarrow \beta * \gamma$.

To proceed, we refer to 5.1, 5.2, 5.5, 6.2, 7.2, 7.3.

8.4 *Definition.* A special variety V which is

8.41 commutative and associative with respect to each of the distinct operations $*$, $\dagger$;

8.42 distributive with respect to the ordered pair $(\dagger, *)$;

8.43 regular with respect to $*$ and having the modulus μ_*, is said to be *ordered* with respect to $\rightarrow$ if and only if the postulates 8.44, 8.45 are satisfied in addition to 8.11, 8.12.

8.44 If α, β are any elements of V such that $\alpha \rightarrow \beta$, and γ is any element of V, then $\alpha * \gamma \rightarrow \beta * \gamma$.

8.45 If α, β, γ are any elements of V such that $\alpha \rightarrow \beta$ and $\gamma \rightarrow \mu_*$, then $\alpha \dagger \gamma \rightarrow \beta \dagger \gamma$.

Part of this is implied by previous definitions and postulates, but on account of its importance for the sequel it has been stated in full.

III. CERTAIN SPECIAL VARIETIES

9. *Named Varieties.* We shall drop the designation "special," and speak, for example, simply of a commutative variety instead of a commutative special variety.

9.1 *Ovum.* A variety defined by 4.1, 5.1, 5.2 is called an *ovum* with respect to $*$.

An ovum is hence a variety commutative and associative with respect to a single operation. Although an ovum is completely defined by 9.1 it may be well for clearness to state some of the properties which can *not* be assumed in it. These are, chiefly, the existence of a modulus, the existence of inverses with respect to $*$, and the cancellation postulate as stated in 9.22. Even when an ovum is contained in a variety which does have one of these properties, that property can not be assumed in the ovum itself. Similar remarks apply to all special varieties.

9.2 *Commutative semigroup.* An ovum V with respect to $*$ which satisfies the postulates 9.21–9.23 is called a commutative semi-group with respect to Σ, $*$.

9.21 Every element of Σ is in V.

9.22 If σ is any element of Σ, and α, β are any elements of V, such that $\alpha * \sigma \sim \beta * \sigma$, then $\alpha \sim \beta$.

9.23 Σ is not null and is the maximal[1] set for which 9.22 is satisfied.

9.24 *Definition.* If Σ in 9.2 coincides with V (that is, if every element of Σ is in V, and every element of V is in Σ), V is called a *commutative semigroup* with respect to $*$.

9.25 *Theorem.* If a commutative semigroup contains a modulus, the modulus is unique.

This is either well known or proved at once from the definitions.

9.3 *Module.* Here we shall define only the simplest modules, in which the composition $*$ is commutative. For this we refer to 7.3, 7.4.

9.31 *Regular Module.* A commutative modular variety which is regular with respect to $*$ is called a *module* with respect to $*$.

9.31 *Irregular Module.* A commutative modular variety which is irregular with respect to $*$, the index of irregularity being $m(\geqq 1)$, is called an *irregular module* of index m.

9.4 *Ray.* This introduces nothing new, and is included in order to take account of the term as it has been used in the literature.

9.41 *Definition.* An irregular module of index 1 is called a ray.

9.5 *Ring.* A variety satisfying the postulates 8.41, 8.42, 8.43 is called a *ring* with respect to $(\dagger, *)$.

A simple example of a ring is the set of all rational integers with respect to $(\times, +)$; the modulus is 0. In the definition it is to be noted particularly that the ring is not postulated to be modular with respect to $\dagger$.

9.6 *Field.* A variety V is called a *field* with respect to $(\dagger, *)$ if the postulates 9.61–9.64 are satisfied.

9.61 V is a ring with respect to $(\dagger, *)$.

9.62 V is a ray with respect to $\dagger$.

9.63 The irregular element of the ray in 9.62 is μ_*, the modulus of the ring in 9.61.

9.64 μ_* is distinct from the modulus $\mu_\dagger$ of the ray in 9.62.

It is readily seen that 9.6 accords with any of the usual complete definitions of a field, when $(\dagger, *) \equiv (\cdot, +)$, $(\mu_\dagger, \mu_*) \equiv (1, 0)$, 1, 0 being the unit and zero elements of the field.

9.7 *Irregular Field.* A variety V is called an *irregular field* with respect to $(\dagger, *)$ if the postulates 9.71–9.74 are satisfied.

9.71 V is a ring with respect to $(\dagger, *)$.

9.72 V is an irregular module with respect to $\dagger$, with index of irregularity $m > 1$.

9.73 μ_* is an irregular element of the module in 9.62.

9.74 μ_*, $\mu_\dagger$ are distinct.

[1] A set S is said to be *maximal* with respect to a property P if every element of S has the property P, and S is not contained in a set having elements not in S which have the property P.

9.8 *Definition.* The *index* of the irregular field in 9.7 is m. An irregular field of index 1 is a field (by definition).

IV. CERTAIN MATRIC VARIETIES

10. *Notation.* For ease in following the isomorphisms between varieties whose elements are real or complex numbers and those considered in this section, also to prepare for an extensive generalization indicated later, we shall use for matrices the signs in the second row following, for real or complex numbers those in third, the signs in any column of the table being instances of the first sign in that column, whose properties have been defined in preceding sections. The meanings of §, ‡, which were not explicitly used, are sufficiently clear as the operations inverse to $*$, † respectively when such inverses exist in the system to which $*$, † refer. The table is (n is an integer >0),

$$\begin{array}{cccccccc} *\,, & \dagger\,, & \S\,, & \ddagger\,, & \sim\,, & \sim'\,, & \rightarrow\,, & \leftarrow\,; \\ (+)_n, & (\cdot)_n, & (-)_n, & (\div)_n, & (=)_n, & (\neq)_n, & (>)_n, & (<)_n; \\ +\,, & \cdot\,, & -\,, & \div\,, & =\,, & \neq\,, & >\,, & <\,. \end{array}$$

In the last row we shall use ab and a/b as customary for $a\cdot b$, $a\div b$ when convenient. The significance of the second row with n omitted is explained in Section 15.

11. *One-rowed Matrices.* A *real, finite one-rowed matrix* of *order n* (n finite) is an ordered set $(x_1, x_2, \cdots, x_n)$ of n real finite numbers $x_1, x_2, \cdots, x_n$. We shall call $(x_1, \cdots, x_n)$ an *M_n-number of order n*, or briefly M_n-number. The *coordinates* of $(x_1, \cdots, x_n)$ are the numbers $x_1, \cdots, x_n$ in this order.

11.1 *Equivalence of M_n-numbers.* We shall say that the M_n-numbers $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$ are *equivalent*, and write

$$(x_1, \cdots, x_n)(=)_n(y_1, \cdots, y_n)$$

when and only when $x_j = y_j (j=1, \cdots, n)$.

11.12 *Theorem.* Equivalence of M_n-numbers is an instance of abstract equivalence, $\sim$, in Section 4, up to and including 4.12.

The postulate 4.14 for $\sim$ will be seen to hold as we proceed. It is evident that the whole of the second line of the table of Section 10 must first be defined for M_n-numbers.

11.2 *Definition.* An M_n-number is said to be *regular* or *irregular* according as none or at least one of its coordinates is zero.

11.3 *Definition.* The M_n-number ζ_n each of whose coordinates is zero, is called the *zero M_n-number.*

11.4 *Definition.* The M_n-number η_n each of whose coordinates is 1 is called the *unit M_n-number.*

12. *Rational Operations upon M_n-Numbers.* Let $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$ be any M_n-numbers, and $(z_1, \cdots, z_n)$ any regular M_n-number. The rational operations upon M_n-numbers are defined by (12.1)–(12.4).

12.1 $$(x_1, \cdots, x_n)(+)_n(y_1, \cdots, y_n)(=)_n(x_1 + y_1, \cdots, x_n + y_n).$$

12.2 $$(x_1, \cdots, x_n)(-)_n(y_1, \cdots, y_n)(=)_n(x_1 - y_1, \cdots, x_n - y_n).$$

12.3 $$(x_1, \cdots, x_n)(\cdot)_n(y_1, \cdots, y_n)(=)_n(x_1y_1, \cdots, x_ny_n).$$

12.4 $$(x_1, \cdots, x_n)(\div)_n(z_1, \cdots, z_n)(=)_n(x_1/z_1, \cdots, x_n/z_n).$$

The following are obvious from the definitions.

12.5 *Theorem.* With respect to each of $(+)_n$, $(\cdot)_n$, the set of all M_n-numbers is an ovum (section 9).

12.6 *Theorem.* With respect to $(+)_n$ the set of all M_n-numbers is a commutative semigroup (9.24), and a module (9.31) in which the element inverse to any element $(x_1, \cdots, x_n)$ is $(-x_1, \cdots, -x_n)$ and the modulus is ζ_n.

12.7 *Theorem.* With respect to $(\cdot)_n$ the set of all regular M_n-numbers is a commutative semigroup (9.24).

12.8 *Theorem.* The set of all M_n-numbers is a ring (9.5) with respect to $((\cdot)_n, (+)_n)$, and an irregular field of index 1 or ∞ according as $n=1$ or $n>1$, the irregular elements being the set of all irregular M_n-numbers (9.7, 9.8).

The case $n=1$ is merely that of the *field* of all real numbers.

13. *Ordered M_n-Numbers.* We refer for the notation to Section 10, and for the definitions and postulates concerning order to Section 8.

13.1 *Definition.* We say that the M_n-number $(x_1, \cdots, x_n)$ has the relation $(>)_n$ to the M-number $(y_1, \cdots, y_n)$, when and only when one, and necessarily only one, of the following n sets of conditions $G_j(x, y)(j=1, \cdots, n)$ is satisfied:

$$G_1(x, y) \equiv x_1 > y_1;$$

$$G_2(x, y) \equiv x_1 = y_1, \ x_2 > y_2;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$G_{n-1}(x, y) \equiv x_1 = y_1, \ x_2 = y_2, \cdots, x_{n-2} = y_{n-2}, \ x_{n-1} > y_{n-1};$$

$$G_n(x, y) \equiv x_1 = y_1, \ y_2 = y_2, \cdots, x_{n-1} = y_{n-1}, \ x_n > y_n.$$

When one of these sets is satisfied, we write

$$(x_1, \cdots, x_n)(>)_n(y_1, \cdots, y_n).$$

13.2 *Definition.* Denote by $L_j(x, y)$ the result of replacing $>$ by $<$ in $G_j(x, y)$. Then we say that the M_n-number $(x_1, \cdots, x_n)$ has the relation $(<)_n$ to the M_n-number $(y_1, \cdots, y_n)$, and write

$$(x_1, \cdots, x_n)(<)_n(y_1, \cdots, y_n),$$

when and only when one (and necessarily only one) of the sets of conditions $L_j(x, y)(j=1, \cdots, n)$ is satisfied.

13.3. *Theorem.* The ring of all M_n-numbers in 12.8 is ordered with respect to $(>)_n$, except for the property 8.45 (exception removed, §19, end).

As the proofs necessary for 13.3 are all similar, we shall give that for only one part, and we shall show that if $(x_1, \cdots, x_n)(>)_n(y_1, \cdots, y_n)$, and

$(y_1, \cdots, y_n)(>)_n(z_1, \cdots, z_n)$, where $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$, $(z_1, \cdots, z_n)$ are any M_n-numbers, then $(x_1, \cdots, x_n)(>)_n(z_1, \cdots, z_n)$. The hypotheses imply the existence of integers i, j, $1 \leqq i, j \leqq n$, such that $G_i(x, y)$ and $G_j(y, z)$ both hold. If $i = j$, everything is proved. If $i \neq j$, let $[i, j]$ denote the smaller of i, j. Then the hypotheses imply $G_{[i,j]}(x, z)$ and the conclusion follows.

Notice that $(x_1, \cdots, x_n)(>)_n(y_1, \cdots, y_n)$ and $(y_1, \cdots, y_n)(<)_n$ $(x_1, \cdots, x_n)$ are indeed, as they should be, identical statements.

14. *Least Equivalent Sets.* The order of an M_n-number is n (defined in section 11, beginning). We have now to consider M_n-numbers of the orders $n = 1, 2, 3, \cdots$, simultaneously. The process may be briefly described as follows. Given any set of such numbers of different orders: the members of the set are replaced by other numbers, all of the same order, and "equivalence" is so defined for the two sets that any relation in either has a unique equivalent correspondent in the other. Operations upon numbers of different orders are thus referred to the corresponding operations upon numbers of the same order. We proceed to make this precise.

Let Ω_j denote a sequence of precisely j terms each of which is zero, and define Ω_0 by $(x_1, \cdots, x_n, \Omega_0) \equiv (x_1, \cdots, x_n)$, where $(x_1, \cdots, x_n)$ is an arbitrary real matrix of any finite order n. Let $X', Y', \cdots, Z'$ be k real matrices, k finite, of the respective orders $n_1, n_2, \cdots, n_k$, and let n be the least integer $\geqq 0$ such that

$$n \geqq n_1, \; n \geqq n_2, \cdots, n \geqq n_k.$$

The *conjoint* (U, V) of $U \equiv (u_1, \cdots, u_a)$, $V \equiv (v_1, \cdots, v_b)$, in this order, is $(u_1, \cdots, u_a, v_1, \cdots, v_b)$.

Define now $X, Y, \cdots, Z$ as the respective conjoints

$$X \equiv (X', \Omega_{n-n_1}), \; Y \equiv (Y', \Omega_{n-n_2}), \cdots, Z \equiv (Z', \Omega_{n-n_k}).$$

Then $X, Y, \cdots, Z$ are all of the same order n, and are said to be *equivalent* respectively to $X', Y', \cdots, Z'$; further, the set $X, Y, \cdots, Z$ is called the *least equivalent set* of $X', Y', \cdots, Z'$. To indicate this equivalence we shall write

14.1 $$[X', Y', \cdots, Z'](\sim)[X, Y, \cdots, Z],$$

and, when convenient,

14.2 $$X'(\sim)X, \; Y'(\sim)Y, \cdots, Z'(\sim)Z.$$

14.3 *Theorem.* $(\sim)$ has the properties 4.12–4.17 of $\sim$.

The process of forming the least equivalent set is in general one of inflation. The complementary process of deflation is of equal importance for our purpose.

Consider first an M_n-number $(x_1, \cdots, x_n)$ different from ζ_n. Then there exists an integer h, $1 \leqq h \leqq n$, such that $x_h \neq 0$, $x_k = 0$, $k > h$, and h is called the *rank* of $(x_1, \cdots, x_n)$. If $x_n \neq 0$ the rank is n. By definition, the rank of ζ_n, for all integers $n > 0$, is 0.

14.4 *Definitions.* An M_n-number of rank $n>0$ is called an *M-number of order n*. The unique M-number of order 0 is (0), the *zero M-number.*

An M-number of order $n>0$ is thus an M_n-number of the form $(x_1, \cdots, x_n)$ with $x_n \neq 0$. Since an M-number of order n is also an M_n-number of order n, the definitions 11.2 of regularity and irregularity apply to M-numbers.

14.5 *Definitions.* If $(x_1, \cdots, x_n)$ is an M_n-number of rank h, the M-number $(x_1, \cdots, x_h)$ is called the *reduced equivalent M-number of* $(x_1, \cdots, x_n)$ or simply the *reduced equivalent* of $(x_1, \cdots, x_n)$. The reduced equivalent of ζ_n is (0).

Let $X', Y', \cdots, Z'$ be any set of k (k finite) M-numbers, and let (see 14.1)

$$[X', Y', \cdots, Z'](\sim)[X, Y, \cdots, Z].$$

Then $X, Y, \cdots, Z$ are all of the same order and the theory of Sections 11–13 can be applied to them.

15. *M-Varieties.* A variety whose elements are M-numbers is called an *M-variety.* The rational operations in an M-variety are defined as follows, with reference to Section 12.

Let $(x_1, \cdots, x_a)$, $(y_1, \cdots, y_b)$ be M-numbers of any orders a, b, (a, b, by the definitions, are finite). Let

$$(x_1, \cdots, x_a)(\sim)(x_1', \cdots, x_c'), \quad (y_1, \cdots, y_b)(\sim)(y_1', \cdots, y_c'),$$

as in 14.2, so that c is a or b if $a=b$ and the greater of a, b if $a \neq b$. The relation $(=)$ for M-numbers is then defined thus:

$$(x_1, \cdots, x_a)(=)(y_1, \cdots, y_b)$$

if and only if $(x_1', \cdots, x_c')(=)_c(y_1', \cdots, y_c')$, and hence if and only if $a=b$.

By 12.1,

$$(x_1', \cdots, x_c')(+)_c(y_1', \cdots, y_c')(=)_c(x_1' + y_1', \cdots, x_c' + y_c').$$

Let the reduced equivalent (14.5) of $(x_1' + y_1', \cdots, x_c' + y_c')$ be $(s_1, \cdots, s_e)$. Then the *M-sum* $(x_1, \cdots, x_a)(+)(y_1, \cdots, y_b)$ of the M-numbers $(x_1, \cdots, x_a)$, $(y_1, \cdots, y_b)$ is defined to be the M-number $(s_1, \cdots, s_e)$, and we write

15.1 $$(x_1, \cdots, x_a)(+)(y_1, \cdots, y_b)(=)(s_1, \cdots, s_e).$$

Similarly, the M-difference is defined from 12.2, and the M-product from 12.3:

(15.2) $$(x_1, \cdots, x_a)(-)(y_1, \cdots, y_b)(=)(d_1, \cdots, d_f);$$

(15.3) $$(x_1, \cdots, x_a)(\cdot)(y_1, \cdots, y_b)(=)(p_1, \cdots, p_g),$$

where $(d_1, \cdots, d_f)$ is the reduced equivalent of $(x_1' - y_1', \cdots, x_c' - y_c')$, and $(p_1, \cdots, p_g)$ is the reduced equivalent of $(x_1' y_1', \cdots, x_c' y_c')$.

It is clear that if we proceed similarly to define M-quotients, such exist when and only when the two M-numbers concerned are of the same order and moreover the divisor $(z_1, \cdots, z_n)$ as in 12.4 is (as there) regular.

(15.4) Division for M-numbers is defined by (12.4) in which $(x_1, \cdots, x_n)$, $(z_1, \cdots, z_n)$ are M-numbers of order n.

The last brings up the important question of "division" for irregular divisors, to which we return in Section 17.

15.5 *Theorem.* The theorems 12.5–12.8 imply the corresponding theorems with M_n, $(\ )_n$, and ζ_n replaced by M, $(\ )$, and (0) respectively.

We have thus constructed ova, commutative semigroups, rays and rings whose elements are M-numbers. These are called *M-varieties.* In other words, we have constructed the commoner algebraic varieties for one-rowed matrices when the numbers of coordinates of the matrices combined according to the rational operations are not necessarily equal. For our purpose it is unnecessary to generalize this to rectangular matrices. The generalization can be attained in several ways, some of which will be considered on another occasion.

The details of the proofs of the several parts of 15.5 are all simple and may be omitted; they are in fact immediate consequences of the definitions. To combine several M-numbers by rational operations, we first replace them by their least equivalent set. Having done so, we operate with the equivalents as in Section 12 and replace the unique M_n-number which results by its reduced equivalent M-number.

The varieties in 15.5 are called *M-varieties.* Thus we shall speak of an M-ring, etc.

16. *Ordered M-varieties.* We may refer here to Sections 13, 15.

16.1 *Definition.* If $(x_1, \cdots, x_a)$, $(y_1, \cdots, y_b)$ are any M-numbers, and

$$(x_1, \cdots, x_a)(\sim)(x_1', \cdots, x_c'),\ (y_1, \cdots, y_b)(\sim)(y_1', \cdots, y_c'),$$

as in Section 15, we say that $(x_1, \cdots, x_a)(>)(y_1, \cdots, y_b)$ if and only if $(x_1', \cdots, x_c')(>)_c(y_1', \cdots, y_c')$; that $(x_1, \cdots, x_a)(<)(y_1, \cdots, y_b)$ if and only if $(x_1', \cdots, x_c')(<)_c(y_1', \cdots, y_c')$, and that $(x_1, \cdots, x_a)(=)(y_1, \cdots, y_b)$ if and only if $(x_1', \cdots, x_c')(=)_c(y_1', \cdots, y_c')$.

The last, $(=)$, implies that $c=a=b$.

16.2 *Theorem.* The ring of all M-numbers in 15.5 is ordered with respect to $(>)$.

The necessary proofs are obvious from the last of Section 15 and 13.3.

17. *Proper Divisibility of M-numbers.* Here we may refer to 15.4.

17.1 *Definitions.* If $(x_1, \cdots, x_n)$ is an irregular M-number other than (0), and if $x_j=0(j=j_1, j_2, \cdots, j_s)$, where $j_1<j_2< \cdots <j_s$, and $x_h\neq 0(h\neq j_1, j_2, \cdots, j_s)$, the regular M-number $(j_1, j_2, \cdots, j_s)$ is called the *index of irregularity* of $(x_1, \cdots, x_n)$.

By definition, the index of irregularity of (0) is (0), and the index of irregularity of a regular M-number is 0.

Irregular M-numbers of the same order having the same (in the sense of $(=)$) index of irregularity, are said to be *co-irregular*, as also are regular M-numbers of the same order.

17.2 *Proper Divisors.* We remark once for all that division by (0) is not

defined. From the definitions 17.1 it is seen that if $(x_1, \cdots, x_n)$ is an irregular M-number different from (0), there exist an infinity of irregular M-numbers $(y_1, \cdots, y_m)$ other than (0) such that $(x_1, \cdots, x_n)(\cdot)(y_1, \cdots, y_n)(=)(0)$. For example, $(0, 2, 3, 0, 1)(\cdot)(y_1, 0, 0, y_2)(=)(0)$, where y_1, y_2 are arbitrary (finite) real numbers. Thus irregular M-numbers are "divisors of zero", the "zero" being the zero element (modulus of addition) of the system. Again, if $(z_1, \cdots, z_n)$ is any irregular M-number other than (0), there exist infinitely many pairs of M-numbers, $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$, neither (0), at least one of which is necessarily irregular, such that $(x_1, \cdots, x_n(\cdot)(y_1, \cdots, y_n)(=)$ $(z_1, \cdots, z_n)$.

17.3 *Theorem.* If $(x_1, \cdots, x_n)$, $(z_1, \cdots, z_n)$ are co-irregular M-numbers different from (0), there exists a unique M-number $(y_1, \cdots, y_n)$ co-irregular with them, such that

$$(x_1, \cdots, x_n)(\cdot)(y_1, \cdots, y_n)(=)(z_1, \cdots, z_n).$$

17.4 *Theorem.* The M-product $(\cdot)$ of two (and hence of any finite number) of co-irregular M-numbers is co-irregular with each of the factors.

17.5 *Definitions.* The irregular M-number $(y_1, \cdots, y_n)$ uniquely determined in 17.3 is called the *M-quotient* of $(z_1, \cdots, z_n)$ by $(x_1, \cdots, x_n)$, and we write

$$(y_1, \cdots, y_n)(=)(z_1, \cdots, z_n)(\div)(x_1, \cdots, x_n);$$

$(x_1, \cdots, x_n)$ is said to divide $(z_1, \cdots, z_n)$ *properly.*

18. *Infinite M-numbers.* Thus far we have considered only one-rowed matrices having a finite number of real, finite coordinates. To develop the theory of unique decomposition for a variety containing a countable infinity of distinct elements it is necessary to introduce matrices having a countable infinity of coordinates. Such matrices fall into two classes, finite and infinite, the second of which, being of no use for our purpose here, will be ignored.

18.1 *Definitions.* A one-rowed matrix $(x_1, x_2, \cdots, x_n, \cdots)$, whose coordinates are real, finite numbers is said to be *finite* if and only if there exists a finite integer m such that $x_h = 0$ for all integers $h > m$. If $(x_1, \cdots, x_n, \cdots)$, is of infinite order and finite, it is called an M_∞-number. The *zero* M_∞-number Ω is the M_∞-number all of whose coordinates are zero.

Operations upon M_∞-numbers are now referred to abstractly identical operations upon M-numbers by making Ω, (0) correspondents, and taking as the correspondent of the M_∞-number $(x_1, \cdots, x_n, \cdots)$, other than Ω the M-number $(x_1, \cdots, x_m)$, where m is the least integer such that $x_m \neq 0$, $x_h = 0$ for all integers $h > m$. Precisely as in defining operations and relations ( ) for M-numbers from the corresponding operations and relations $(\ )_n$ for M_n-numbers, we do the like for $(\ )_\infty$ and M_∞-numbers with reference to ( ) and M-numbers.

18.2 *Definitions.* The *reduced equivalent* of Ω is the M-number (0); the *reduced equivalent* of the M_∞-number $(x_1, \cdots, x_n, \cdots)$, other than Ω is the M-number $(x_1, \cdots, x_m)$, where m is the least integer such that $x_m \neq 0$, $x_h = 0$ for all integers $h > m$. We write $\Omega(\sim)_\infty(0)$,

$$(x_1, \cdots, x_n, \cdots)(\sim)_\infty(x_1, \cdots, x_m).$$

We define the relations $(\gtreqless)_\infty$ by

$$(x_1, \cdots, x_n, \cdots)(\gtreqless)_\infty(y_1, \cdots, y_n, \cdots)$$

when and only when

$$(x_1, \cdots, x_n, \cdots)(\sim)_\infty(x_1, \cdots, x_m), \ (y_1, \cdots, y_n, \cdots)(\sim)_\infty(y_1, \cdots, y_s),$$
$$(x_1, \cdots, x_m)(\gtreqless)(y_1, \cdots, y_s),$$

respectively.

Refer now to 15.1–15.4 and let the respective reduced equivalents of the M_∞-numbers,

$$(x_1, \cdots, x_a, \cdots), \cdots, (p_1, \cdots, p_g, \cdots), \ (x_1, \cdots, x_n, \cdots), (z_1, \cdots, z_n, \cdots)$$

be the M-numbers,

$$(x_1, \cdots, x_a), \cdots, (p_1, \cdots, p_g), \cdots, (x_1, \cdots, x_n), \ (z_1, \cdots, z_n).$$

Then the rational operations upon M_∞-numbers are defined thus:

$$(x_1, \cdots, x_a, \cdots)(+)_\infty(y_1, \cdots, y_b, \cdots)(=)_\infty(s_1, \cdots, s_e, \cdots)$$

when and only when 15.1 holds, and similarly for $(-)_\infty$ and 15.2 $(\cdot)_\infty$ and 15.3, $(\div)_\infty$ and 15.4.

18.3 *Theorem.* The theorems in 15.5 referring to M-numbers and operations ( ) imply the corresponding theorems for M_∞-numbers and operations $(\ )_\infty$, with M, ( ), (0) replaced by M_∞, $(\ )_\infty$ and Ω respectively, and any M-number $(x_1, \cdots, x_n)$ other than (0) replaced by the M_∞-number $(x_1, \cdots, x_n, \cdots)$.

In the same way from 16.2 we have

18.4 *Theorem.* The ring of all M_∞-numbers is ordered with respect to $(>)_\infty$.

18.5 *Definition.* Two or more M_∞-numbers are said to be *co-irregular* when their reduced equivalents (M-numbers) are co-irregular (Section 17); the *index of irregularity* of an M_∞-number is defined to be the index of irregularity of its reduced equivalent.

18.6 *Theorem.* If $(x_1, \cdots, x_n, \cdots)$, $(z_1, \cdots, z_n, \cdots)$, are co-irregular M_∞-numbers different from Ω, there exists a unique M_∞-number $(y_1, \cdots, y_n, \cdots)$, co-irregular with them, such that

$$(x_1, \cdots, x_n, \cdots)(\cdot)(y_1, \cdots y_n, \cdots)(=)_\infty(z_1, \cdots, z_n, \cdots).$$

The M_∞-product of any finite number of co-irregular M_∞-numbers is co-irregular with each of the factors.

This is evident from 18.5, 17.3, 17.4.

18.7 *Definitions.* The irregular M_∞-number $(y_1, \cdots, y_n, \cdots)$, uniquely determined in 18.6 is called the M_∞-quotient of $(z_1, \cdots, z_n, \cdots)$, by $(x_1, \cdots, x_n, \cdots)$, and we write

$$(y_1, \cdots, y_n, \cdots)(=)_\infty(z_1, \cdots, z_n, \cdots)(\div)_\infty(x_1, \cdots, x_n, \cdots);$$

$(x_1, \cdots, x_n, \cdots)$, is said to divide $(z_1, \cdots, z_n, \cdots)$, *properly.*

18.8 *Scalar Products, Derivatives, Etc.* Although what follows is not required for the arithmetical problem in view, it is included to complete the algebraic part of the theory. As the developments thus far for M_n-, M-, and M_∞- numbers are abstractly identical, we may discuss all simultaneously, and lay down the following comprehensive definition of the subsequent notation.

18.81 *Definition.* N shall refer to any one of M_n, M, M_∞, the corresponding operations and relations $(\ )_n$, $(\ )$, $(\ )_\infty$ in the three cases being subsumed under the common one in the third line of the table in Section 10, with any one of the three interpretations $(\ )_n$, $(\ )$, $(\ )_\infty$ understood throughout a given context. The elements of any one of the three species of varieties will be called *N-numbers*, and will be denoted by capital Latin letters, $A, B, \cdots, X, Y, \cdots$. The modulus of the operation now denoted by $+$ will be denoted by 0 (with, of course, the interpretation appropriate to the particular species concerned).

18.82 *Definition.* If c is a real or complex number, the scalar product cX of c and the N-number X is the matrix obtained from X by multiplying each of the coordinates of X by c. If c is pure imaginary, cX is called an *imaginary N-number.* (If c is real, cX is an N-number).

18.83 *Theorem.* Scalar products as in 18.82 obey all the laws of combination of scalar products in a linear algebra.

For example, $(a+b)X=aX+bX$, where $a+b$ is ordinary addition, and, by 18.81, the $+$ on the right is interpreted in the N-variety.

Let $[X', Y'](\sim)[X, Y]$ as in 14.1, so that X, Y are of the same order n. Then if the jth coordinate y_j of Y is a function of the jth coordinate x_j of $X(j=1, \cdots, n)$, Y' is said to be a function of X'.

If Y' is a function of X', and if the derivatives dy_j/dx_j exist $(j=1, \cdots, n)$, the *derivative of Y' with respect to X'* is the matrix whose jth coordinate is dy_j/dx_j. Anti-derivatives, integrals as sums, derivatives of complex N-functions and their integrals are defined in a similar manner; namely, by means of equivalent pairs of N-numbers, the definition from the real or complex field being applied to the individual coordinates of the pair having the same order. We shall not stop to state the procedure for irregular N-numbers where it differs from that for regular, as it is sufficiently evident from the like for proper division as already constructed.

The analyses then defined are abstractly identical with those of the real and complex number systems.

V. ARITHMETIC OF N-NUMBERS

19. *N-integers.* For the significance of N see 18.81; for the definition of integers and primes, see 3.1–3.3, and for proper divisibility, 17.5. From these the complete rational arithmetic, isomorphic to that of the rational integers, of N-numbers is readily constructed. It will be sufficient to lay down the cardinal definitions and to state a few of the principal elementary theorems.

19.1 *Definitions.* An N-number all of whose coordinates are rational integers is called an N-integer.

If A, B are N-integers, A is said to *divide* B, written $A \mid B$, if and only if B is properly divisible by A (17.5) and the N-quotient (17.5) of B by A is an N-integer. If $A \mid B$, A is called an *N-divisor* of B.

This is the definition of *arithmetical* divisibility as distinguished from mere *algebraic* divisibility as in 17.5.

An N integer, other than the zero N-integer, each of whose non-zero coordinates $=1$ in absolute value, is called an *N-unit.*

Each of the coordinates of an N-unit is therefore one of the numbers $-1, 0, 1$, the case where each is 0 excluded.

We now separate N-integers into four mutually exclusive classes:

19.11 The zero *N-integer* (0).

19.12 *Positive N-integers*: an N-integer is said to be *positive* if each of its coordinates is >0.

19.13 *Mixed N-integers*: an N-integer is said to be *mixed* if at least one, and not all, of its coordinates are negative.

19.14 *Negative N-integers*: an N-integer is said to be *negative* if each of its coordinates is <0.

19.2 *Theorem.* If X is an N-integer other than (0), there exists precisely one N-unit U such that the N-quotient of X by U is positive.

Thus it suffices to discuss divisibility of N-integers for positive N-integers only.

The *N-greatest common divisor* (abbreviated N. G. C. D) of the positive N-integers, A, B is that positive divisor which is divisible by every common positive divisor of A, B; the *N-least common multiple* (N. L. C. M) of A, B is that positive common multiple of A, B which divides every common multiple of A, B.

It is easily verified that the terms greatest and least have here the significance abstractly identical with the like for G. C. D and L. C. M in rational arithmetic: if G is the N. G. C. D of A, B and D is any common positive divisor, then (see 18.82), $G \geqq D$; if L is the N. L. C. M, and M is any common positive multiple of A, B, then $L \leqq M$. Further, all the properties of G. C. D and L. C. M have their N-isomorphs which can be written down at once from the comprehensive correspondence established between N-integers and rational integers; for example $AB = GL$.

If P is an N-integer other than the N-zero or an N-unit, and if P is positive and has only the divisors P and an N-unit, P is called a (positive) N-prime.

19.3 *Theorem.* If an N-prime divides the product of two positive N-integers it divides at least one of the factors.

19.4 *Theorem.* An N-integer is divisible by only a finite number of N-primes.

19.5 *Theorem.* A positive N-integer is, apart from permutations of the factors, uniquely a product of positive N-primes.

If the N. G. C. D of the positive N-integers A, B is an N-unity, A, B are said to be *coprime.* Let $x_j (j = i_1, i_2, \cdots, i_s)$ be all those coordinates of the positive N-integer X which are >0 so that $\nu(X) \equiv x_{i_1} \cdots x_{i_s}$ is the *norm* of X. Each of

the $\nu(X)$ N-integers X' obtained from X by substituting for x_j in turn the rational integers x_j', where $0<x_j' \leqq x_j(j=i_1, \cdots, i_s)$, is co-irregular with X. Consider the special case where each $x_j>1$, and let η be the positive N-unit co-irregular with X. Then, if ϕ denotes Euler's function, there are precisely $\phi(x_{i_1}) \cdots \phi(x_{i_s})$, say $\Phi(X)$ of the $\nu(X)$ integers X' co-irregular and coprime with X. Let A be any positive N-integer coprime with X. Then, if A^n denotes the N-product of precisely n factors each A, the N-difference of $A^{\Phi(X)}$ and η is N-divisible by X, which is the analogue for N-numbers of Euler's extension of Fermat's theorem. The complete theory of congruences requires several details on account of irregularity into which we need not enter here.

It is to be noticed in the ordering of N-numbers that $\gamma \rightarrow \mu_*$ in 8.45 is insufficient to ensure $\alpha \dagger \gamma \rightarrow \beta \dagger \gamma$ *unless* it be interpreted in the *narrow* sense that γ is *positive*. Since a rational integer which is positive in the ordinary sense is an N-integer which is positive as defined in 19.12, the isomorphism is complete. Similarly the isomorphism is exact for all N-numbers when in 19.12 for N-integer we substitute N-number (as a definition).

VI. ARITHMETIZATION OF A COMMUTATIVE VARIETY

20. *Finite Decomposition.* Let Γ be a commutative variety (Section 4) having a countable (Section 2, end) subset Σ of elements (Γ, Σ may or may not be identical), such that there exists in Σ a subset Π of elements having with respect to the composition $*$ of Γ the following properties.

20.1 No element of Π is a composite of elements of Π.

20.2 Every element of Σ not in Π is a composite of a finite number of elements of Π in one way only apart from permutations of elements.

In 20.2 we presuppose the convention

20.3 If Γ has the modulus μ_*, and α is any element of Γ, $\mu_* * \alpha$ (and hence also $\alpha * \mu_*$) is to be replaced by α wherever it occurs.

Since Γ is countable, it may be ordered. Assume any ordering of its elements —one usually will be given by some special property of Γ which is simpler than the others, of which an infinity are possible if Γ contains an infinity (countable) of distinct elements. Hence also the elements $\pi_j(j=1, 2, \cdots)$, of Π are ordered; say that $\pi_j \rightarrow \pi_k$ (see §8) when $j>k$.

20.4 We shall say that the element α of Σ has a *finite* decomposition if and only if

$$\alpha \sim \pi_{j_1} * \pi_{j_2} * \cdots * \pi_{j_s},$$

where each of $j_1, \cdots, j_s$ is finite and s is finite; and decomposition shall mean *finite decomposition.*

A composite of precisely n elements ($n<0$) each of which is β will be written β^n, and β^0 shall denote μ_*. With these conventions the unique decomposition of α may be written

$$\alpha \sim \pi_1^{x_1} * \pi_2^{x_2} * \cdots * \pi_n^{x_n},$$

where the $x_j(j=1, \cdots, n, \cdots)$ are rational integers $\geqq 0$, not all 0. We shall

call the N-number $(x_1, x_2, \cdots, x_n, \cdots)$, the *index* of α, and we shall write $I(\alpha) \equiv (x_1, \cdots, x_n, \cdots)$; $I(\mu_*) = 0$. (See 18.82.)

21. *Construction of Varieties from* Γ. Let small Greek letters $\alpha, \beta, \cdots$, denote elements of Γ, and let ∇ denote any one of the symbols of operations and relations in a given N-variety (18.87). Then the variety Γ_N, whose elements coincide with those of Γ, is completely defined by the following definition of ∇_N:

21.1 $\alpha \nabla_N \beta$ when and only when $I(\alpha) \nabla I(\beta)$.

22. *Arithmetic of* Γ. This is completely defined by

22.1 The integers of Γ are all those elements α of Γ for which $I(\alpha)$ is an N-integer. (This is implied by previous definitions.)

22.2 A property $P(\alpha, \beta, \cdots)$, of integers of Γ subsists when and only when in the N-variety in 21.1 the property $P(I(\alpha), I(\beta), \cdots)$, subsists between the N-integers $I(\alpha), I(\beta), \cdots$.

For example, $\alpha | \beta$, that is, α divides β arithmetically in Γ when and only when $I(\alpha) | I(\beta)$ in the N-variety; $\alpha > \beta$ in Γ when and only when $I(\alpha) > I(\beta)$; $\alpha + \beta$ in Γ is equal to γ, where γ is the uniquely determined element of Γ such that $I(\gamma) = I(\alpha) + I(\beta)$ in the N-variety.

Owing to the length of this paper we must omit applications. If the N-variety is a ring, addition in Γ is multiplication in the ring; multiplication in Γ is a new process in the ring, closely connected with the G. C. D and L. C. M in the ring and with the properties of elements of the ring considered as decomposed into powers of coprime products of distinct primes. It is interesting that the arithmetical properties (Fermat's theorem, quadratic reciprocity, etc.) of this new process are abstractly identical with those of multiplication and arithmetical divisibility.

SEMI-SERIAL ORDER

By ALBERT A. BENNETT, Brown University

The subjects of serial order and of the calculus of classes have proved particularly attractive to postulationalists. In each case by use of relatively few postulates the complete logical basis for an entire mathematical discipline may be exhibited. Excellent (but entirely unrelated) discussions of both of these topics have been provided by E. V. Huntington.[1] The two subjects differ considerably but both may be developed by use of a common symbol, $<$, of order relation. Some other important systems differing from both show also an essentially analogous use of a symbol of dyadic ordinal relation (e.g. the system of idempotent elements in some algebras, see example 12, below). It appears therefore worth noting that a body of common relations found in these various basic mathematical studies has hitherto escaped a common formulation.

[1] *The Continuum and Other Types of Serial Order*, 2nd Edition, 1917, Harvard University Press, *Postulates for the Algebra of Logic*, Transactions of the American Mathematical Society, vol. 5 (1904), pp. 288–309. (In particular, "The second set," pp. 297–305.)

A class K with at least two distinct elements will be said to have *semi-serial order* if it satisfies postulates I–V given below. In many interesting cases postulates VI–VIII are also satisfied. The wording used, save for VIII, will be essentially that of E. V. Huntington, for corresponding postulates in the algebra of logic. However by the omission of certain postulates there given but here extraneous and by introducing VIII an essentially new system of more extensive application is obtained. The arrangement of the postulates here used differs from Huntington's arrangement.

K is a class of elements. $\leqq$ is a dyadic relation among elements of K.

I. (Reflexiveness.) $a \leqq a$ whenever a belongs to the class.

II. (Restricted Symmetry.) If $a \leqq b$, and also $b \leqq a$, then $a = b$.

III. (Transitivity.) If $a \leqq b$, and $b \leqq c$, then $a \leqq c$.

IV. (Predecessor.) If $a \neq b$, and if neither $a \leqq b$, nor $b \leqq a$, then there is an element m, such that

(1) $m \leqq a$, and $m \leqq b$,

(2) if $x \neq m$, is such that $x \leqq a$, and $x \leqq b$, then $x \leqq m$.

V. (Successor.) If $a \neq b$, and if neither $a \leqq b$, nor $b \leqq a$, then there is an element n, such that

(1) $a \leqq n$, and $b \leqq n$, and

(2) if $y \neq n$, is such that $a \leqq y$, and $b \leqq y$, then $n \leqq y$.

VI. (Initial Element.) There is an element i such that for each $a \neq i$, $i \leqq a$.

VII. (Ultimate Element.) There is an element u such that for each $a \neq u$, $a \leqq u$.

VIII. (Primitive Element.) If there is an element i such that for each $a \neq i$, $i \leqq a$, then there is at least one element p, $\neq i$, such that if $t \leqq p$, then $t = i$ or $t = p$.

That these postulates I–VII are independent has been shown by Huntington for a different purpose. That VIII is at least ordinally independent of I–VII is seen by taking the example of the set of rational numbers between zero and unity, ends inclusive.

Deductions[2] *from postulates* I–V:

Th. 1. If $a = b$, $a \leqq b$. (I)

Def. 1. If $a \leqq b$, and $a \neq b$, then the notation $a < b$ may be used.

Th. 2. If $a \neq b$ then not both $a \leqq b$, and $b \leqq a$ are possible simultaneously. (II, Def. 1)

Th. 3. If simultaneously $a_1 \leqq a_2, a_2 \leqq a_3, \cdots, a_{k-1} \leqq a_k$, ($k$ finite) then $a_1 \leqq a_k$. (III)

Cor. If $a_1 \leqq a_2$, $a_2 \leqq a_3, \cdots, a_{k-1} \leqq a_k$ but not $a_1 < a_k$, then $a_1 = a_2 = \cdots = a_{k-1} = a_k$ (II, III)

Th. 4. The m of IV is unique. (II, IV)

Hence the rule of combination, $\wedge$, may be defined[3] as follows:

[2] These save for necessary rearrangements and a few additions are essentially those of Huntington, l.c.

[3] This notation in the case of subalgebras is traditional. Cf. L. E. Dickson, *Algebras and their Arithmetics*. University of Chicago, 1923, p. 26. It suggests the $\frown$ used by Peano, Russell, and later writers for logical product of classes.

Def. 2. If $a \leqq b$, the element $a \wedge b$ is defined as a. If $b \leqq a$ the element $a \wedge b$ is defined as b. (If $b = a$, then $a \wedge a$ is defined as a.) Otherwise $a \wedge b$ is defined as the m of IV.

Cor. 1. $a \wedge b = b \wedge c$. (II, IV)

Cor. 2. $(a \wedge b) \leqq a$, $(a \wedge b) \leqq b$. (I, II, IV)

Cor. 3. If $x \leqq a$, and $x \leqq b$, then $x \leqq (a \wedge b)$. (I, II, IV)

Th. 5. The n of V is unique. (II, V)

Hence the rule of combination, $\vee$, may be defined as follows.[4]

Def. 3. If $a \leqq b$, the element $a \vee b$, is defined as b. If $b \leqq a$, the element $a \vee b$ is defined as b. (If $b = a$, then $a \vee a$ is defined as a.) Otherwise $a \vee b$ is defined as the n of V.

Cor. 1. $a \vee b = b \vee a$. (II, V)

Cor. 2. $a \leqq (a \vee b)$, $b \leqq (a \vee b)$. (I, II, V)

Cor. 3. If $a \leqq y$, and $b \leqq y$, then $(a \vee b) \leqq y$. (I, II, V)

Th. 6. If $x \leqq (a \wedge b)$, then $x \leqq a$, and $x \leqq b$. (I, II, III, IV)

Th. 7. If $(a \vee b) \leqq y$, then $a \leqq y$, and $b \leqq y$. (I, II, III, V)

Th. 8. If $a \leqq b$, and $x \leqq y$, then $(a \wedge x) \leqq (b \wedge y)$. (I, II, III, IV)

In particular if $x \leqq y$, then $(a \wedge x) \leqq (a \wedge y)$.

Th. 9. If $a \leqq b$, and $x \leqq y$, then $(a \vee x) \leqq (b \vee y)$. (I, II, III, V)

In particular if $x \leqq y$, then $(a \vee x) \leqq (b \vee y)$.

Th. 10. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$. (I, II, III, IV)

Def. 4. The unique common element $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ may be denoted by $a \wedge b \wedge c$.

Th. 11. $(a \vee b) \vee c = a \vee (b \vee c)$. (I, II, III, V)

Def. 5. The unique common element $(a \vee b) \vee c = a \vee (b \vee c)$ may be denoted by $a \vee b \vee c$.

Th. 12. If $a_1, a_2, \cdots, a_k$ are any k elements of the class there is a unique element m which may be denoted by $a_1 \wedge a_2 \wedge \cdots \wedge a_k$ and which is such that

(1) $m \leqq a_1$, $m \leqq a_2, \cdots, m \leqq a_k$ and

(2) if x is such that $x \leqq a_1$, $x \leqq a_2, \cdots, x \leqq a_k$, then $x \leqq m$.

(I, II, III, IV)

Th. 13. If $a_1, a_2, \cdots, a_k$ are any k elements of the class there is a unique element n which may be denoted by $a_1 \vee a_2 \vee \cdots \vee a_k$, and which is such that

(1) $a_1 \leqq n$, $a_2 \leqq n, \cdots, a_k \leqq n$, and

(2) if y is such that $a_1 \leqq y$, $a_2 \leqq y, \cdots, a_k \leqq y$, then $n \leqq y$. (I, II, III, V)

Proofs of Theorems 8, 10:

The other theorems given above follow immediately from the postulates. The proofs of theorems 8 and 10 are here given as by Huntington. Theorems 9, 11, may be proved by symmetry.

Proof of Th. 8: By Th. 6, $(a \wedge x) \leqq a$, and $(a \wedge x) \leqq x$. From $a \wedge x \leqq a$, and $a \leqq b$, follows by III that $(a \wedge x) \leqq b$. From $(a \wedge x) \leqq x$ and $x \leqq y$ follows by III that $(a \wedge x) \leqq y$. Hence by Th. 4, Cor. 3, $(a \wedge x) \leqq (b \wedge y)$, as desired.

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[4] Cf. $\cup$ used by Peano, Russell, and later writers for logical sum of classes.

Proof of Th. 10: Let $(a\wedge b)=d$, and $d\wedge c=e$. Then by Th. 4, Cor. 2, $e\leqq d$, $e\leqq c$. Also $d\leqq a$, $d\leqq b$. Hence $e\leqq a$, $e\leqq b$, $e\leqq c$. Hence by Th. 4, Cor. 3, $e\leqq(b\wedge c)$, and hence again $e\leqq a\wedge(b\wedge c)$. Thus $(a\wedge b)\wedge c\leqq a\wedge(b\wedge c)$. Similarly $a\wedge(b\wedge c)\leqq(a\wedge b)\wedge c$. Hence the theorem.

As stated previously any system with at least two elements, satisfying I–V, is said to have *semi-serial order*. In particular it may be noted that these postulates are satisfied by each system having serial order. Indeed for postulates of serial order we may take the following due to Huntington (l.c.).

0. The class K is not an empty class, nor a class containing merely a single element.

1. If a and b are distinct elements of K, then either $a<b$, or $b<a$.
2. If $a<b$, then a and b are distinct.
3. If $a<b$, and $b<c$, then $a<c$.

It may be noted that while I–VIII include the assumption that K contains at least two distinct elements, I–V fails to assure this property of K, here covered by 0. Postulates 2, and 3, are essentially equivalent to II and III. Postulate 1 is too strong for general semi-serial order. The postulates IV and V hold for serial order but in view of 1 they are only vacuously true. However the set of theorems 1–13, given above which for the general case make explicit use of IV and V, not only continue to hold in the special case of serial order but are not in most cases vacuous or trivial.

Further Deductions from I–VII:

Th. 14. The element i of VI is unique. (II, VI)
Th. 15. For each a, $i\leqq a$, and if $x\leqq i$, then $x=i$. (I, II, VI)
Th. 16. The element u of VII is unique. (II, VII)
Th. 17. For each a, $a\leqq u$, and if $u\leqq y$, then $y=u$. (I, II, VII)
Th. 18. $a\wedge i=i$, and $a\wedge u=a$. (I, II, IV, VI, VII)
Th. 19. $a\vee i=a$, and $a\vee u=u$. (I, II, V, VI, VII)

The proofs of these are again immediate. Hitherto no use has been made of VIII, which is also the only postulate whose dual statement, (interchanging left and right members, and interchanging i and u), is not asserted. One conclusion, using only a small part of the content of VIII, is

Th. 20. $i\neq u$. (II, VI, VII, VIII)

The chief significance in VIII lies in the fact that it also is satisfied by the more interesting of the examples of semi-serial order as given below. Other analogous examples can be obtained with ease.

Some Examples Satisfying I-VIII:

1. The system of natural numbers 1, 2, 3, $\cdots$ together with ω, the first transfinite ordinal. Here $i=1$, $p=2$, $u=\omega$. By $a\leqq b$ is meant the usual order relation of magnitude.

2. The system of real non-negative numbers together with -1, and $+\infty$. Here $i=-1$, $p=0$, $u=+\infty$. By $a\leqq b$ is meant the usual order relation of magnitude.

3. The system of non-negative rational integers. Here $i=1$, p is any prime number $(\neq 1)$, $u=0$. By $a \leqq b$, is meant a is a divisor of b. In particular unity is counted as a divisor of each number of the system, and zero is regarded as divisible by each such number. Here $a \wedge b$ is the greatest common divisor, and $a \vee b$, the least common multiple of a and b.

4. The subclasses of a given class u. By i is meant the null class and by p, any class of but a single element. By $a \leqq b$, is meant that a is a subclass of b. By $a \wedge b$ is meant the logical product and by $a \vee b$ is meant the logical sum of a and b.

5. The linear projective subspaces of a given space u of N dimensions. By i is meant the null space. By p is meant a point. By $a \leqq b$ is meant that a is a linear subspace of b. By $a \wedge b$ is meant the intersection of a and b, by $a \vee b$ is meant the linear space of least dimensionality containing both a and b. Note that while $a \wedge b$ is the logical product $a \vee b$ is not the logical sum of the spaces regarded as classes of points. It may be remarked that the theory of the intersections of projective subspaces[5] may be regarded as more general than that of the theory of the projective space obtained by adjoining a suitable infinite region to a Euclidean space of N dimensions. The projective space may be the domain of a "finite geometry" in the sense of Veblen and Young.[6] While Veblen and Young explicitly stipulate that a projective line shall contain at least three points, the assumption that it shall contain only two points is not completely trivial. In fact the finite N-dimensional geometry where each line contains exactly two points is exactly the finite "logical field" discussed on its own account by Huntington. This relationship is however left unnoted by Veblen and Young, and by Huntington. Thus a complete theory for a finite geometry of N dimensions with $p+1$ points on a line, if it be so stated as to be applicable for $p=1$, gives in this special case the algebra of logic applied to a class of $N+1$ objects.

The transversal axiom assures that any generalized real projective space must contain in each real line a dense set of points. This notion of density is not essential to semi-serial order.

6. The set of closed intervals on a line, counting among intervals the null interval, i, each isolated point, p, and the entire line, u. By $a \leqq b$, is meant that a is a subinterval of b. By $a \wedge b$ is meant the common subinterval of a and b. By $a \vee b$ is meant the smallest interval containing both a and b. This is not in general the logical sum of the given intervals. One may note that for a corresponding system of segments (without end points) VIII will not be satisfied.

7. The set of all convex regions in a plane, counting among convex regions the null region, i, each isolated point p, and the entire plane u.

8. The set of all submodules of a given arithmetic module.[7]

[5] Cf. P. H. Schoute, *Mehrdimensionale Geometrie*. 1. Teil, *Die Linearen Räume*. Leipzig, 1902. Eugenio Bertini, *Introduzione alla Geometria Proiettiva degli Iperspazi*. Piza, 1907.

[6] *Projective Geometry*, Vol. 1, 1910.

[7] Cf. F. S. Macaulay, *Modular systems*, Cambridge Tracts, No. 19, 1916.

9. The class of all linear sets[8] which are subsets of a given linear set.

10. The class of all regions in the plane each of which is bounded by a circle; including as circular region in particular the null region i, each region comprising but a single point p, and the entire plane u. Here $a\wedge b$ is the largest circular region contained entirely within the two given circular regions, and $a\vee b$ is the smallest circular region entirely covering the two given circular regions. Here $a\wedge b$ is less than the logical product of the two point-sets.

11. The system of subgroups of a given group u. By i is meant the identical element. By p is meant any cyclic subgroup of prime order. By $a\leqq b$ is meant a is a subgroup of b. By $a\wedge b$ is meant the maximal common subgroup of a and b, and by $a\wedge b$ is meant the minimal common supergroup of a and b. Since the entire discussion takes place within the group u, this minimal common supergroup always exists.

12. The set of idempotent elements (together with zero) in certain algebras of finite basis, containing a modulus as these terms are used by Dickson, (l.c.), where by $a\leqq b$ is meant that simultaneously $a\cdot a=a$, and $a\cdot b=b\cdot a=a$. By i is here meant the element 0, such that $0\cdot 0=0$ and for each a, $a\cdot 0=0\cdot a=0$. By u is meant the element 1, (modulus) such that $1\cdot 1=1$ and for each a, $a\cdot 1=1\cdot a=a$. The elements here called primitive are those called primitive by Dickson (l.c. page 55).

COMBINING CONSTANT PROBABILITY FUNCTIONS[1]

By WILLIAM DOWELL BATEN, University of Michigan

The object of this paper is to show some interesting results which arise in combining probability functions which are constant in a particular interval and zero elsewhere, and to show how the probability function for the sum of n independent variables is formed and constructed. This will be treated for n finite and n infinite. The probability function for the sum, by a proper change of the variable, approaches the Gaussian function. The rapidity of approach is surprising for there is slight difference between the function for the sum of four variables and the Gaussian function.

The particular function treated comes up in considering the errors made in using any logarithmic table or any numerical table where the numbers are rounded-off numbers; also in considering measurements of time when the watch records time in units of seconds, tenths of seconds, etc. In geometric probability a constant probability function may arise. This particular function is a constant, that is regardless of the values of the variable the probability is the same in $(0, k)$ and the same for $(-\infty, 0)$ and (k, ∞). In a measure the probability is independent of the variable. Two applications are given at the close of the paper.

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[8] Used in the sense of *Dickson*, l.c., Chap. II.

[1] Read before the Michigan Section of the Mathematical Association of America, March 22, 1930.

As by-products, expressions for $n!$ and n^h, where $h=1, 2, \cdots, n-1$, are found. These expressions are certain summations which involve only integers.

The derivatives of the nth order will be used to indicate just how the different portions of the function for the sum differ at the ends of the intervals considered. Polynomials arise from which the general law for the sum is obtained.

Case 1

Let $f_1(x_1)dx_1$ be the probability that x_1 lies in the interval (x_1, x_1+dx_1), and $f_2(x_2)dx_2$ be the probability that x_2 lies in the interval (x_2, x_2+dx_2), where

$$f_1(x_1) = 0 \text{ in } (-\infty, 0); \qquad f_2(x_2) = 0 \text{ in } (-\infty, 0);$$

$$f_1(x_1) = \frac{1}{k} \text{ in } (0, k); \qquad f_2(x_2) = \frac{1}{k} \text{ in } (0, k);$$

$$f_1(x_1) = 0 \text{ in } (k, \infty); \qquad f_2(x_2) = 0 \text{ in } (k, \infty).$$

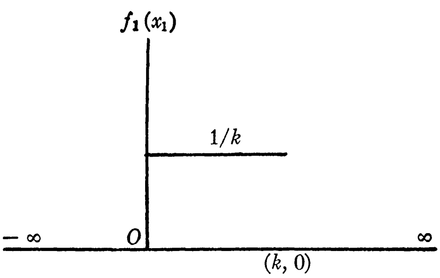

FIG. a

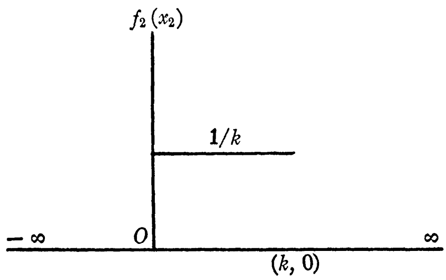

FIG. b

Let it be required to find the probability that $x_1+x_2=z$ lies in the interval $(z, z+dz)$. This can be secured by employing the following theorem which is well known:[1] *If $f(x)dx$ is the probability that x lies in the interval $(x, x+dx)$ and $g(y)dy$ is the probability that y lies in the interval $(y, y+dy)$, then the probability that $x+y=z$ lies in the interval $(z, z+dz)$ is*

$$F(z)dz = dz \int_{-\infty}^{\infty} f(x)g(z-x)dx.$$

There are two intervals which give results other than zero; they are the interval $0 \leqq z \leqq k$ and the interval $k \leqq z \leqq 2k$. According to the above theorem for $0 \leqq z \leqq k$,

$$F_2(z) = \int_{-\infty}^{0} 0 \cdot 0 dx_1 + \int_0^z \frac{1}{k} \cdot \frac{1}{k} dx_1 + \int_z^{\infty} \frac{1}{k} \cdot 0 dx_1 = \frac{z}{k^2};$$

[1] *Wahrscheinlichkeitsfunktionen und ihre Anwendungen*, by Von Karl Mayr in Wien. Monatshefte für Mathematik und Physik, vol. 30, art. 4, and *Calcul des Probabilités*, by H. Poincaré.

and for $k \leqq z \leqq 2k$,

$$F_2(z) = \int_{-\infty}^{0} 0 \cdot 0 dx_1 + \int_{0}^{z-k} \frac{1}{k} \cdot 0 dx_1 + \int_{z-k}^{k} \frac{1}{k} \cdot \frac{1}{k} dx_1 + \int_{k}^{\infty} 0 \cdot \frac{1}{k} dx_1 = \frac{2k-z}{k^2},$$

from which

$$\begin{aligned} F_2(z) &= 0 \text{ in } (-\infty, 0), \\ &= \frac{z}{k^2} \text{ in } (0, k), \\ &= \frac{2k-z}{k^2} \text{ in } (k, 2k), \\ &= 0 \text{ in } (2k, \infty). \end{aligned}$$

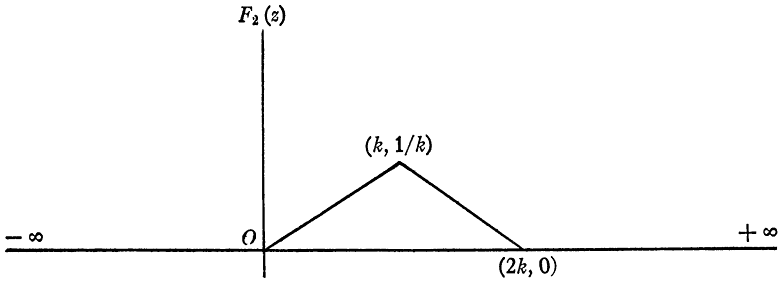

FIG. C

$F_2(z)$ is continuous for all values of z but its first derivative is discontinuous at $(0, 0)$, $(k, 1/k)$, and $(2k, 0)$. f_1 and f_2 are discontinuous at the origin and at the point with abscissa k, yet the probability function for the sum is continuous.

Let the probability function for the variable x_3 be $f_3(x_3)$ and similar to f_1 and f_2, that is equal to $1/k$ in $(0, k)$, and zero elsewhere. Then the probability function for the sum $x_1+x_2+x_3=z$ is given in five intervals: For $0 \leqq z \leqq k$,

$$F_3(z) = \int_{-\infty}^{0} 0 \cdot 0 dy + \int_{0}^{z} \frac{1}{k} \cdot \frac{z-y}{k^2} dy + \int_{z}^{\infty} \frac{1}{k} \cdot 0 dy = \frac{z^2}{2k^3};$$

for $k \leqq z \leqq 2k$,

$$F_3(z) = \int_{-\infty}^{0} 0 \cdot 0 dy + \int_{0}^{z-k} \frac{1}{k} \cdot \frac{2k-z+y}{k^2} dy + \int_{z-k}^{k} \frac{1}{k} \cdot \frac{z-y}{k^2} dy + \int_{k}^{\infty} 0 \cdot \frac{1}{k} dy$$
$$= \frac{6kz - 3k^2 - 2z^2}{2k^3};$$

for $2k \leqq z \leqq 3k$,

$$F_3(z) = \int_{-\infty}^{0} 0\cdot 0dy + \int_{0}^{z-2k} 0\cdot(\ )dy + \int_{z-2k}^{k} \frac{1}{k}\cdot\frac{2k-z+y}{k^2}dy$$
$$+ \int_{k}^{\infty} 0\cdot(\ )dy = \frac{9k^2-6kz+z^2}{2k^3};$$

where $y=x_1+x_2$; from which

$$\begin{aligned} F_3(z) &= 0, \text{ in } (-\infty, 0), \\ &= z^2/2k^3, \text{ in } (0, k), \\ &= (-2z^2+6kz-3k^2)/2k^3, \text{ in } (k, 2k), \\ &= (z^2-6kz+9k^2)/2k^3, \text{ in } (2k, 3k), \\ &= 0, \text{ in } (3k, \infty). \end{aligned}$$

$F_3(z)$ is continuous and its first derivative is continuous for every value of z, yet its second derivative is discontinuous at the points whose abscissas are 0, k, $2k$, and $3k$.

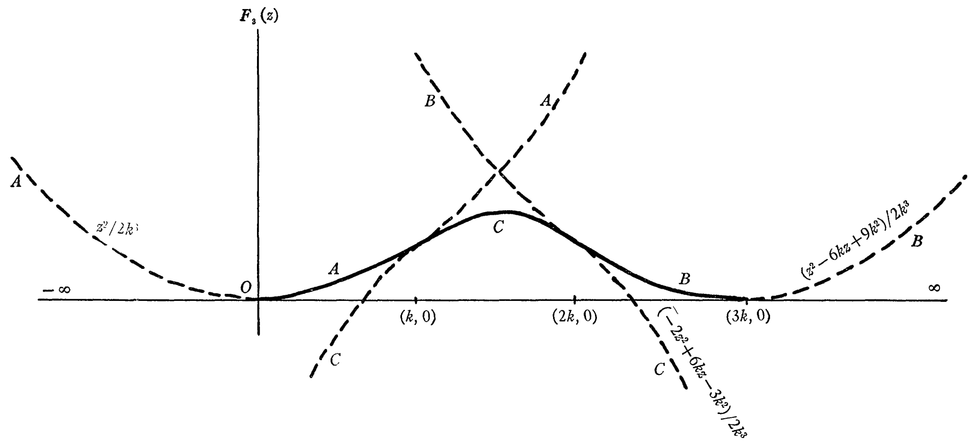

FIG. 1

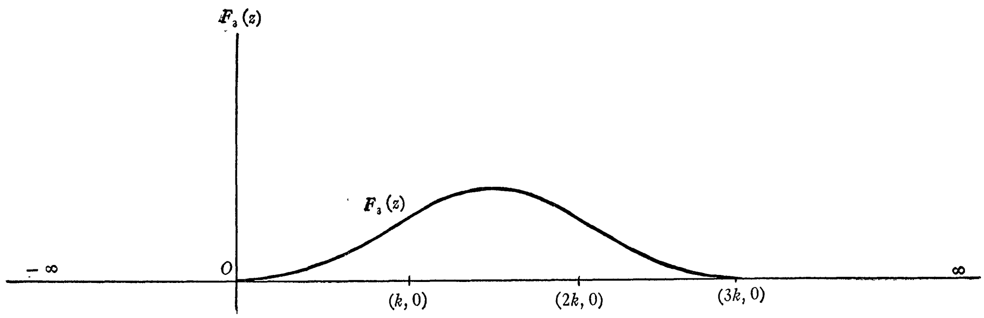

FIG. 2

Figure 1 shows the curves, parts of which form $F_3(z)$; while figure 2 shows $F_3(z)$. The dotted lines in figure 1 are not used to make up $F_3(z)$. At the points $(0, 0)$, $(k, 1/2k)$, $(2k, 1/2k)$, and $(3k, 0)$ the separate parts or curves are tangent.

If $f_4(x_4)$ is similar to f_1, then the probability function for the sum $x_1+x_2+x_3+x_4=z$, according to the theorem, is

$$
\begin{aligned}
F_4(z) &= \frac{1}{3!k^4}(z^3) \text{ in } (0, k),\\
&= \frac{1}{3!k^4}(-3z^3+12kz^2-12k^2z+4k^3) \text{ in } (k, 2k),\\
&= \frac{1}{3!k^4}(3z^3-24kz^2+60k^2z-44k^3) \text{ in } (2k, 3k),\\
&= \frac{1}{3!k^4}(-z^3+12kz^2-48k^2z+64k^3) \text{ in } (3k, 4k),\\
&= 0 \text{ for } (-\infty, 0) \text{ and } (4k, \infty).
\end{aligned}
$$

$F_4(z)$ here is made up of six parts each represented by a different equation. $F_4(z)$ is continuous, its first and second derivatives are continuous for all values of z, but its third derivative is discontinuous at the points whose abscissas are 0, k, $2k$, $3k$, and $4k$. The different parts deviate less from the adjacent parts than was found in the probability function for the sum of three variables, because here the function and its first two derivatives are continuous for all values of z. The following figures shows how $F_4(z)$ is formed and $F_4(z)$ respectively. The

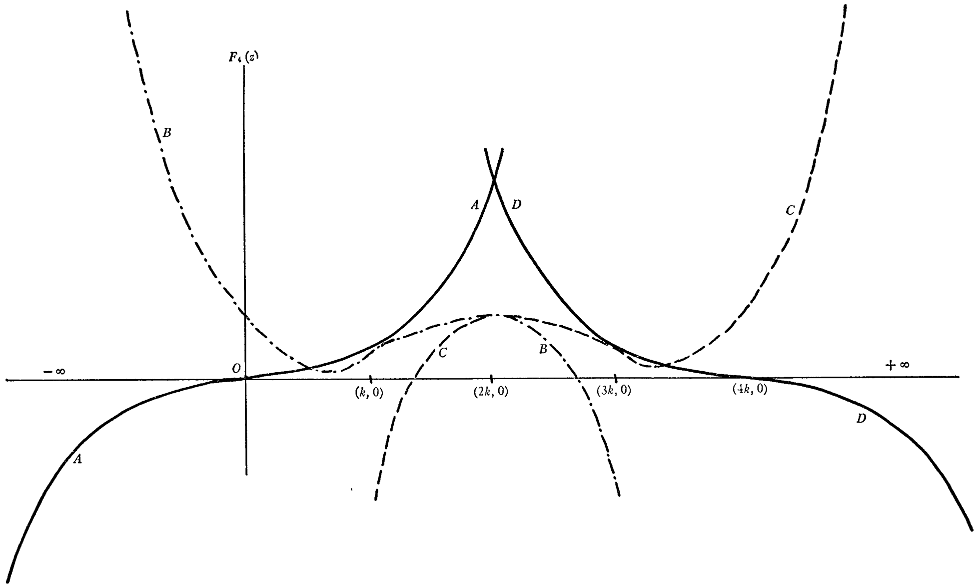

FIG. 3

parts coincide almost so completely that it is difficult to draw and indicate in the illustration how they fail to coincide perfectly. Figure 3 shows in a measure just this fact, while figure 4 appears to be one curve with all of its derivatives continuous.

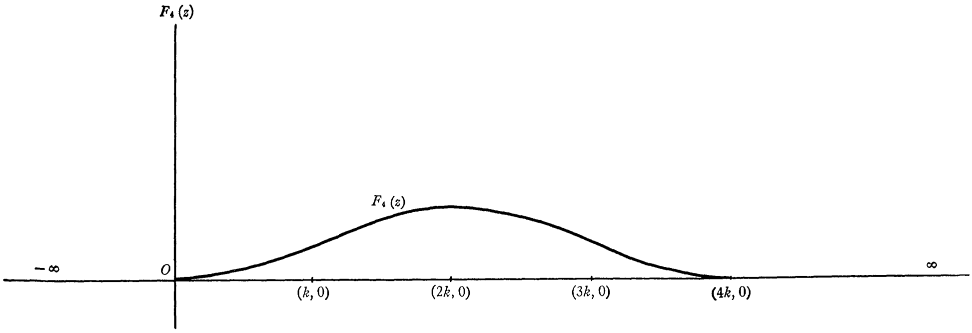

FIG. 4

Summarizing the results it is seen that for the proper intervals:

$$F_1(z) = \left\{\frac{1}{k}\right\}$$

$$F_2(z) = \left\{\begin{matrix} \dfrac{1}{k^2}(z) \\ \dfrac{1}{k^2}(-z+2k) \end{matrix}\right\}$$

$$F_3(z) = \left\{\begin{matrix} \dfrac{1}{2!k^3}(z^2) \\ \dfrac{1}{2!k^3}(-2z^2+6kz-3k^2) \\ \dfrac{1}{2!k^3}(z^2-6kz+9k^2) \end{matrix}\right\}$$

$$F_4(z) = \left\{\begin{matrix} \dfrac{1}{3!k^4}(z^3) \\ \dfrac{1}{3!k^4}(-3z^3+12kz^2-12k^2z+4k^3) \\ \dfrac{1}{3!k^4}(3z^3-24kz^2+60k^2z-44k^3) \\ \dfrac{1}{3!k^4}(-z^3+12kz^2-48k^2z+64k^3) \end{matrix}\right\}$$

Consider $F_4(z)$. Add the polynomials; the sum of the coefficients of each column is zero except in the last column in which it is 4! when the denominators are omitted. Subtract the first polynomial z^3 from the second polynomial and factor out -4, and this becomes $-4(z-k)^3$. From the third polynomial subtract the second and factor out 6, and this becomes $6(z-2k)^3$. Finally subtract from the fourth the third and factor out -4, and this gives $-4(z-3k)^3$. In $F_4(z)$, as written above, it is seen that for $(0, k)$, $F_4(z)=z^3$, (omitting denominators); for $(k, 2k)$, $F_4(z)=z^3-4(z-k)^3$; for $(2k, 3k)$, $F_4(z)=z^3-4(z-k)^3+6(z-2k)^3$; for each new interval another parenthesis is added or subtracted. Hence $F_4(z)$ and also the preceding F's can be written as follows:

$$F_1(z) = \left\{\begin{array}{l} \dfrac{1}{k}[1 \qquad] \\[2ex] \dfrac{1}{k}[1 - 1] \end{array}\right\} = \sum_{r=0}^{s}(-1)^r\binom{1}{r}(z - rk)^0, \text{ where } s = 0, 1.$$

$$F_2(z) = \left\{\begin{array}{l} \dfrac{1}{k^2}[z \qquad\qquad\qquad\qquad\quad] \\[2ex] \dfrac{1}{k^2}[z - 2(z - k) \qquad\qquad\quad] \\[2ex] \dfrac{1}{k^2}[z - 2(z - k) + (z - 2k)] \end{array}\right\} = \frac{1}{k^2}\sum_{r=0}^{s}(-1)^r\binom{2}{r}(z - rk)^1, \quad s = 0, 1, 2.$$

$$F_3(z) = \left\{\begin{array}{l} \dfrac{1}{2!k^3}[z^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad] \\[2ex] \dfrac{1}{2!k^3}[z^2 - 3(z - k)^2 \qquad\qquad\qquad\qquad\qquad\quad] \\[2ex] \dfrac{1}{2!k^3}[z^2 - 3(z - k)^2 + 3(z - 2k)^2 \qquad\qquad\quad] \\[2ex] \dfrac{1}{2!k^3}[z^2 - 3(z - k)^2 + 3(z - 2k)^2 - (z - 3k)^2] \end{array}\right\}$$

$$= \frac{1}{2!k^3}\sum_{r=0}^{s}(-1)^r\binom{3}{r}(z - rk)^2, \quad s = 0, 1, 2, 3.$$

$$F_4(z) = \begin{cases} \dfrac{1}{3!k^4}[z^3] \\ \dfrac{1}{3!k^4}[z^3 - 4(z-k)^3] \\ \dfrac{1}{3!k^4}[z^3 - 4(z-k)^3 + 6(z-2k)^3] \\ \dfrac{1}{3!k^4}[z^3 - 4(z-k)^3 + 6(z-2k)^3 - 4(z-3k)^3] \\ \dfrac{1}{3!k^4}[z^3 - 4(z-k)^3 + 6(z-2k)^3 - 4(z-3k)^3 + (z-4k)^3] \end{cases}$$

$$= \frac{1}{3!k^4} \sum_{r=0}^{s} (-1)^r \binom{4}{r} (z - rk)^3, \quad s = 0, 1, 2, 3, 4.$$

The part of $F_4(z)$ in $(4k, \infty)$ is zero and can be written as

$$\sum_{r=0}^{r=4} (-1)^r \binom{4}{r} (z - rk)^3.$$

In general it can be proven by mathematical iuduction that

$$F_n(z) = \begin{cases} \dfrac{1}{(n-1)!k^n}[z^{n-1}] \\ \dfrac{1}{(n-1)!k^n}\left[z^{n-1} - \binom{n}{1}(z-k)^{n-1}\right] \\ \dfrac{1}{(n-1)!k^n}\left[z^{n-1} - \binom{n}{1}(z-k)^{n-1} + \binom{n}{2}(z-2k)^{n-1}\right] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{1}{(n-1)!k^n}\left[z^{n-1} - \binom{n}{1}(z-k)^{n-1} + \cdots + (-1)^{n-1}\binom{n}{n-1}(z-\overline{n-1}\cdot k)^{n-1}\right] \\ \dfrac{1}{(n-1)!k^n}\left[z^{n-1} - \binom{n}{1}(z-k)^{n-1} + \cdots + (-1)^n\binom{n}{n}(z-nk)^{n-1}\right] \end{cases}$$

$$= \frac{1}{(n-1)!k^n} \sum_{r=0}^{s} (-1)^r \binom{n}{r} (z - rk)^{n-1}, \quad s = 0, 1, 2, \cdots, n.$$

$F_n(z)$ is made up of $(n+2)$ different curves, where $F_n(z)$ is zero for $(-\infty, 0)$ and is shown above for the other interval between 0 and nk and the interval

(nk, ∞). It is not difficult to find a law for the coefficients of the polynomials on pages 428, 429. The above may be briefly written as

$$F_n(z) = \begin{cases} F_{n,0}(z) = 0 \text{ for } (-\infty, 0) \\ F_{n,t}(z) = \dfrac{1}{(n-1)!k^n}\sum_{r=0}^{t}(-1)^r\binom{n}{r}(z-rk)^{n-1}, \text{ for } [tk, (t+1)k], \end{cases}$$

where t runs from 0 to n. The interval $[nk, (n+1)k]$ must here mean the interval (nk, ∞).

Consider

$$F_{n,t-1}(z) = \frac{1}{(n-1)!k^n}\sum_{r=0}^{t-1}(-1)^r\binom{n}{r}(z-rk)^{n-1},$$

$$F_{n,t}(z) = \frac{1}{(n-1)!k^n}\sum_{r=0}^{t}(-1)^r\binom{n}{r}(z-rk)^{n-1},$$

$$F_{n,t+1}(z) = \frac{1}{(n-1)!k^n}\sum_{r=0}^{t+1}(-1)^r\binom{n}{r}(z-rk)^{n-1}.$$

It is seen that the first $(n-2)$ derivatives of $F_{n,t-1}(z)$ and $F_{n,t}(z)$ are the same at the point whose abscissa is tk, and also the $(n-2)$ first derivatives of $F_{n,t}(z)$ and $F_{n,t+1}(z)$ are the same at the point whose abscissa is $(t+1)k$. The $(n-1)$th derivatives at these points are discontinuous, which shows that the different curves tend to continue into the adjacent curves completely as continuation of themselves.

Case 2

By Liapounoff's theorem, [2] $F_n(z)$ for large values of n approaches uniformly the Gaussian function $\pi^{-1/2}e^{-z^2}$. By this theorem the probability that the sum of n independent quantities lies between $t_1(2B_n)^{1/2}$ and t_2 $(2B_n)^{1/2}$, or that $t_1(2B_n)^{1/2} < z < t_2(2B_n)^{1/2}$, approaches uniformly

$$\pi^{-1/2}\int_{t_1}^{t_2} e^{-t^2}dt$$

provided that

$$\lim_{n\to\infty}\left[(B_n)^{-3/2}\sum_{i=1}^{n} c_i\right] = 0,$$

[2] *Sur l'extension du théoremè limite du calcul des probabilités aux sommes de quantités dépendantes*, by S. Bernstein. Mathematische Annalen, vol. 97 (1926). *Sur une Proposition de la Théorie des Probabilités*, by A. Liapounoff, Bull. de L Acad. Imp. des Sc. de St. Petersbourg, (5), vol. 13 (1900), pages 358–386.

where B_n is the sum of the expected values of the x's raised to the second power and c_i is the expected value of $|x_i|^3$. If the origin is translated to the point $(k/2, 0)$ it is easily seen that the limit is zero and that the probability that $t_1(2B_n)^{1/2} < z < t_2(2B_n)^{1/2}$ approaches

$$\pi^{-1/2} \int_{t_1}^{t_2} e^{-t^2} dt$$

uniformly, or that

$$\lim_{n \to \infty} \int_{t_1(2B_n)^{1/2}}^{t_2(2B_n)^{1/2}} F_n(z)dz = \pi^{-1/2} \int_{t_1}^{t_2} e^{-t^2} dt$$

or that

$$\lim_{n \to \infty} \int_{t_1}^{t_2} (2B_n)^{1/2} \cdot F_n[(2B_n)^{1/2} \cdot z]dz = \pi^{-1/2} \int_{t_1}^{t_2} e^{-t^2} dt.$$

But since t_1 and t_2 can take on any values it follows that

$$(2B_n)^{1/2} \cdot F_n[(2B_n)^{1/2} \cdot z] \to \pi^{-1/2} e^{-t^2}.$$

Figure 5 shows $\pi^{-1/2}e^{-z^2}$ and $(2B_4)^{1/2} \cdot F_4[(2B_4)^{1/2} \cdot z]$ plotted together. There is less difference between them than the figure indicates, especially at the ends.

By-Products

Since the sum of the polynomials in $F_4(z)$ is $4!$, then by adding them as written on page 430 it follows that (omitting denominators)

$$4\{z^3 - 3(z-k)^3 + 3(z-2k)^3 - (z-3k)^3\} = 4!k^3.$$

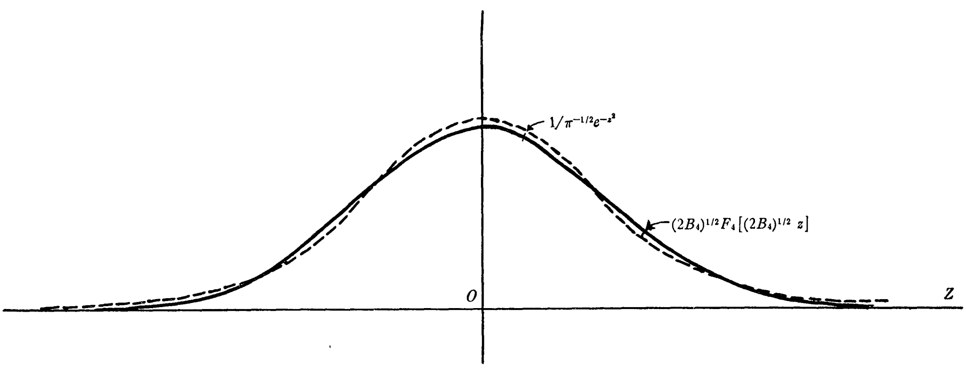

FIG. 5

But since all of the coefficients of the different powers of the z's vanish it follows that

$$(0)^3 - 3(-k)^3 + 3(-2k)^3 - (-3k)^3 = 3!k^3.$$

Let $k=1$ and this becomes

$$(0)^3 - 3(-1)^3 + 3(-2)^3 - (-3)^3 = 3!$$

The coefficients here are the coefficients of the binomial $(a-b)^3$. In general this becomes

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}(-rk)^n = n!k^n \text{ and } \sum_{r=0}^{n}(-1)^r\binom{n}{r}(-r)^n = n!$$

Since the constant term in $F_{n,n-1}(z)$ is $(nk)^{n-1}$, the following relation is obtained if the constant term in $F_{n,n-1}(z)$ as written on page 431 is equated to $(nk)^{n-1}$:

$$\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}(-rk)^{n-1} = (nk)^{n-1}.$$

If $k=1$ and $1/n$, respectively, then

$$\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}(-r)^{n-1} = n^{n-1},$$

$$\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}\left(-\frac{r}{n}\right)^{n-1} = 1.$$

$F_{n,n-1}(z)$ is always equal to $(-z+nk)^{n-1}$. By comparing the coefficients of this with those of $F_{n,n-1}(z)$ on page 431 it is seen that

$$\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}(-rk)^h = (-1)^{n-1-h}(nk)^h.$$

If $k=1$, then

$$\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}(-r)^h = n^h(-1)^{n-1-h},$$

where h is an integer and less than or equal to $n-1$ and $n>2$. It will be noted that n^h is an algebraic sum of integers to $n-1$ all raised to the power h with proper coefficients. The coefficients of these integers raised to the power h are the coefficients of the bionomial $(a-b)^n$ with the exception of the last one.

Applications

The mantissas in a five place table of logarithms are rounded-off numbers. For example the mantissa of log 152 is .18184. From what is in the table it is not known whether the 4 came from .1818362, or .1818449, or .18183901, or

.18184001. The 4 is accepted as being the correct fifth digit. The value of the mantissa as written in this table might have been any one of the numbers lying between .181835 and .181845. Any number in this interval as far as is known from the table is equally likely and the probability that the fifth digit came from numbers in this interval is unity and hence the probability that the error lies in the interval $(-.5, +.5)$ is unity and the probability that the error lies outside of this interval is zero. This gives a probability function similar to the ones discussed in this paper. If numbers are multiplied their logarithms are added and it is desirable to know the probability function or error law for the sum of n such functions. $F_n(z)$ developed in this paper gives such a probability function; in fact it gives a probability function for the sum of the errors arising in using any logarithmic table.

A scientist wishes to measure the duration of a certain event and uses a watch which gives the time in seconds. The hand of the watch jumps every second and makes the jump at the moment the particular second is reached. He finds that the duration of a certain event was t seconds. The actual duration might have been t.001 seconds or t.99999 seconds or any one of the infinite numbers lying between t and $t+1$ seconds. The law of error for the individual event is a constant in the interval $(-.5, .5)$ and zero elsewhere. If the average of n such experiments is obtained the error law for this average is given by a function similar to $F_n(z)$

THE SOLUTION OF CERTAIN DYNAMICAL PROBLEMS

By R. C. COLWELL, Department of Physics, West Virginia University

The type of dynamical problems to be solved in this article is discussed in Routh's *Rigid Dynamics*, (Part 1, page 171) but the method of solution is somewhat different. Routh states that in many dynamical problems the relative motion of the different bodies is all that is required and it is not necessary to find the absolute motion of each body in space. His fundamental theorem is that if any dynamical system is referred to a moving point C, we may reduce C to rest by applying to every element of the system an acceleration equal and opposite to that of C. It is also necessary to suppose that an initial velocity equal and opposite to that of C has been applied to each element. In the method outlined here, it is first assumed that the point C is fixed in space; but no reversed forces are applied to C. After the solution is obtained with C fixed the forces acting are calculated for C in motion and the necessary correction applied directly to the first solution. This method will be exemplified first with the following problem given by Routh. A circular hoop whose weight is nw is free to move on a smooth horizontal plane. It carries on its circumference a small ring weight w, the coefficient of friction between the two being μ. Initially the hoop is at rest and the ring has an angular velocity ω about the center of the hoop. Show that the ring

will be at rest on the hoop after a time $(1+n)/\mu\omega$. First suppose that the center of the hoop is fixed in space so that the hoop has a rotation but no translation and is thus compelled to rotate about its center by the friction of the ring w. Solve for the time under these conditions. Let ω_b be the angular velocity of the ring at any time t and ω_h that of the hoop. Then, by equating momenta,

$$I_1\omega = I_1\omega_b + I_2\omega_h; \tag{1}$$

also, from the energies,

$$\tfrac{1}{2}I_1\omega^2 = \tfrac{1}{2}I_1\omega_b^2 + \tfrac{1}{2}I_2\omega_h^2 + \text{Loss in friction}. \tag{2}$$

Now the centrifugal force acting is $\mu m v_b^2/r$ and the distance passed over along the hoop in time dt is $r(\omega_b - \omega_h)dt$; hence the loss in friction is

$$\int_0^t \mu m r^2 \omega_b^2(\omega_b - \omega_h)dt. \tag{3}$$

Substituting (3) in (2), we get

$$\tfrac{1}{2}I_1\omega^2 = \tfrac{1}{2}I_1\omega_b^2 + \tfrac{1}{2}I_2\omega_h^2 + \int_0^t \mu m r^2 \omega_b^2(\omega_b - \omega_h)dt. \tag{4}$$

Differentiating (4) and using (1), we obtain the differential equation

$$\frac{d\omega_h}{dt} - \frac{\mu m r^2}{I_2}\left\{\frac{I_1\omega - I_2\omega_h}{I_1}\right\}^2 = 0. \tag{5}$$

With the initial conditions $t=0$, $\omega_h=0$, $\omega_b=\omega$ the solution of (5) is

$$\frac{I_2\omega_h}{\omega\{I_1\omega - I_2\omega_h\}} = \frac{\mu m r^2}{I_1}t. \tag{6}$$

At end of time t,

$$I_1\omega = I_1\omega_b + I_2\omega_h. \tag{7}$$

With (7) substituted in (6), the solution becomes

$$t = \frac{I_2}{I_1\mu\omega} = \frac{n}{\mu\omega}\,\cdot \tag{8}$$

So far the hoop has been regarded as fixed upon a central axis. If, however, it is free to move, the actual acceleration instead of being v^2/r will be given by

$$(nw + w)a' = wv^2/r$$

or

$$a' = \frac{v^2}{[r(n+1)]}\,\cdot \tag{9}$$

The relative acceleration of the hoop and ring is then

$$a - a' = v^2/r - v^2/[r(n+1)]$$

or

$$a - a' = (v^2/r)\{n/(n+1)\}. \tag{10}$$

The force is weakened in the same ratio and the time lengthened in the inverse ratio.
Hence

$$t = (n/\mu\omega) \times (n+1)/n = (n+1)/\mu\omega. \tag{11}$$

A second problem of the same general type is this: A flat circular disk in two parts (A and B) is lying upon a smooth table. A block m is started in rotation about A with angular velocity ω and coefficient of friction μ(Fig. 1). The part A which is concentric with the part B rotates around B with a frictional couple

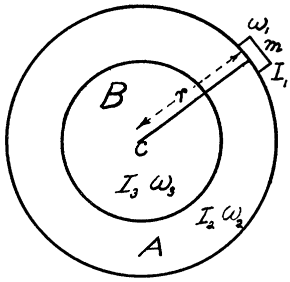

FIG. 1

K. How long before m and A have the same angular velocity? First suppose that the disk is fixed at C so that there is rotation without translation. Then the momental equation gives

$$I_1\omega = I_1\omega_1 + I_2\omega_2 + I_3\omega_3. \tag{12}$$

The loss in friction between m and A is

$$\int_0^t \mu m r^2 \omega_1^2 (\omega_1 - \omega_2) dt.$$

The loss in friction between A and B is

$$\int_0^t K(\omega_2 - \omega_3) dt,$$

where K represents the work done by the frictional couple in turning through

one radian and the friction does not change with the velocity. The energy equation is then

$$(13)\quad \tfrac{1}{2}I_1\omega^2 = \tfrac{1}{2}I_1\omega_1^2 + \tfrac{1}{2}I_2\omega_2^2 + \tfrac{1}{2}I_3\omega_3^2 + \int_0^t \mu m r^2\omega_1^2(\omega_1 - \omega_2)dt + \int_0^t K(\omega_2 - \omega_3)dt.$$

The third equation is

$$(14)\quad K = I_3(d\omega_3/dt).$$

The solution of (12), (13) and (14) for ω_1 gives the differential equation

$$(15)\quad I_1(d\omega_1/dt) + \mu m r^2\omega_1^2 = 0,$$

from which

$$(16)\quad \omega_1 = I_1\omega/(I_1 + \mu m r^2\omega t).$$

The three equations also give for ω_2

$$(17)\quad I_2\frac{d\omega_2}{dt} = \frac{\mu m r^2\omega^2 I_1^2}{(I_1 + \mu m r^2\omega t)^2} - K.$$

Integrating,

$$(18)\quad I_2\omega_2 = -Kt + I_1\omega - I_1^2\omega/(I_1 + \mu m r^2\omega t).$$

Equation (14) gives

$$(19)\quad I_3\omega_3 = Kt.$$

With some loss in generality the three differential equations of motion could have been written down at once from the formula $L = I\alpha$:

$$(20)\quad \mu m v^2 = -I_1(d\omega_1/dt),$$

$$(21)\quad \mu m v^2 - K = I_2(d\omega_2/dt),$$

$$(22)\quad K = I_3(d\omega_3/dt).$$

To find the time which elapses before $\omega_1 = \omega_2$, use equations (16) and (18). From these

$$(23)\quad t^2(\mu m r^2\omega K) + I_1(K - \mu m r^2\omega^2)t + I_1I_2\omega = 0.$$

If as in this case $mr^2 = I_1$ equation (23) reduces to

$$(24)\quad t^2(\mu\omega K) + (K - \mu m r^2\omega^2)t + I_2\omega = 0.$$

From this equation the time t may be found for any given case when the center C is fixed. If, however, the center is not fixed and the whole disk M slides over a

perfectly smooth surface, it is only necessary as before to write $mM/(M+m)$ for m in equation (24). Thus the transition from fixed to moving bodies becomes extremely simple.

It is easy to show that fixing the center of mass of the whole system gives another simple and complete solution of this type of problem. The reason for this is that when a mass m moves around the circumference of a circle of mass M and radius a we may regard the system as a rigid body, because m and M must always remain the same distance apart. Hence if no outer forces are acting upon the system, the center of mass of the complete system must move in a straight line. The energy due to the motion of the center of mass may then be subtracted from the total energy and the center of mass regarded as fixed in space. Suppose that the hoop M is lying upon a smooth table and the mass m moves upon the circumference of the hoop without friction. Initially the mass m is given a

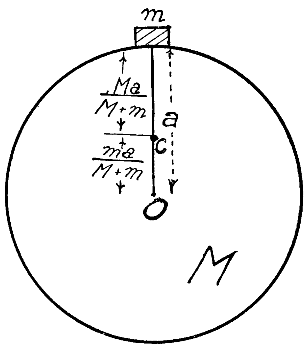

FIG. 2

velocity ω about the center of the hoop; let us find the subsequent motion. The total kinetic energy at once divides into three parts; the energy of m about the center of gravity C, the energy of M about C, and the kinetic energy of both masses regarded as located at the center of gravity. Since $(m+M)V'=mv$, where V' equals the linear velocity of the center of mass, we have

$$\frac{1}{2}m\omega^2a^2 = \frac{1}{2}m\left\{\frac{Ma}{M+m}\right\}^2\omega^2 + \frac{1}{2}M\left\{\frac{ma}{M+m}\right\}^2\omega^2 + \frac{1}{2}\,\frac{(ma\omega)^2}{M+m}\,.$$

Thus the center of gravity of the system moves in a straight line parallel to the direction of the original impulse with a velocity $mv/(M+m)$ while the mass m and the center of the circle rotate about it with uniform angular velocity ω.

If, however, there is a coefficient of friction μ between the block and the hoop, the block and hoop lose angular velocity about the center of gravity, but the hoop gains angular velocity about its own center. Let I_{bc} be the moment of inertia of the block about the center of mass of the system, I_{hc} the moment of inertia of the hoop about the same point, I_h the moment of inertia of the hoop

about its own center, ω_b and ω_h the angular velocities of the block and hoop respectively. Then regarding the center of mass C as fixed and omitting its energy from our equations we have

$$I\omega - I\omega_b = I_h\omega_h. \tag{25}$$

In this equation $I = I_{bc} + I_{hc}$ since the two masses M and m move as a rigid body.

The centrifugal force of the rotating block is

$$F = mMa\omega_b^2/(M + m),$$

and the frictional force is

$$\mu mMa\omega_b^2/(M + m);$$

the frictional couple about the center of the hoop is

$$\mu mMa^2\omega_b^2/(M + m) = K'\omega_b^2,$$

where K' is written for the constant $\mu mMa^2/(M+m)$.

The energy equation so far as rotation is concerned is

$$\tfrac{1}{2}I\omega^2 - \tfrac{1}{2}I\omega_b^2 = \tfrac{1}{2}I_h\omega_h^2 + \int_0^t K'\omega_b^2(\omega_b - \omega_h)dt. \tag{26}$$

The last term in (26) is the energy lost in friction.

From these two equations, we obtain

$$I_h(d\omega_h/dt) = K'\omega_b^2 \text{ and } I(d\omega_b/dt) = -K'\omega_b^2. \tag{27}$$

Initially $t = 0$, $\omega_b = \omega$, and finally

$$\omega_h = \omega_b. \tag{28}$$

Using (27) and (28),

$$\omega_b = \frac{I\omega}{K'\omega t + I}, \quad \omega_h = \frac{I\omega}{I_h} - \frac{I^2\omega}{I_h(K'\omega t + I)}. \tag{29}$$

Equating the two parts of (29), $t = I_h/K'\omega$, which is the time required for the block to cease sliding along the circumference of the hoop. Putting in the value of K'

$$t = (M + m)/um\omega. \tag{30}$$

In this method of solution it is possible to keep track of the whole energy at any time t. Let us calculate the energies at the time when the block just ceases to slide along the circumference of the hoop, that is when $t = I_h/K'\omega$. First the kinetic energy of the center of gravity due to its linear velocity is

$$m^2a^2\omega^2/2(M + m). \tag{31}$$

The kinetic energy of the hoop about the center of gravity of the system is

$$I_{hc}\omega_b^2/2. \tag{32}$$

The kinetic energy of the block about the center of gravity of the system is

$$I_{bc}\omega_b^2/2. \tag{33}$$

The kinetic energy of the hoop about its own center is

$$\frac{1}{2}Ma^2\omega_b^2 = \frac{1}{2}Ma^2\left\{\frac{I^2\omega^2}{(I+I_h)^2}\right\}. \tag{34}$$

The energy lost in friction is

$$\int_0^t K'\omega_b^2(\omega_b - \omega_h)dt = \frac{K'I^3\omega^3}{I_h}\int_0^t \frac{I_h - K'\omega t}{(I+K'\omega t)^3}dt \tag{35}$$
$$= \frac{1}{2}\,\frac{II_h}{I+I_h}\omega^2.$$

The sum of (32), (33), (34) and (35) gives the value $I\omega^2/2$ which is the amount of energy originally going into the rotation. Adding to this the linear kinetic energy (31), we obtain the complete energy $ma^2\omega^2/2$.

QUESTIONS AND DISCUSSIONS

EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A REPLY TO QUESTION NO. 58[1]

By WM. FITCH CHENEY, JR., Connecticut Agricultural College

QUESTION NO. 58[1]: Does the exponential with general base occur frequently enough in mathematics to warrant the general adoption of a special new notation for it, and if so, is Professor Ransom's suggested symbol[2] likely to meet with sufficient approval to be generally adopted?

Some time ago I had occasion to prepare and type a paper involving a great many exponents, with various bases, and I adopted for my own convenience the use of the dollar sign, placed between the base and the exponent. It is

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[1] This Monthly, vol. 37 (1930), p. 188.

[2] This Monthly, vol. 37 (1930), pp. 187–188.

easy to remember, in that anyone will agree that the dollar sign denotes power, and hence such a complicated expression as

$$A^{x^2} - B^{2x-3} + \sqrt{y}$$

may be written all on one line as

$$A\$x\$2 - B\$(2x - 3) + y\$\tfrac{1}{2},$$

which could not be confused with any existing notation. A judicious use of parentheses is the one essential. This has an advantage over the notation proposed by Professor Ransom, in that but one symbol is ordinarily necessary, while his suggested symbol, $\between$ requires two strokes with a back-space between them.

In my opinion, the exponential with general base does not occur frequently enough in mathematics to warrant the general adoption of a special new notation for it, but in any paper which does involve its frequent use, I advocate the use of some special notation, specially defined in that paper.

The Arc of the Ellipse

By R. Goormaghtigh, Bruges, Belgium

In the April, 1930 issue[1] of this Monthly, R. A. Johnson has given the following expression for the approximate length S of the ellipse whose semi-axes are a; b:

$$S_0 = \frac{\pi}{2}\left[a + b + 2\left(\frac{a^2 + b^2}{2}\right)^{1/2}\right].$$

Expanding in terms of a and the eccentricity e, we have

$$S_0 = 2\pi a\left[1 - \frac{1}{2^2}e^2 - \frac{3}{2^6}e^4 - \frac{5}{2^8}e^6 - \frac{180}{2^{14}}e^8 \cdots\right],$$

whereas

$$S = 2\pi a\left[1 - \frac{1}{2^2}e^2 - \frac{3}{2^6}e^4 - \frac{5}{2^8}e^6 - \frac{175}{2^{14}}e^8 - \frac{441}{2^{16}}e^{10} - \frac{4851}{2^{20}}e^{12} \cdots\right].$$

Up to the *sixth* power of e, the two expansions are the same.

Another remarkable and probably new approximation is

$$S_1 = \frac{\pi}{8}\left[9(a + b) - 5(ab)^{1/2} + 3\left(\frac{a^2 + b^2}{2}\right)^{1/2}\right].$$

[1] Vol. 37 (1930), p. 188.

The expansion is the same as for S, up to the *tenth* power of e:

$$S_1 = 2\pi a\left[1 - \frac{1}{2^2}e^2 - \frac{3}{2^6}e^4 - \frac{5}{2^8}e^6 - \frac{175}{2^{14}}e^8 - \frac{441}{2^{16}}e^{10} - \frac{4844}{2^{20}}e^{12} \cdots \right].$$

A Note by the Editor

It may be of interest to emphasize the degree of accuracy of the preceding formula by applying it to an ellipse having the general size and shape of the earth's orbit (which in Newtonian mechanics would be a true ellipse if one neglects, as we shall do, the perturbations due to heavenly bodies other than the sun). Assuming, as at present it seems proper to, that the age of the earth is less than one hundred billion years, then the total distance the earth has traveled since its creation would differ from the approximate distance given by the preceding formula by about the thickness of the sheet of paper on which this is written.

On Strophoidal Curves

By R. Goormaghtigh, Bruges, Belgium

1. In the December, 1928 issue of this Monthly was published a paper by R. M. Mathews and Otto Dunkel,[1] giving a solution of the following question:

A line drawn through the end F of the loop of a right strophoid or a folium of Descartes cuts the curve again in Q and R such that the segment QR subtends a right angle at the node O. Are these results particular cases of a general result for a certain class of cubic curves?[2]

This question suggests a general problem: To determine algebraic plane curves Γ of the n^{th} order, having chords QR passing through a fixed point F and subtending a right angle at another given point O.

2. Consider the polar curve Γ' of Γ, with respect to a circle C having its center at O; then the orthoptic curve of Γ' is a straight line Δ, the polar of F. If Γ'' is the polar curve of Γ' with respect to a circle C' having its center O' on Δ, the chords of Γ'' subtending a right angle at O' are parallel.

Consider rectangular axes with the origin at O', and let the y-axis be parallel to these chords. The equation of the curve Γ'' may be written:

$$\sum_{i=0}^{i=k} A_{k-i}(x)y^i = 0, \qquad k \leqq n$$

where $A_{k-i}(x)$ is a polynomial in x of degree not exceeding $n-i$.

[1] Vol. 35 (1928), pp. 544–547.

[2] This question was proposed as Problem 2928 in this Monthly for the year 1921. (Vol. 28, p. 466.)

Since the degree of Γ' is n there must be at least one of these polynomials with the maximum degree. If we set in turn $y=tx$ and $y=-t^{-1}x$, we shall obtain two equations,

$$\sum_{i=0}^{i=k} A_{k-i}(x)x^i t^i = 0, \qquad \sum_{i=0}^{i=k} A_i(x)x^{k-i}(-1)^{k-i}t^i = 0,$$

which must have the same roots for a given x. Hence

$$A_{k-i}(x)x^i = c(-1)^{k-i}A_i(x)x^{k-i},$$

where c is a constant for the given x. Replacing i by $k-i$ we have also

$$A_i(x)x^{k-i} = c(-1)^i A_{k-i}(x)x^i = c^2(-1)^k A_i(x)x^{k-i}.$$

Since not all of the $A_i(x)$ polynomials can be identically zero, we must have $c^2=1$ and $k=2m$. It will now be convenient to take the coefficient of y^m as $2A_m(x)$.

Thus the equations of the Γ'' curves have the form,

$$\sum_{i=0}^{i=m} A_i(x)\left[y^{2m-i} + c(-1)^i x^{2m-2i}y^i\right] = 0, \ c = +1 \text{ or } c = -1, \ 2m \leqq n,$$

where the polynomials $A_i(x)$ are quite arbitrary, except that there must be at least one, say $A_i(x)$, of degree $n-2m+i$ if the degree of the curve is n.

The initial remark regarding the relation between Γ and Γ'' enables us to pass from the equation of one to the equation of the other.

Let α, β be the coordinates of θ, r the radius of C, and R the radius of C'. Let X, Y be a point on Γ, and x, y a point on Γ''.

An easy calculation gives the equations of transformation

$$zx = X - \alpha, \quad zy = Y - \beta, \quad R^2 z = \alpha(X-\alpha) + \beta(Y-\beta) + r^2.$$

If we take O as the origin of parallel axes for Γ, we have the simpler equations

$$2x = x', \qquad 2y = y', \qquad R^2 x = \alpha x' + \beta y' + r^2.$$

If we insert these values of x and y in the equation for Γ'', the result, after suppressing the accents, may be written thus:

$$\sum_{i=0}^{i=m} A_i(x, 2)\left[y^{2m-i} + c(-1)^i x^{2m-2i}y^i\right] = 0,$$

$$2 = \alpha x + \beta y + r^2, \ c = +1 \text{ or } c = -1,$$

where $B_i(x, z)$ is an arbitrary homogeneous polynomial in x and z of degree $n-2m+i$; the R^2 of the transformation has been absorbed into the arbitrary constants of $B_i(x, z)$. The coordinates of the point F are 0, $-r^2/\beta$, this point being the pole of Δ with respect to C.

We shall now apply these general results to cubics and quartics.

Cubics: Setting $n=3$, $m=1$, we find

$$A_0(x)[y^2 \pm x^2] + A_1(x)[y \mp y] = 0.$$

The only case of interest is then

$$(x+a)(y^2 - x^2) + (bx^2 + cx + d)y = 0;$$

the cubics have an infinity of chords parallel to $O'y$ and subtending a right angle at O'.

When $d=0$, O' is the double point of the cubic, and the tangents at this point to the two branches of Γ'' are perpendicular. Therefore the tangents at the double point O of the corresponding curve are also perpendicular; since the cubic passes through the point at infinity on $O'y$, F lies on Γ:

If a cubic has a node with perpendicular tangents at this point, the chords subtending a right angle at the node pass through a fixed point lying on the curve.

Quartics: Here $n=4$, $m=1$, 2. Disregarding certain degenerate cases, we find

$$(x^2 + ax + b)(y^2 - x^2) + (cx^3 + dx^2 + cx + f)y = 0$$

and

$$y^4 + x^4 + (ax + b)(y^3 - x^2y) + (cx^2 + dx + e)y^2 = 0.$$

For instance, when $a=d=e=o$, $c=2$, this last equation represents the trifolium of de Longchamps, and when $a=c=d=e=o$, Cramer's trifolium. The properties of the parallel chords of these curves, subtending a right angle at the node, are well known.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Introduction to the Theory of Fourier's Series and Integrals. Third Edition. By H. S. Carslaw, Macmillan and Co., Limited, London, England, twenty shillings net. Macmillan and Co., New York, seven dollars. 1930.

The first and second editions of this book are too well known to be given space for a review. The newly appearing third edition differs but little from the second. There are certain enlargements amounting to twenty-eight pages in text and exercises, exclusive of appendices. There are also occasional improved wordings. The historical introduction is brought more nearly up-to-date,

and accounts for four pages of the increase. The portion of the book dealing directly with Fourier series contains a few theorems proved with increased generality. Parseval's theorem is a noteworthy addition. Appendix II on Lebesgue integrals is new. This is a readable simple introduction to this subject in English and was much needed. Brief applications to Fourier series are given.

No bibliography is given at the end of the volume in the third as in the second edition. References, however, are given at the ends of chapters.

Persons owning the second edition will probably not find it worth while to buy the third unless wanting it for a class text. Large libraries should have the latest edition of so widely used a book.

TOMLINSON FORT

The History and Significance of Certain Standard Problems in Algebra. By Vera Sanford. Teachers College, Columbia University, 1927. Teachers College Contributions to Education, No. 251. viii+102 pages.

This little book contains a careful and interesting study of the standard verbal problems which appear in textbooks on algebra. These problems the author classifies under three heads; genuine problems, or those arising from real situations; puzzles, or recreations; and an intermediate class called pseudo-real problems in which natural phenomena are combined with unnatural conditions.

There is a discussion of the reasons why problems are taught, the characteristics of good problems, efforts to make problems interesting and real to the student, and some of the ingenious methods formerly used for solving problems.

A chapter is devoted to the history of commercial problems, showing the striking way in which the genuine problems of business arithmetic reflect contemporary conditions and offer material of value in compiling a commercial history of Europe.

Problems from scientific mathematics form another class of genuine problems to which a chapter is devoted. The striking thing about the history of problems of this type is their late appearance. Even so simple an idea as the principle of the lever was overlooked by the makers of problems until the sixteenth century and instances of scientific problems were rare until the time of Newton.

Problems of the recreational type, on the other hand, are for the most part handed down to us from antiquity and have seen little or no development in modern times. Many interesting illustrations of puzzle problems are given both in ancient and in modern dress. Of particular interest are those which are based on progressions; and the author suggests how the teaching of that subject might be improved by the use of such colorful problems instead of the lifeless ones of the usual textbook.

Problems of the pseudo-real type are defended on the ground that they provide a varied material for practice in analysis, and that in many cases

they are more real to the student than problems which actually occur in real situations.

In the closing chapters the author discusses changes in problem material, reasons for the disappearance of certain types of problems, and the survival of others, and finally makes some suggestions as to the lines along which the future development of problems is likely to proceed. In conclusion she says that "in the future puzzle problems will be used because human beings enjoy them; genuine problems will remain because men demand specific and immediate applications of the theory they study; and pseudo-real problems will supplement the other two in providing opportunity for the analytic thinking that forms the most important reason for teaching algebra."

ROSA L. JACKSON

Manual of Trigonometry, for Colleges, Universities, and Technical Schools. By E. C. Kennedy. New York, The Macmillan Co., 1930. 90 pages $8\frac{1}{2}\times 11$ inches. $1.10

This is a collection of 126 problems and exercises in Trigonometry, so arranged as to be available for laboratory work. For the most part there is one exercise to each page, but the last nineteen exercises are listed on one page with no space for solutions. Some attention is paid to the difficulties of solving triangles of extreme forms, and the tables at the end of the book include the common Maclaurin series. There is also a table of the functions of small angles; there are conversion tables for degrees and radians, and tables of the values of constants to fifteen places.

R. A. J.

PROBLEMS AND SOLUTIONS

Edited by B. F. Finkel, Otto Dunkel, and H. L. Olson

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3448. *Proposed by Oliver D. Kellogg, Harvard University.*

Prove that whenever the infinite series, with positive terms,

$$u_1 + u_2 + u_3 + \cdots$$

converges, the series

$$\frac{u_1}{r_1} + \frac{u_2}{r_2} + \frac{u_3}{r_3} + \cdots$$

diverges,

$$r_n = u_n + u_{n+1} + u_{n+2} + \cdots$$

being the remainder of the first series after $n-1$ terms.

3449. *Proposed by W. E. Buker, Leetsdale, Pa.*

(1) Find a number X such that X^2+5 and X^2-5 are each square numbers.

Note: This problem was proposed by John of Palermo and solved by Leonardo of Pisa about 1220 A. D. A solution is $X=41/12$; for $(41/12)^2+5=(49/12)^2$; $(41/12)^2-5=(31/12)^2$. How did he arrive at his result?

Reference: Florian Cajori, *History of Mathematics*, pp. 124.

3450. *Proposed by the Late G. B. M. Zerr.*

A starts from Washington towards Baltimore, a distance of 40 miles, and travels at the rate of 10 miles per hour. Before he arrives at Baltimore, a snowstorm starts at Washington and at all places occupying a certain unknown distance towards, but not reaching beyond Baltimore; if A is caught in this storm he must stop until it is over; he is also to receive for this journey a number of dollars inversely proportional to the time occupied in it, at the rate of 50 dollars for one hour. The time when the storm commences is unknown, but all events are equally probable; show that A's expectation is \$11.66.

3451. *Proposed by R. Goormaghtigh, Bruges (Belgium).*

If two conics circumscribed to a triangle are orthogonal at a given point M, their tangents at M being the axes of the conic conjugate to the triangle and having M as center, the product of their normal chords at M is equal to four times the product of their radii of curvature at M.

3452. *Proposed by Charles K. Robbins, Purdue University.*

Solve the functional equation

$$\lambda(x, y) \cdot \mu(x, z) + \nu(x, y) = \omega(y, z).$$

x, y, and z are independent variables and $\partial\omega/\partial y \neq 0$ and $\partial\mu/\partial z \neq 0$.

3453. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Given the radical axis of two circles, their circle of similitude, and the length of their line of centers, construct the two circles.

3454. *Proposed by M. S. Knebelman, Princeton Universiy.*

Deduce the explicit expression for $P(m, n)$ from the relation

$$P(m, n) = P(n, m) = \sum_{i=1}^{i=m} P(i, n-1), \ P(1, 1) = 1,$$

where m and n are positive integers.

3455. *Proposed by Solomon Kullback, Brooklyn, N. Y.*

A triangle ABC, with AB constant, is inscribed in a circle. On AC and BC,

construct equilateral triangles, ACD and BCE. Find the locus of the midpoint of ED as C moves along the arc ACB.

SOLUTIONS

3403 [1929, 543]. *Proposed by E. A. Whitman, Carnegie Institute of Technology.*

If in problem 3379, proposed in May, 1929, \$12 and 12 are replaced by "a" dollars and a respectively, for what values of "a" is there a solution?

Problem 3379 reads: Two men own jointly x cows which they sell for x dollars per head, and with the returns buy sheep at \$12 per head. As their income from the cows is not divisible by 12 they purchase a lamb with the remainder. Later they divided the flock so that each had the same number of animals. How much money was due the man with the lamb by the other man?

Solution by William I. Miller, University of Pittsburgh.

Let the number of sheep, which must be odd, be $2n+1$, and let k represent the price of the lamb in dollars. Then

$$x^2 = a(2n+1) + k,\ 0 < k < a,$$

or

$$x^2 \equiv a + k \pmod{2a},\ 0 < k < a.$$

The problem is to determine a so that the congruence has solutions for just one value of k. By trial, $a=3$, 6, 8, 10, or 12. It is not necessary to try values for a greater than 18, for if $a \geqq 19$, there will always be at least two perfect squares between a and $2a$, so that k cannot be unique.

To prove the last statement, suppose we can choose n so that $2a>(n+1)^2$ where $n^2>a\geqq(n-1)^2$. This will be possible if $2(n-1)^2>(n+1)^2$, or if $n\geqq 6$, and hence if $a\geqq 25$. The statement is obviously true if $19\leqq a\leqq 24$.

Also solved by W. E. Buker, and the Proposer.

Note by the Editors: If a is odd it may be shown that k is not unique for $a>3$. For, set $x=ma+i$, where m is an odd positive integer and $i=1$ or 2. Then $x^2=a[am^2+2mi]+i^2$ and the brackets contain an odd integer. Hence k may be taken as 1 or 4. This reduces the trials above to $a=3$ and to the even integers.

3406 [1930, 37]. *Proposed by William P. Parker, Pyengyang, Chosen, Korea.*

Find the condition that $(\alpha x+\beta y+\gamma z)\ (\alpha_1 x+\beta_1 y+\gamma_1 z)-(\alpha_2 x+\beta_2 y+\gamma_2 z)^2$ may be resolved into two linear factors.

Solution by E. D. Rainville, University of Colorado.

The expression under consideration is a ternary quadratic form. By a well-known theorem, (Maxime Bôcher, *Introduction to Higher Algebra*, page 137), the necessary and sufficient condition that this expression be reducible is that

$$\Delta = \begin{vmatrix} 2(\alpha\alpha_1-\alpha_2^2) & \alpha\beta_1+\beta\alpha_1-2\alpha_2\beta_2 & \alpha\gamma_1+\gamma\alpha_1-2\alpha_2\gamma_2 \\ \alpha\beta_1+\beta\alpha_1-2\alpha_2\beta_2 & 2(\beta\beta_1-\beta_2^2) & \beta\gamma_1+\gamma\beta_1-2\beta_2\gamma_2 \\ \alpha\gamma_1+\gamma\alpha_1-2\alpha_2\gamma_2 & \beta\gamma_1+\gamma\beta_1-2\beta_2\gamma_2 & 2(\gamma\gamma_1-\gamma_2^2) \end{vmatrix} = 0.$$

If we expand Δ into 27 third order determinants in the usual way, it is immediately obvious that

$$\Delta = 2\begin{vmatrix} \alpha & \alpha_1 & \alpha_2 \\ \beta & \beta_1 & \beta_2 \\ \gamma & \gamma_1 & \gamma_2 \end{vmatrix}^2 = 0.$$

That is, the necessary and sufficient condition desired is

$$\begin{vmatrix} \alpha & \alpha_1 & \alpha_2 \\ \beta & \beta_1 & \beta_2 \\ \gamma & \gamma_1 & \gamma_2 \end{vmatrix} = 0.$$

Note by Otto Dunkel. Geometric considerations enable us to discover the required condition. If λ, μ, ν denote homogeneous linear expressions in x, y, z, then $\lambda\mu-\nu^2$ set equal to zero gives a conic for which $\nu=0$ is the chord of contact of the two tangents $\lambda=0$, $\mu=0$. If the conic degenerates into two straight lines, then these two straight lines pass through a point common to the three lines $\lambda=0$, $\mu=0$, $\nu=0$. The converse may also be considered geometrically.

An analytical proof may be given which depends mainly upon the simple theorems for linear dependence. We shall prove that:

A necessary and sufficient condition that

$$\lambda\mu - \nu^2 \tag{1}$$

where λ, μ, ν are homogeneous linear expressions in x, y, z, shall reduce to the product of two linear expressions, is that λ, μ and ν shall be linearly dependent, that is that

$$a\lambda + b\mu + c\nu \equiv 0, \tag{2}$$

where a, b, c are constants not all of which are zero.

If (2) is true, then we may solve for at least one of the three expressions, say λ, in terms of the other two, μ and ν. Inserting this expression for λ in (1), we shall obtain a homogeneous quadratic expression in μ and ν. Such an expression can always be factored, and hence (1) reduces to the product of two linear factors.

Suppose now that

$$\lambda\mu - \nu^2 \equiv \alpha\beta, \tag{3}$$

where α and β are linear in x, y, z. Suppose first that $\alpha\equiv c\nu$, $\beta\equiv c'\nu$, where c and c' are constants. Since $\nu=0$ for an infinite set of values of x, y, z, either λ or μ must vanish for more than one set of these values. Suppose it is λ, then λ and ν are linearly dependent; consequently, λ, μ, ν are linearly dependent. We may now suppose that not both of the above identities are true, say α is not linearly dependent upon ν. Since λ and α vanish simultaneously for one set of values of x, y, z, this set of values must make ν also vanish. Hence λ, ν, α are linearly dependent. If the coefficient of α in this linear relation is zero, then λ and ν are

linearly dependent, and, therefore, λ, μ, ν are linearly dependent. Suppose now the coefficient of α is not zero, we may then write $a\lambda+b\nu\equiv\alpha$ where a is not zero since this case is excluded here. By a similar argument we arrive at the result $a'u+b'\nu\equiv\alpha$. Hence $a\lambda-a'\mu+(b-b')\nu\equiv 0$, where neither a nor a' is zero. Since this exhausts the possible cases the theorem is proved.

Also solved by W. E. Buker, E. F. Cox, Raymond Garver, J. H. Neelley, A. Pelletier, . J. ORamler.

3407 [1930, 37]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

A ray of light emanating from a fixed source L is reflected by a flat mirror at a point M so that the reflected ray passes through a given point I. Find the locus of M when the mirror revolves about a fixed axis s.

Solution by A. Pelletier, Montreal, Canada.

Let s be the z-axis, and let the coordinates of I and L be $(a, 0, 0)$ and (b, c, d), respectively. For any position of the mirror, LM passes through I_1, the point symmetric to I with regard to the mirror in that position. Thus the locus of I_1, is a circle of radius $OI=a$ which lies in the xy-plane. Hence the point M lies on a cone with the vertex L and with this circle as a base. The equation of this cone is

$$(1) \qquad (dx - bz)^2 + (dy - cz)^2 = a^2(d - z)^2.$$

Next let us consider the projection of the path of M on the xy-plane. Let ON be the trace of the mirror on the xy-plane, and let α be the angle which ON makes with the x-axis. Then the coordinates of I_1 are $a \cos 2\alpha$, $a \sin 2\alpha$; and the equation of the projection of this path is obtained by eliminating α from

$$(2) \qquad y = x \tan \alpha, \quad (y - c)/(x - b) = (c - a \sin 2\alpha)/(b - a \cos 2\alpha).$$

The elimination of α gives

$$(3) \qquad (by - cx)(x^2 + y^2) + a(c - y)(x^2 - y^2) + 2a(x - b)xy = 0.$$

The equations (1) and (3) determine the locus of M in space. The points L and I are the extremities of the actual path of M.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Harvard University has conferred an honorary doctorate on K. T. Compton, the newly elected president of the Massachusetts Institute of Technology.

Indiana University has conferred honorary doctorates on Dr. C. O. Lampland and Dr. V. M. Slipher of the Lowell Observatory, Flagstaff.

Lake Forest College has conferred an honorary doctorate on Professor W. D. MacMillan, of the University of Chicago.

Professors P. Franklin and D. J. Struick, of the Massachusetts Institute of Technology, have been elected Fellows of the American Academy of Arts and Sciences.

A portrait of Professor E. H. Moore, of the University of Chicago, painted by Mr. Ralph Clarkson, has been presented to that University by Professor Moore's former students. It will be hung in the new Eckhart Hall of Mathematics, Physics, and Astronomy.

Dr. W. F. Willcox, of Cornell University, represented the United States at the meeting of the International Institute of Statistics which was held at Tokyo in September, 1930.

Dr. V. W. Adkisson has been appointed assistant professor of mathematics at the University of Arkansas.

Mr. J. G. Adshead has been appointed assistant professor of mathematics at Dalhousie University.

Assistant Professor J. P. Ballantine, of the University of Washington, has been promoted to an associate professorship of mathematics.

Dr. P. M. Batchelder has been promoted to an associate professorship of mathematics at the University of Texas.

Leon H. Bunyan, who has been an instructor at the University of Wisconsin, Extension Division, Milwaukee, Wisconsin, has been appointed assistant professor in the department of mathematics of Rutgers University, New Brunswick, N. J.

Associate Professor C. W. Cobb, of Amherst College, has been promoted to a professorship of mathematics.

Dr. H. P. Evans has been promoted to an assistant professorship of mathematics at the University of Wisconsin.

Professor J. W. Glover, of the University of Michigan, has been appointed president of the Teachers Insurance and Annuity Association of America.

Assistant Professor Cornelius Gouwens, of Iowa State College, has been promoted to an associate professorship of mathematics.

Dr. Lois W. Griffiths, of Northwestern University, has been promoted to an assistant professorship of mathematics.

Assistant Professor Einar Hille, of Princeton University, has been promoted to an associate professorship of mathematics.

Dr. J. J. L. Hinrichsen has been appointed assistant professor of mathematics at Iowa State College.

Assistant Professor Clyde M. Huber, of Rutgers University, has been appointed professor and head of the department of mathematics at Atlantic University, Virginia Beach, Va.

Dr. B. W. Jones has been appointed assistant professor of mathematics at Cornell University.

Dr. A. Marguerite Lehr, of Bryn Mawr College, has been promoted to be an associate in mathematics.

Dr. C. O. Oakley has been promoted to an assistant professorship of mathematics at Brown University.

Associate Professor J. F. Reilly, of the State University of Iowa, has been promoted to a professorship of mathematics.

Assistant Professor C. A. Rupp has been promoted to an associate professorship at Pennsylvania State College.

Professor C. D. Smith, of Louisiana College, has been appointed professor and head of the department of mathematics at the Mississippi Agricultural and Mechanical College.

Assistant Professor W. M. Whyburn has been promoted to an associate professorship of mathematics at the University of California at Los Angeles.

Dr. B. C. Wong has been promoted to an assistant professorship of mathematics at the University of California at Berkeley.

A Correction

There is a misprint in line 7 of J. P. Ballantine's article entitled *A peculiar Function*, in this Monthly, May 1930, p. 250. Instead of

$$r = R\cos\tfrac{1}{2}\theta \quad \text{when} \quad 90 < \theta \leq 180$$

read

$$r = R\sin\tfrac{1}{2}\theta \quad \text{when} \quad 90 < \theta \leq 180.$$

CONTENTS

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.
INDIANA, Earlham College, May 2-3.
IOWA, Ames, Iowa, May 2-3.
KANSAS, February 15.
KENTUCKY, Lexington, Ky., April 5.
LOUISIANA-MISSISSIPPI, Cleveland, Miss., March 7-8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, May 10.
MICHIGAN, Ann Arbor, Mich., March 22.
MINNESOTA, Carleton College, May 17.
MISSOURI.
NEBRASKA, Peru, Neb., May 9.
OHIO, Columbus, Ohio, April 3.
PHILADELPHIA, Philadelphia, Pa., November 29.
ROCKY MOUNTAIN, Denver, Colo., April 11-12.
SOUTHEASTERN, Atlanta, Ga., May 2-3.
SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, Calif., March 8.
TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE FOURTEENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The fourteenth summer meeting of the Mathematical Association of America was held, by invitation, at Brown University, Providence, R.I., on Monday and Tuesday, September 8–9, 1930, in conjunction with the summer meeting and colloquium of the American Mathematical Society. Three hundred five were present at the meetings, including the following one hundred seventy members of the Association.

C. R. Adams, Brown University
Ethel M. Anderton, Mount Holyoke College
R. C. Archibald, Brown University

Clara L. Bacon, Goucher College
W. D. Baten, University of Michigan
A. A. Bennett, Brown University
Theodore Bennett, University of Wisconsin
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G. D. Birkhoff, Harvard University
H. F. Blichfeldt, Stanford University
G. A. Bliss, University of Chicago
M. G. Boyce, Western Reserve University
R. W. Brink, University of Minnesota
Gertrude L. Brockett, Brown University
J. A. Bullard, University of Vermont
L. H. Bunyan, Rutgers University
W. H. Bussey, University of Minnesota
A. D. Butterfield, University of Vermont

S. S. Cairns, Yale University
W. D. Cairns, Oberlin College
B. H. Camp, Wesleyan University
C. C. Camp, University of Nebraska
A. D. Campbell, Syracuse University
G. A. Campbell, Amer. Tel. and Tel. Co., New York, N. Y.
A. B. Chace, Brown University
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J. W. Davis, Dorchester High School for Boys
F. F. Decker, Syracuse University
L. S. Dederick, Aberdeen Proving Ground
Arnold Dresden, Swarthmore College
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L. A. Dye, Cornell University

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H. P. Evans, University of Wisconsin
D. A. Flanders, New York University

Philip Franklin, Massachusetts Institute of Technology
Orrin Frink, Jr., Pennsylvania State College
T. C. Fry, Bell Telephone Laboratories, New York, N. Y.

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H. A. Garabedian, John Hancock Mutual Life Ins. Co., Boston Mass.
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D. C. Gillespie, Cornell University
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J. W. Glover, Teachers Ins. and Annuity Assn., New York, N. Y.
W. C. Graustein, Harvard University

E. S. Hammond, Bowdoin College
E. R. Hedrick, University of California at Los Angeles
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W. A. WILSON, Yale University
RUTH G. WOOD, Smith College
F. S. WOODS, Massachusetts Institute of Technology
FRANCES M. WRIGHT, Elmira College
J. W. YOUNG, Dartmouth College

The meeting at Providence was characterized by the very extensive plans formed for the entertainment of the guests. They were comfortably housed in the dormitories of Pembroke College of the University, and enjoyed the convenience of having their meals together in the dining room of Alumnae Hall. The parlors of this hall were continuously used as meeting places throughout each day and evening.

On Monday afternoon a large party was shown about Providence under the guidance of Mr. Norman M. Isham, an authority on Colonial architecture and furniture; numerous examples of New England architecture were visited. Monday evening President and Mrs. Clarence A. Barbour with Vice President and Mrs. Albert D. Mead received the guests in the parlors of Alumnae Hall, a welcoming speech was made by Doctor Barbour and delectable refreshments were afterward served. On Tuesday afternoon the ladies of the department of mathematics of Brown University served tea to the visiting guests.

Wednesday afternoon was observed as a half holiday, giving opportunity for the guests to go in automobiles or by bus to Newport, where they followed first the Shore Drive past beautiful estates, then the Cliff Walk of three and one half miles to the Casino, where a real shore dinner was served. On Thursday afternoon an opportunity was given to drive about the city, particularly to the beautiful Roger Williams Park and other scenic parts of Providence. The mathematicians were indebted to Miss Margaret Stillwell, custodian of the Annmary Brown Memorial, and to Mr. Lawrence Wroth, librarian of the John Carter Brown Library, for interesting special exhibits and many courtesies. Professor R. C. Archibald was frequently on duty throughout the week to show visitors the very remarkable mathematical library situated on the second floor of Wilson Hall, and cordial welcome was extended to the mathematicians to work in the library before or after the meetings; numerous visitors availed themselves of the opportunity to visit and to use the library. A resolution was adopted by rising vote at the close of the sessions of the Association expressing our appreciation of the fine hospitality shown by the administration and the department of mathematics of Brown University, and of the efficient service rendered by the committee on arrangements and the program committee.

The American Mathematical Society held its thirty-sixth summer meeting and fourteenth colloquium from Tuesday to Friday, with lectures by Professor Solomon Lefschetz of Princeton University on "Topology." A very gratifying increase in enrollment is to be noted as the colloquium lectures follow from summer to summer. Addresses were also given by Professor T. H. Hildebrandt of the University of Michigan and Professor J. D. Tamarkin of Brown Uni-

versity on topics in the general theory of linear operations and integral equations. Sessions for the reading of papers were held on Tuesday afternoon, Wednesday morning, and Thursday morning and afternoon.

The Mathematical Association held sessions on Monday afternoon and Tuesday morning, President Young presiding at the first session and Professor Kempner at the second session. An excellent program had been arranged by a committee consisting of Professors Marston Morse (chairman), A. A. Bennett, Lennie P. Copeland, and E. P. Lane. Abstracts of some of the papers are given, numbered in accordance with the numbers of the papers.

First Session of the Association

(1) "Heaviside operational calculus" by Dr. J. J. Smith, General Electric Company.

(2) "Random sampling" (illustrated) by Dr. W. A. Shewhart, Bell Telephone Laboratories.

(3) "Teaching and research in mathematics" by Professor W. A. Hurwitz, Cornell University.

1. The paper of Dr. Smith will appear in an early issue of the Monthly.

2. Dr. Shewhart's paper will also appear soon in the Monthly.

Second Session of the Association

(4) "Italian contributions to modern mathematics" by Professor Enrico Bompiani, Rome, Italy.

(5) "A mathematical theory of harmony and melody" (illustrated) by Professor G. D. Birkhoff, Harvard University.

(6) "The summer school for teachers of mathematics" by Professor H. P. Hammond, Director of summer school, Society for the Promotion of Engineering Education.

4. The Mathematical Association was greatly honored in having Professor Bompiani accept its invitation to speak at this summer meeting. He had been lecturing the past summer quarter at the University of Chicago, and is lecturing for a part of the present academic year in eastern universities. His presence at the meetings of the mathematical organizations brought with it much pleasure and he won many new friends by his geniality and pleasing personality. The paper will appear in a later issue of the Monthly.

5. Professor Birkhoff, assisted by Professor Marston Morse at the piano, developed a theory of harmony in which a comparison of chords and sequences of chords was effected by means of an analysis of the underlying aesthetic factors and application of a formula presented by him before the International Mathematical Congress of Bologna of 1928. The possibility of a similar treatment of simple melodies was also outlined. These conclusions will be contained in a book on the "Mathematical Elements in Art" which Professor Birkhoff is preparing.

6. The Summer School for Engineering Teachers, of the Society for the Promotion of Engineering Education, holds a particular interest this year for

teachers of mathematics, since it is probable that one of the sessions to be held during the summer of 1931 will be devoted to the study of methods of teaching mathematics to engineering students.

The Summer School had its inception in the general investigation of Engineering Education, conducted by the Society for the Promotion of Engineering Education in the years 1924 to 1929, inclusive. It is a continuing enterprise which has as its purpose concrete measures for the improvement of instruction in engineering. When it was established in 1927 there were no precedents for such an enterprise in higher education in America, and but one similar undertaking is now conducted, namely—the Summer School for Teachers of Engineering of Great Britain.

Since its establishment seven sessions of the Summer School have been held, as follows:

In 1927—Mechanics, at Cornell University and at the University of Wisconsin.

In 1928—Physics, at the Massachusetts Institute of Technology, and Electrical Engineering at the University of Pittsburgh.

In 1929—Mechanical Engineering at Purdue University.

In 1930—Engineering Drawing and Descriptive Geometry at the Carnegie Institute of Technology and Civil Engineering at Yale University.

From small beginnings, with forty teachers in attendance at each session the Summer School has grown to considerable proportions, the session of 1930 having been attended by a total of 190 teachers. Since its beginning, nearly 500 teachers have attended the sessions and the staffs have numbered nearly 200.

An idea of the nature of one of these sessions can be secured from a brief account of one of them. The program of the physics session was divided into a number of subdivisions:

No. 1. Educational principles and practices; for the study of the general principles of teaching. This division was under the direction of Professor Walter F. Dearborn, of the Graduate School of Education of Harvard University.

No. 2. Methods of teaching and subject matter of college courses in physics for engineers. This was the principal division of the program and was conducted under the direction of a full-time teaching staff, which included Professors W. S. Franklin, of Massachusetts Institute of Technology, O. M. Stewart, of the University of Missouri, and A. W. Duff, of Worcester Polytechnic Institute.

No. 3. The correlation of physics with other subjects of instruction, conducted under the leadership chiefly of teachers of engineering and of subjects related closely to physics.

No. 4. Applications of physical laws, a division in charge of practicing engineers and scientists.

No. 5. Advanced physics, a series of lectures conducted by prominent physicists from colleges and industries.

No. 6. Committee reports prepared by groups of members of the session on problems of importance in the teaching of physics.

No. 7. General lectures and recreation.

In accordance with the general plan, the Summer School sessions on different subjects or divisions of the curriculum are conducted each year in different institutions throughout the country. It is planned in this way, during a course of years, to cover all of the important divisions of engineering education and so to locate the sessions as to have them readily accessible to teachers in different parts of the country. It is notable, however, that all of the sessions thus far have been attended by men from widely separated localities. The session on civil engineering, in 1930, for example, numbered among its attendants representatives of thirty states and provinces of Canada and from one foreign country. Sixty different collegiate institutions were represented.

Members of the sessions are also widely representative of the different teaching grades and degrees of experiences, one of the sessions numbering, as an illustration, three deans, thirty professors, seventeen associate professors thirty assistant professors, and fourteen instructors.

The staffs of the sessions are recruited from among the foremost teachers, scientists and engineers of the country. They read, in fact, like a section from "Who's Who in Engineering Education."

The meetings are conducted through formal lectures, discussion periods, seminars, model teaching exercises, and committee work.

Of equal importance with the regular meetings are the informal gatherings of the teachers throughout the sessions for the discussion of their mutual problems. Arrangements are made at each session so that the entire group can live together for the period of the session, generally in one of the dormitories of the institution. Organized recreational features are provided.

When the Summer School was begun in 1927, the initiative for holding the sessions came from the Board and staff of the Society for the Promotion of Engineering Education, in charge of the work. For the past two or three years, however, the sessions have been held as the result of invitations received from groups or organizations of teachers. The American Mathematical Society and the Mathematical Association of America have thus expressed a desire to have one of the sessions devoted to the teaching of mathematics, and the Society for the Promotion of Engineering Education, through action of its Council, has approved the holding of such a session if the necessary arrangements can be made. The University of Minnesota has extended the invitation to hold this meeting at Minneapolis just prior to the meetings of the two mathematical societies in September, 1930. It is probable that the session will open on August 24 and close on September 5. The program will be devoted to educational principles, to the content and method of teaching mathematics, correlation with other subjects of the engineering curriculum, to advanced mathematics, and a number of other sub-divisions.

Applications to attend this session may be addressed to H. P. Hammond, Director of Summer School, Society for the Promotion of Engineering Education, 99 Livingston St., Brooklyn, N. Y.

Meeting of the Board of Trustees

Eight trustees were present at the meeting on Monday evening, Professors Hedrick, Huntington and Jackson being also present by invitation.

The trustees appointed Professor Lao G. Simons of Hunter College as a trustee to fill the vacancy occasioned by the death of Professor Cajori; she will serve until January 1932. A suitable resolution recognizing the Association's loss in Professor Cajori's death will be offered at the time of the annual meeting in December.

The following thirty-eight persons were elected to membership on applications duly certified:

To Individual Membership

C. E. Amos, B.Sc. (Denison). Grad. Asst., Ohio State Univ., Columbus, Ohio.

Brother Anselm, A.B. (Mt. St. Joseph's Coll.). Teacher, St. Xavier High School, Louisville, Ky.

W. E. Bobertz, M.S.E. (Michigan). Research Dept., Westinghouse Elec. and Mfg. Co., Wilkinsburg, Pa.

Anthony Braun, A.M. (Catholic Univ.). Monastery, Washington, D.C.

J. O. Chellevold, A.M. (Northwestern). Asst. Prof., St. Paul-Luther Coll., St. Paul, Minn.

H. V. Craig, Ph.D. (Wisconsin). Adj. Prof., Univ. of Texas, Austin, Tex.

H. B. Curry, Ph.D. (G ttingen). Asst. Prof., Pennsylvania State Coll., State College, Pa.

J. D. Dawson, M.S. (Ohio State). Prof., Antioch Coll., Yellow Springs, Ohio.

J. M. Feld, A.B., Chem. E. (Columbia). Instr., Columbia Coll., New York, N. Y.

Orrin Frink, Jr., Ph.D. (Columbia). Asso. Prof., Pennsylvania State Coll., State College Pa.

H. A. Garabedian, B.S. (Tufts). (F.A.S. 1930). Asst. Mathematician, John Hancock Mut. Life Ins. Co., Boston, Mass.

Grace T. Gunn, A.M. (Northwestern). Univ. of Omaha, Omaha, Nebr.

J. D. Hill, A.B. (California at L.A.). Asst., Univ. of California at Los Angeles, Los Angeles, Calif.

J. A. Hurry, A.M. (California). Prof. and Head of Dept., Western State Coll., Gunnison, Colo.

C. E. Kellam, A.M. (Chicago). Head of Dept., East Chicago High Schools, East Chicago, Ind.

V. V. Latshaw, Ph.D. (Indiana). Head of Dept., State Teachers Coll., Hays, Kans.

Ralph Lefever, A.M. (Nebraska). Strang, Nebr.

M. M. Lemme, A.M. (Indiana). Instr., Univ. of City of Toledo, Toledo, Ohio.

R. H. MacCullough, M.S. (Lafayette). Prof., Defiance Coll., Defiance, Ohio.

I. J. MacDonald, M.S. (St. Bonaventure's Coll.). Oratory Prep. School, Summit, N. J.

H. B. MacDougal, M.S. (Iowa). South Dakota State Coll., Brookings, S. Dak.

Edith A. McDougle, A.B. (Delaware). Instr., Women's Coll., Univ. of Delaware, Newark, Del.

James McGiffert, Ph.D. (Columbia). Prof., Grad. Dept., Rensselaer Polytech. Inst., Troy, N. Y.

H. E. Menke, A.M. (Ohio State). Asst. Prof., Otterbein Coll., Westerville, Ohio.

T. C. Morehouse, Mgr. College Dept., The Macmillan Co., New York, N. Y.

J. D. Novak, B.S. (Gettysburg). Univ. of Chicago, Chicago, Ill.

Rufus Oldenburger, M.S. (Chicago). Instr., Univ. of Michigan, Ann Arbor, Mich.

Oystein Ore, Ph.D. Prof., Yale Univ., New Haven, Conn.

Brother Paulinus, A.B. (Mt. St. Joseph's Coll.). Instr., St. Xavier High School, Louisville, Ky.

Mary S. Paxton, A.M. (Indiana). Teacher, S. S. High School, Fort Wayne, Ind.

James Pierpont, Ph.D. (Vienna). Prof., Yale Univ., New Haven, Conn.

Winnifred F. Pine, A.B. (Brown). Teacher, St. Margaret's School, Waterbury, Conn.

Ruth B. Rasmusen, M.S. (Chicago). Instr., South Dakota State Coll., Brookings, S. Dak.

C. G. Reen, M.S.E. (Michigan). Asst. Prof., Civil Eng., Gettysburg Coll., Gettysburg, Pa.
H. F. Ries, A.B. (Michigan). Asst. Actuary, Guaranty Life Ins. Co., Davenport, Iowa.
Flora M. Streetman, A.M. (Rice Inst.). Houston, Tex.
H. A. Toops, Ph.D. (Columbia). Prof., Psychology, Ohio State Univ., Columbus, Ohio.
Marquess Wallace, A.B. (Westminster Coll., Fulton, Mo.). Prin., and head of Dept., Missouri Milit. Acad., Mexico, Mo.

Aside from routine business of the trustees, a report was presented by Professor Dunham Jackson, chairman, of the Committee on Geometry, a joint committee of the Mathematical Association and the National Council of Teachers of Mathematics. This is to be sent to the trustees in mimeographed form with a view to action at the December meeting.

W. D. Cairns, *Secretary*

FLORIAN CAJORI

By LAO G. SIMONS, Hunter College of the City of New York

The death of Professor Florian Cajori occurred on August 14, 1930. He had undergone an operation toward the end of last year, but had satisfactorily recovered. The suddenness of the event increased the regret that followed upon its announcement.

Professor Cajori was born at St. Aignan near Thusis, Canton of Grisons, Switzerland, on February 28, 1859. He came to the United States in 1875 and in the course of time became a naturalized American citizen. His early education was obtained in Zillis, Thusis, then at the Kantonsschule in Chur (Grisons) and later at the State Normal School in Whitewater, Wisconsin. He received the degree of B.S. in 1883 and M.S. in 1886, both from the University of Wisconsin, (spending one year at this time as a student at Johns Hopkins University) and the degree of Ph.D. from Tulane University in 1894. The honorary degree of LL.D. was conferred upon him by Colorado College in 1913, that of Sc.D. by the University of Wisconsin in 1913, and the LL.D. again by the University of California in 1930.

Professor Cajori was associated professionally with Tulane University and Colorado College from 1885 to 1918 except for one year spent with the United States Bureau of Education. He was appointed professor of the history of mathematics at the University of California on July 1, 1918 and served in that capacity until July 1, 1929, when he retired with the title of professor of the history of mathematics, emeritus. He was a member of many scientific societies, among them: the Association of University Professors, the American Association for the Advancement of Science, the History of Science Society, the American Mathematical Society, the Mathematical Association of America, the National Council of Teachers of Mathematics, the Deutsche Mathematiker Vereinigung, and the Circolo Matematico di Palermo.

As already stated in the October issue of this Monthly, "he was a charter

member of the Mathematical Association of America and was one of an original group of four (later enlarged to twelve) representatives of mid-western universities and colleges who made possible the re-establishment of the American Mathematical Monthly on a sound financial basis." He served as the Association's president during 1917.

He was a member of the "National Conference of Ten" of the National Education Association "Committee of Ten", 1892, and a member of the "Geometry Syllabus Committee of Fifteen" 1910–1913, vice-president of the History of Science Society, 1924–1925, the first year of its existence. The last recognition that came to him from historical organizations was his election in 1929 as president of the Comité International d'Histoire des Sciences.

He was a familiar figure at meetings until recent years when he felt that the time consumed in this way must be spent on his own researches and writings. His last personal appearance on a program was at the History of Science Society meeting in Durham, North Carolina, last Christmas week when he read a paper on "A Century of American Geodesy."

Professor Cajori's special field was the history of mathematics. He was a pioneer in it in the United States and an indefatigable worker from as early as 1890, when his book on *The Teaching and History of Mathematics in the United States* was published by the U. S. Bureau of Education, until his death. A remarkable variety of articles and reviews in magazines flowed from his pen, the number from 1918 to 1930 exceeding one hundred-thirty. In addition, he published a number of books. His work entitled *History of Mathematics* appeared first in 1894 and *A History of Elementary Mathematics with Hints on Methods of Teaching* in 1896. (A later edition of the latter was translated into Japanese and published in Tokyo in 1929.) These were followed by *History of Physics*, 1899; Abschnitt XX in M. Cantor's *Vorlesungen über Geschichte der Mathematik*, 1908; *History of the Logarithmic Slide Rule*, 1909; *William Oughtred*, 1916; *History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse*, 1918; *The Early Mathematical Sciences in North and South America*, 1928; *Mathematics in Liberal Education*, 1928; *Checquered Career of Ferdinand Rudolph Hassler*, 1929; *History of Mathematical Notations*, in two volumes, 1928–1929.

The following books were about ready for publication at the time of Professor Cajori's death and will be printed in the near future: *Newton's "Principia"* and *Newton's "Optics."*

Particularly valuable contributions by Professor Cajori are found in *Sir Isaac Newton, 1727–1927*, published in 1928 and *A Source Book in Mathematics*, 1930. Interspersed during these years are several elementary and college textbooks. The words on the memorial over the tomb of Sir Christopher Wren in St. Paul's Cathedral, London, "Lector, si monumentum requiris, circumspice", may well be transferred from stone to living words in the case of this historian of mathematics.

The ability of Professor Cajori to do continuous, concentrated work was

phenomenal. The painstaking effort of years went into his monumental work *A History of Mathematical Notations*. Such a result as this is the outcome of endless search, powers of observation and an almost miraculous recognition of possible sources of information. He was like a magnet in his chosen field working to draw items of interest to himself, and when he came to combine these items into the written work, his acute observations and deductions made his writings no mere recital of facts, valuable as this might have been. Instead there are the suggestions as to improvement along many lines which a study of history would provide.

Professor Cajori had a charm of personality that made him an endearing character to those with whom he came in contact. He was a fine type of gentleman, interesting to listen to and interesting to talk to. His influence was far-reaching extending to his students and colleagues, to his associates around the world in many lines of research, and to a wide circle of friends. That influence will continue to be felt through the presence of the man in his historical writings.

A GENERALIZED FOURIER SERIES REPRESENTATION OF A FUNCTION

By W. O. PENNELL, St. Louis, Mo.

Introduction

This paper describes a general method of obtaining, by operational analysis, a quasi Fourier series $S(x)$ representing a function $f(x)$ as follows:

$$S(x) = b^n f(x - na), \quad na < x < (n + 1)a, \tag{1}$$

where n takes on positive and negative integral values including zero, b is any real constant, and $f(x)$ is a function defined in the interval $0<x<a$. If unity is substituted for b the series reduces to the classical Fourier series for $f(x)$. $0<x<a$.

It will be noted that if $S_1(x)$ denote the Fourier series for $f(x)b^{-x/a}$ for the interval $0<x<a$ then

$$S_1(x) = f(x - na)b^{(-x-na)/a}, \quad na < x < (n + 1)a \tag{2}$$

where n as in (1) takes on positive and negative integral values including 0. By multiplying both sides of (2) by $b^{x/a}$ we get

$$S_1(x)b^{x/a} = b^n f(x - na)$$

which is the same as (1). It is obvious then that the series (1) may be obtained by multiplying by $b^{x/a}$ the classical Fourier series for $f(x)b^{-x/a}$, $0<x<a$.

Heaviside, in his *Electromagnetic Theory*, vol. 2, discussed operational methods of obtaining Fourier series for certain functions, but as far as the author is aware no general method of obtaining Fourier series for a function by operational means has been published.

Deduction of Formula

There is an operator ϵ^{-ap} known as the translation[1] operator which has the property of shifting the operand to the right or to the left a distance a, according to whether a is positive or negative. That is $\epsilon^{-ap}f(x)=f(x-a)$. This is true whether $f(x)$ (subject to certain restriction) is continuous or discontinuous. The proof of this action of the operator may be obtained for positive values of x rather easily from an application of Carson's integral equation[2] and for negative values of x from an application of a modified form of Carson's equation.

Now if we write

$$\epsilon^{ap}(\epsilon^{ap}+b)^{-1}f(x)P(x),$$

where $f(x)$ is the function for which we wish to obtain the generalized or quasi Fourier series and where $P(x)$ is a "pulse" defined as follows:

$$\begin{aligned} P(x) &= 1, \quad 0 < x < a, \\ &= \tfrac{1}{2}, \quad x = 0 \text{ and } \quad x = a, \\ &= 0, \quad x > a, \\ &= 0, \quad x < 0, \end{aligned}$$

we get by expanding the operator by long division:

$$\begin{aligned} (3)\quad \epsilon^{ap}(\epsilon^{ap}+b)^{-1}f(x)P(x) &= [1 - b\epsilon^{-ap} + b^2\epsilon^{-2ap} - \cdots]f(x)P(x) \\ &= f(x)P(x) - bf(x-a)P(x-a) + b^2f(x-2a)P(x-2a) - \cdots \end{aligned}$$

which is evidently the right half of the series we wish or that portion of (1) for which $n \geqq 0$.

(3) can be expressed in operational form as follows:

$$f(x)P(x) = f(x)\ [1(x) - 1(x-a)], \quad \text{where } 1(x) \text{ is the unit}^3 \text{ function,}$$

$$\text{that is where } 1(x) = 1, \quad x > 0, \qquad = 0, \quad x < 0, \qquad = \tfrac{1}{2}, \quad x = 0,$$

$$\begin{aligned} &= f(x)1(x) - \epsilon^{-ap}f(x+a)1(x), \\ &= [F(p) - \epsilon^{-ap}\psi(p)]1(x) \end{aligned}$$

where $F(p)$ and $\psi(p)$ are the operational equivalents[4] of $f(x)$ and $f(x+a)$ respectively.

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[1] This name was apparently first given to this operator by Oliver Heaviside in his *Electromagnetic Theory*, Vol. 2, page 111.

[2] See *Electric Circuit Theory and the Operational Calculus*, by J. C. Carson. (Published by the McGraw-Hill Book Co.)

[3] Many writers have defined the unit function as having all values between 0 and 1 for $x=0$. The definition given satisfies Dirichlet's condition and conforms with the results obtained at the points of discontinuity by a Fourier series.

[4] See *Heaviside's Operational Calculus*, by Berg (McGraw-Hill Co., 1929), *Electric Circuit*

So

$$\epsilon^{ap}(\epsilon^{ap} + b)^{-1}f(x)P(x) = [\epsilon^{ap}F(p) - \psi(p)](\epsilon^{ap} + b)^{-1}1(x). \tag{4}$$

The left hand half of the series or that part of (1) for which n is negative is evidently given by

$$\begin{aligned} -\epsilon^{ap}(b + \epsilon^{ap})^{-1}f(x)P(x) = &- b^{-1}f(x + a)P(x + a) \\ &+ b^{-2}f(x + 2a)P(x + 2a) - b^{-3}f(x + 3a)P(x + 3a) + \cdots \end{aligned} \tag{5}$$

(5) may be expressed in operational form as follows:

$$\begin{aligned} f(x)P(x) &= f(x)[1(-x + a) - 1(-x)] \\ &= \epsilon^{-ap}f(x + a)1(-x) - f(x)1(-x) \\ &= [\epsilon^{-ap}\psi(p) - F(p)]1(-x). \end{aligned}$$

So

$$-\epsilon^{ap}(b + \epsilon^{ap})^{-1}f(x)P(x) = [\epsilon^{ap}F(p) - \psi(p)](b + \epsilon^{ap})^{-1}1(-x). \tag{6}$$

(4) is 0 if $x<0$ and represents the series of $x>0$.
(6) is 0 if $x>0$ and represents the series of $x<0$.
(4)+(6) represents the series for all values of x.
But

$$\begin{aligned} (4) + (6) &= [\epsilon^{ap}F(p) - \psi(p)](b + \epsilon^{ap})^{-1}[1(x) + 1(-x)] \\ &= [\epsilon^{ap}F(p) - \psi(p)](b + \epsilon^{ap})^{-1}, \end{aligned} \tag{7}$$

since $1(x)+1(-x)=1$.

If expression (7) is evaluated by the Heaviside expansion theorem or some other recognized operational method the generalized Fourier series is obtained.

It has not been rigorously demonstrated that the expansion theorem applies to an expression like $(\epsilon^{ap}+b)^{-1}$ or $(\epsilon^{ap}-b)^{-1}$. Regarding p as an algebraic quantity and by starting from the known[5] equality,

$$\frac{\epsilon^{-ap} - \epsilon^{ap}}{\epsilon^{-ap} + \epsilon^{ap}} = \sum_{n=1,2\ldots}^{\infty} \frac{-8ap}{(2n-1)^2\pi^2 + 4a^2p^2}$$

by rather simple algebraic manipulations we get

$$\frac{1}{\epsilon^{ap} + b} = \frac{1}{2b} - \frac{2}{b}\sum_{n=1,3,5\ldots}^{\infty} \frac{ap - \log_\epsilon b}{n^2\pi^2 + (ap - \log_\epsilon b)^2}, \quad b > 0. \tag{8}$$

Theory & Operational Calculus, by J. R. Carson, (McGraw-Hill Co., 1926, *Operational Circuit Analysis*, by Bush (John Wiley and Sons, 1929), *The Practical Application of the Fourier Integral*, by Campbell (Bell System Technical Journal, October, 1928), *Generalization of Heaviside's Expansion Theorem*, by Pennell (Bell System Technical Journal, August, 1929), *Heaviside's Electric Circuit Theory*, by Cohen (McGraw-Hill Co., 1928), *Electromagnetic Theory*, by Oliver Heaviside, Vols. 2 & 3.

[5] Chrystal's *Algebra*, Vol. 2, page 338.

Similarly by starting with the known expansion of

$$(\epsilon^{ap}+\epsilon^{-ap})(\epsilon^{ap}-\epsilon^{-ap})^{-1},$$

it can be shown that

$$(9)\quad \frac{1}{\epsilon^{ap}-b}=-\frac{1}{2b}+\frac{1}{b(ap-\log_\epsilon b)}+\frac{2}{b}\sum_{n=1,2,3\cdots}^{\infty}\frac{ap-\log_\epsilon b}{4n^2\pi^2+(ap-\log_\epsilon b)^2},\quad b>0.$$

Applying the Heaviside expansion theorem to the partial fractions on the right of (8) and (9) we get

$$(10)\quad \frac{1}{\epsilon^{ap}+b}=\frac{1}{1+b}-\frac{1}{b}\sum_{n=1,3,5\cdots}^{\infty}\frac{2n\pi b^{x/a}\sin n\pi x/a+2(\log_\epsilon b)b^{x/a}\cos n\pi x/a}{\log_\epsilon^2 b+n^2\pi^2},\quad b>0$$

and

$$(11)\quad \frac{1}{\epsilon^{ap}-b}=\frac{1}{1-b}+\frac{b^{(x/a)-1}}{\log_\epsilon b}$$
$$+\frac{2}{b}\sum_{n=1,2,3\cdots}^{\infty}\frac{b^{x/a}\log_\epsilon b\cos 2n\pi x/a+2n\pi b^{x/a}\sin 2n\pi x/a}{\log_\epsilon^2 b+4n^2\pi^2}.$$

If we apply the expansion theorem directly to $(\epsilon^{ap}+b)^{-1}$, $b>0$ we get the same result as by (10).

If, however, we apply the expansion theorem directly to $(\epsilon^{ap}-b)^{-1}$, $b>0$, we get the same result as given by (11) except that the term $b^{(x/a)-1}/\log_\epsilon b$ is missing.

The expansion theorem cannot then be applied directly to an expression $\epsilon^{ap}(\epsilon^{ap}-b)^{-1}$, $b>0$ unless the extra term is added to the result.

We will now obtain the generalized Fourier series for several functions.

Example 1: Find a series to represent $f(x)$ such that

$$\begin{aligned} f(x) &= 1, && 0<x<a,\\ &= 1-b, && a<x<2a,\\ &= 1-b+b^2, && 2a<x<3a, \quad \text{etc.}\end{aligned}$$

Here the pulse is an infinite one, namely $1(x)$. The operational expression is

$$\begin{aligned}\epsilon^{ap}(\epsilon^{ap}+b)^{-1}1(x) &= 1(x)-b\epsilon^{-ap}1(x)+b^2\epsilon^{-2ap}1(x)\cdots\\ &= 1(x)-b1(x-a)+b^21(x-2a)\cdots.\end{aligned}$$

To evaluate $\epsilon^{ap}1(x)(\epsilon^{ap}+b)^{-1}$ remember that the expansion theorem is given by

$$(12)\quad \frac{Y(p)}{Z(p)}=\frac{Y(0)}{Z(0)}+\sum_{p_n}\frac{Y(p_n)}{p_nZ'(p_n)}\epsilon^{p_nx},$$

where p_n = the roots of $Z(p)=0$. In this case $Y(p)=\epsilon^{ap}$, $pZ'p=ap\epsilon^{ap}$, and

$$\begin{aligned} p_n &= \text{the roots of } (\epsilon^{ap}=-b)\\ &= n\pi i/a+(\log_\epsilon b)/a,\\ n &= \pm 1,\ \pm 3,\ \cdots.\end{aligned}$$

So

$$f(x)=\frac{\epsilon^{ap}}{\epsilon^{ap}+b}=\frac{1}{1+b}+\sum\frac{\epsilon^{ap_n}}{ap_n\epsilon^{ap_n}}\epsilon^{p_nx}$$

$$=\frac{1}{1+b}+\sum_{\pm 1,\pm 3\ldots}^{\infty}\frac{\epsilon^{n\pi ix/a+x/a\log_\epsilon b}}{n\pi i+\log_\epsilon b}.$$

(13) $$f(x)=\frac{1}{1+b}+2b^{x/a}\log_\epsilon b\sum_{n=1,3,5\ldots}^{\infty}\frac{\cos n\pi x/a}{n^2\pi^2+\log_\epsilon^2 b}$$

$$+2\pi b^{x/a}\sum_{n=1,3,5\ldots}^{\infty}\frac{n\sin n\pi x/a}{n^2\pi^2+\log_\epsilon^2 b}.$$

In case $b=1$ (13) reduces to

(14) $$f(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{1,3,5,\ldots}\frac{\sin n\pi x/a}{n},$$

which is the classical Fourier series for

$$\begin{aligned} f(x) &= 1, && 0<x<a,\\ &= 0, && a<x<2a,\\ &= 1, && 2a<x<3a, \quad \text{etc.}\end{aligned}$$

The values of (13) for $b=\frac{1}{2}$, $a=1$, and for the first three approximation curves have been calculated and plotted on Fig. I. The Gibbs effect will be seen at $x=0, 1, 2, 3$, etc.

The first wave between $x=0$ and $x=1$ is obviously the same as for the corresponding wave in the classical series where $b=1$. On Fig. I-A has been plotted and superimposed upon the generalized curve, the first wave of the Fourier curve, and it will be seen that the two waves are practically identical.

Example 2: Find the series for $f(x)$ when

$$\begin{aligned} f(x) &= \epsilon^x, && 0<x<c,\\ &= -b\epsilon^{x-c}, && c<x<2c,\\ &= +b^2\epsilon^{x-2c}, && 2c<x<3c, \quad \text{etc.}\end{aligned}$$

The pulse is

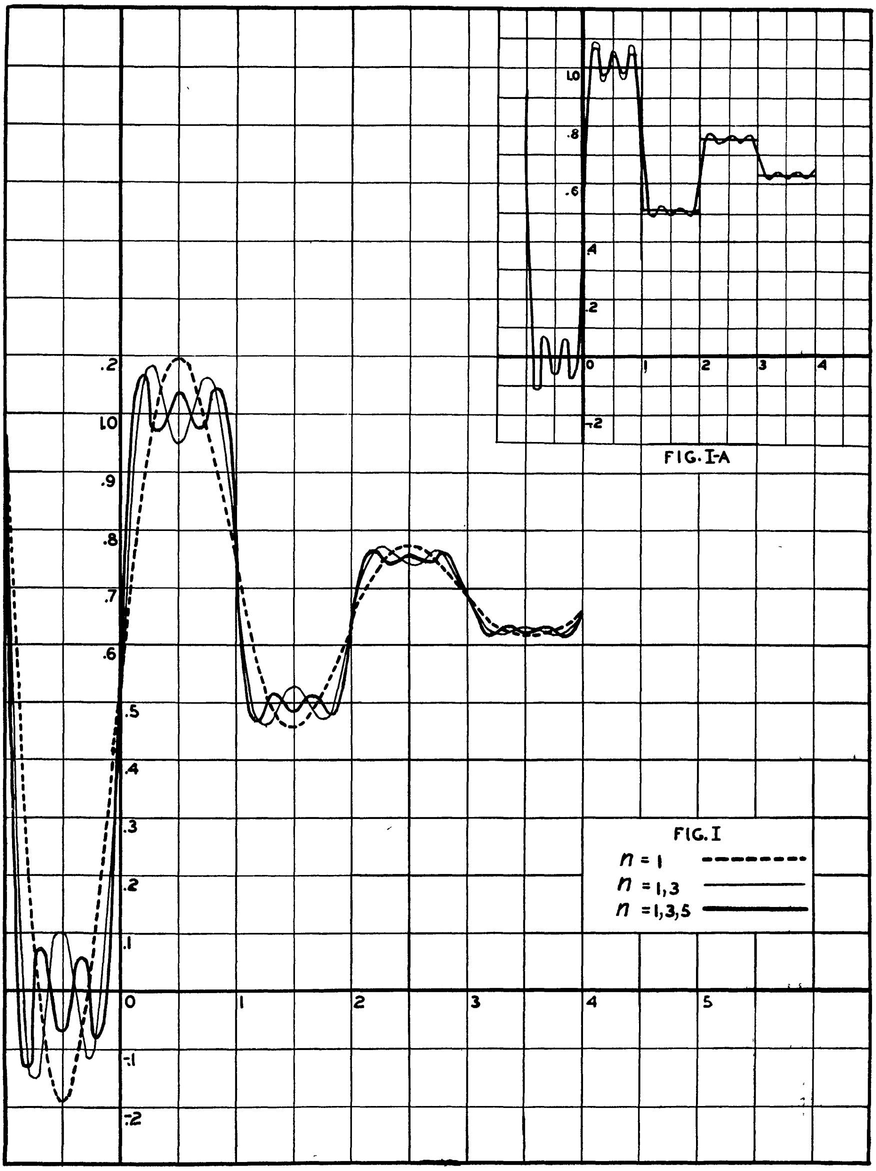
$$\begin{aligned} P(x) &= \epsilon^x[1(x)-1(x-c)]\\ &= p(p-1)^{-1}1(x)-\epsilon^{-cp}\epsilon^{x+c}1(x)\\ &= [p(p-1)^{-1}-p(p-1)^{-1}\epsilon^{-cp}\epsilon^c]1(x), \text{ since } \epsilon^x=p(p-1)^{-1}.\end{aligned}$$

So the operational expression for the series is

(15) $$\frac{\epsilon^{cp}}{\epsilon^{cp}+b}\left[\frac{p}{p-1}-\frac{p\epsilon^{-cp}\epsilon^c}{p-1}\right]1(x)=\left[\frac{p}{(p-1)}\frac{\epsilon^{cp}}{\epsilon^{cp}+b}-\frac{p\epsilon^c}{(p-1)(\epsilon^{cp}+b)}\right]1(x).$$

Evaluating (15) by the expansion theorem we get

$$(16)\qquad f(x) = 2(b + \epsilon^c)(\log_\epsilon b - c)b^{(x/c)-1} \sum_{n=1,3,5\ldots}^{\infty} \frac{\cos (n\pi x/c)}{n^2\pi^2 + (\log_\epsilon b - c)^2}$$
$$+ 2(b + \epsilon^c)\pi b^{(x/c)-1} \sum_{n=1,3,5\ldots}^{\infty} \frac{n \sin (n\pi x/c)}{n^2\pi^2 + (\log_\epsilon b - c)^2}.$$



If $b=1$, (16) reduces to the classical Fourier series

$$(17)\; f(x) = 2\pi(1 + \epsilon^c) \sum_{n=1,3,5\ldots}^{\infty} \frac{n \sin (n\pi x/c)}{n^2\pi^2 + c^2} - 2c(1 + \epsilon^c) \sum_{n=1,3,5\ldots}^{\infty} \frac{\cos(n\pi x/c)}{n^2\pi^2 + c^2},$$

where

$$
\begin{aligned}
f(x) &= \epsilon^{x}, & 0 < x < c, \\
&= -\epsilon^{x-c}, & c < x < 2c, \\
&= \epsilon^{x-2c}, & 2c < x < 3c, \text{ etc.}
\end{aligned}
$$

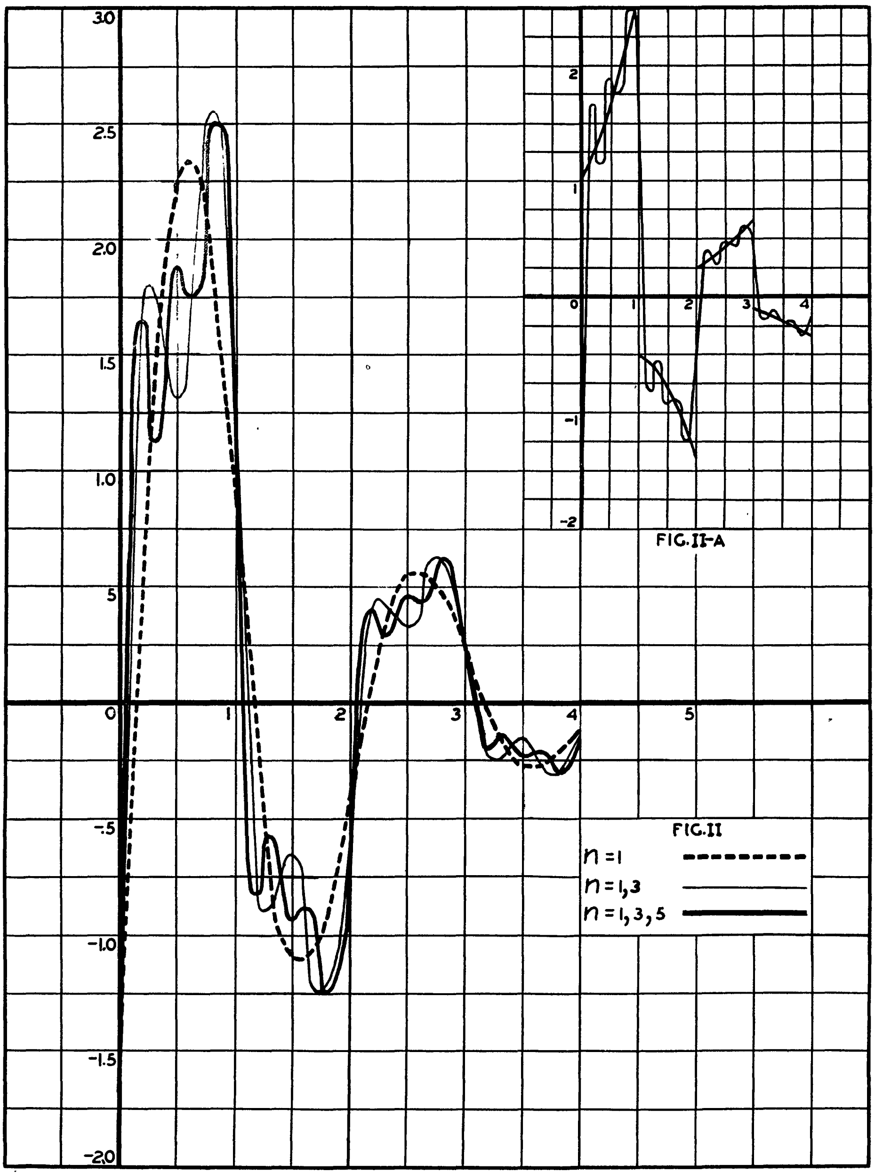


Fig. II shows the first 3 approximation curves. Here also the Gibbs effect is seen. Fig. II-A shows the third approximation curve superimposed upon the ultimate curve.

Example 3: Find the series for

$$\begin{aligned} f(x) &= \cos x, & -\frac{\pi}{2} < x < \frac{\pi}{2}, \\ &= b\cos x, & \frac{\pi}{2} < x < \frac{3\pi}{2}, \\ &= b^2\cos x, & \frac{3\pi}{2} < x < \frac{5\pi}{2}, \quad \text{etc.} \end{aligned}$$

The pulse is

$$\begin{aligned} P(x) &= (\cos x)1(x+\tfrac{1}{2}\pi) - 1(x-\tfrac{1}{2}\pi) \\ &= [\epsilon^{(\pi/2)p}\cos(x-\tfrac{1}{2}\pi) - \epsilon^{(-\pi/2)p}\cos(x+\tfrac{1}{2}\pi)]1(x) \\ &= \epsilon^{(\pi/2)p}(\sin x)1(x) + \epsilon^{(-\pi/2)p}(\sin x)1(x) \\ &= \epsilon^{(\pi/2)p}p(p^2+1)^{-1}1(x) + \epsilon^{(-\pi/2)p}p(p^2+1)^{-1}1(x) \end{aligned}$$

$$\text{since } \sin x = p(p^2+1)^{-1}.$$

The operational expression for the wave is

$$\frac{\epsilon^{\pi p}}{\epsilon^{\pi p}+b}P(x) = \frac{p}{(p^2+1)}\,\frac{\epsilon^{(3\pi/2)p}}{(\epsilon^{\pi p}+b)}1(x) + \frac{p}{(p^2+1)}\,\frac{\epsilon^{(\pi/2)p}}{(\epsilon^{\pi p}+b)}1(x). \tag{18}$$

Solving (18) by the expansion theorem we get

$$f(x) = b^{x/\pi}\frac{(b^{1/2}-b^{-1/2})}{\pi}\sum_{n=1,3,5\cdots}^{\infty}\frac{2(1-n^2+\pi^{-2}\log_e^2 b)\cos n(x+\frac{1}{2}\pi) + 4n\pi^{-1}\log_e b\sin n(x+\frac{1}{2}\pi)}{[(n+1)^2+\pi^{-2}\log_e^2 b][(n-1)^2+\pi^{-2}\log_e^2 b]}. \tag{19}$$

The three first approximation curves are shown on Fig. III for the value $b=\frac{1}{2}$. Each half of the wave is a perfect sine curve, with the same wave length, but the amplitude of the 2nd half wave is one half that of the first. This is an entirely different curve from one in which the damping factor is exponential as for instance $\epsilon^{-ax}\cos x$.

When $b=1$, (19) becomes equal to $\cos x$ which is the same result as obtained by Fourier series.

Example 4: Find the series for

$$\begin{aligned} f(x) &= x, & 0 < x < c, \quad & f(x) = b^{-1}(x+c), & -c < x < 0, \\ &= b(x-c), & c < x < 2c, \quad & = b^{-2}(x+2c), & -c < x < -2c, \\ &= b^2(x-2c), & 2c < x < 3c, \quad & = b^{-3}(x+3c), & -2c < x < -3c, \\ &\text{etc.} & & \text{etc.} \end{aligned}$$

The pulse is

$$x[1(x)-1(x-c)] = [p^{-1}-p^{-1}\epsilon^{-cp}-c\epsilon^{-cp}]1(x).$$

The operational equivalent is

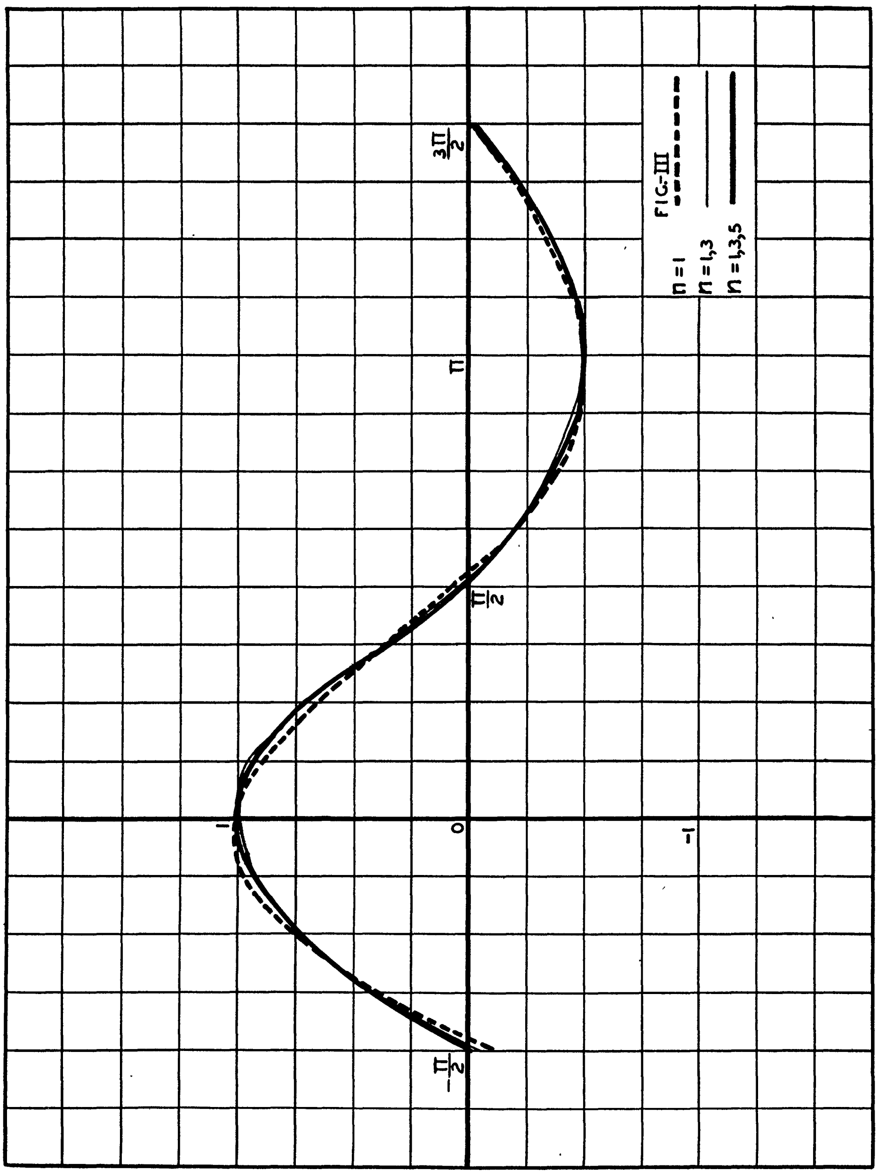
$$\epsilon^{cp}/(\epsilon^{cp}-b)^{-1}[p^{-1}-p^{-1}\epsilon^{-cp}-c\epsilon^{-cp}]1(x).$$

Solving this we get

$$(20)\ f(x) = \frac{cb^{x/c}[b - 1 - \log_\epsilon b]}{b \log_\epsilon^2 b}$$

$$+ \frac{2(b-1)}{b} \sum_{n=1,2,3\cdots} \frac{cb^{x/c}(\log_\epsilon^2 b)\cos\left(\frac{2n\pi x}{c}\right) + 4cn\pi b^{x/c}(\log_\epsilon b)\sin\left(\frac{2n\pi x}{c}\right) - 4cn^2\pi^2 b^{x/c}\cos\left(\frac{2n\pi x}{c}\right)}{[\log_\epsilon^2 b + 4n^2\pi^2]^2}$$

$$+ \frac{2}{b} \sum_{n=1,2,3\cdots} \frac{-cb^{x/c}\log_\epsilon b \cos\left(\frac{2n\pi x}{c}\right) - 2cn\pi b^{x/c}\sin\left(\frac{2n\pi x}{c}\right)}{[\log_\epsilon^2 b + 4n^2\pi^2]}.$$

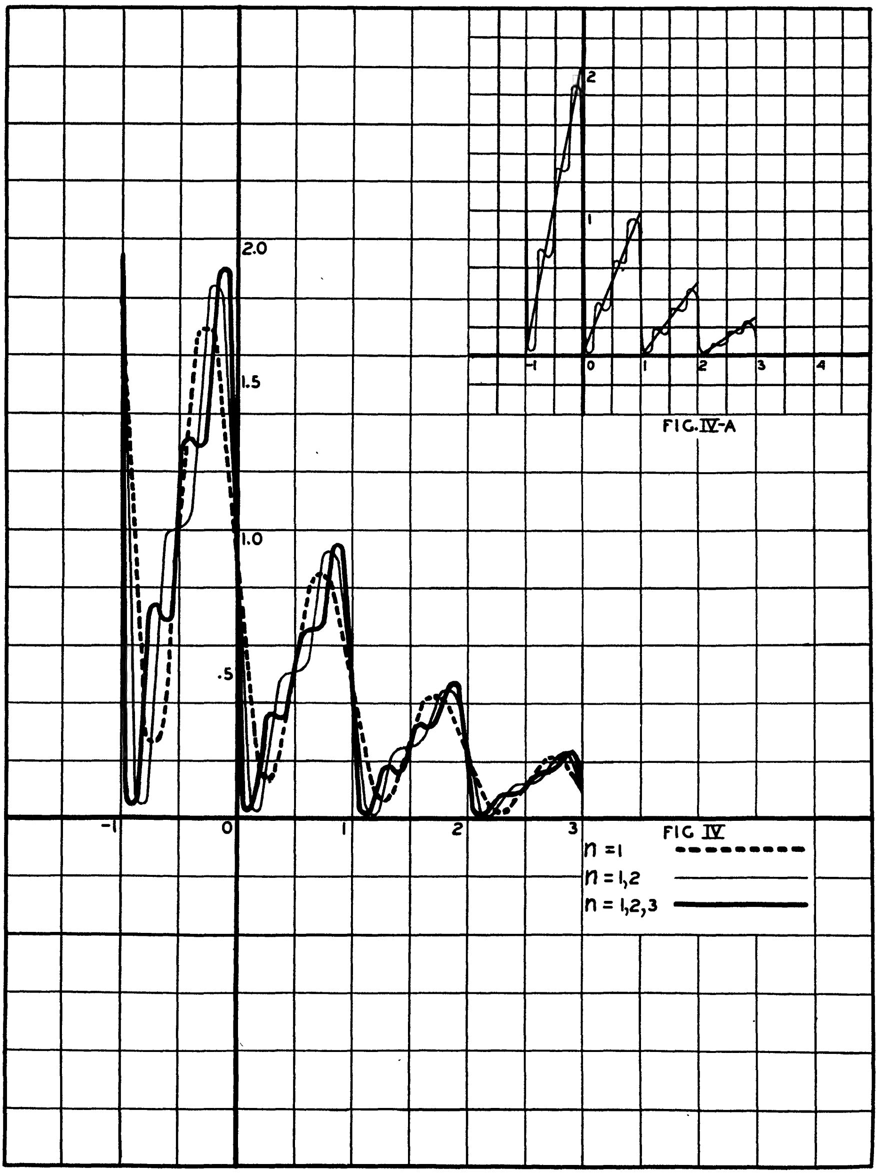


In (20) if $b=1$ we get

$$f(x) = \tfrac{1}{2}c - c\pi^{-1} \sum_{n=1,2,3\cdots} n^{-1} \sin (2n\pi x/c),$$

which is the Fourier expansion for

$$\begin{aligned} f(x) &= x, & 0 < x < c \\ &= x - c, & c < x < 2c, \\ &= x - 2c, & 2c < x < 3c. \end{aligned}$$



The curve represented by (20) when $c=1$ and $b=\frac{1}{2}$ and the first three approximation curves are shown on Figs. IV-A and IV.

Final Remarks

The operational process explained in this paper can be used, by starting with $b=1$, to derive directly, the classical Fourier series for a function. In general there is no saving in labor by deducing a Fourier series in this manner. For one accustomed to the operational methods it may be easier than the classical way, but the reverse is undoubtedly true for one used to the classical way.

The operational method does, naturally, open up the generalized series, which, while they can be obtained from the classical theory, as has been explained, are not generally known, at least the author has seen no reference to them.

In all the examples given the amount of shift in the translation operator is the same as the wave length of the pulse. This equality is not essential and by taking the amount of shift more or less than the wave length more complicated waves are produced and their sine and cosine series may be obtained.

It will be noted that the expansion is of the form $S(x)=A+Bb^{x/a}+Cb^{x/a}\sum^{\infty}[D_n \sin (n\pi x/a)+E_n \cos (n\pi x/a)]$, $n=1, 2, \cdots$, although in some cases some of the costants are zero.

The author wishes to acknowledge his indebtedness to Miss Elizabeth Harris who calculated and plotted the approximation curves, an undertaking of very considerable labor.

ON A CLASS OF FINITE SUMS[1]

By LEONARD CARLITZ, University of Pennsylvania

1. *Introduction.* The present paper grew out of an attempt to evaluate the sum of the n-th powers of the natural numbers from one to m. The expression most easily arrived at is in terms of the so-called Stirling[2] numbers and coincides with that as given, for example, by Schwatt.[3] Schwatt shows how useful the operator $(xd/dx)^n$ may be in this and like summations. Here we employ the generalized operator $(x^\lambda d^\lambda/dx^\lambda)^n$; this enables us to sum

$$\sum_{m=1}^{m}[m(m-1)\cdots(m-\lambda+1)]^n.$$

The sum involves a set of numbers that are a direct generalization of the Stirling numbers; it consists of $\lambda(n-1)+1$ terms (see equation (5)).

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[1] Presented to the American Mathematical Society, April 18, 1930.

[2] Stirling, *Methodus Differentialis*, p. 8.

[3] I. J. Schwatt, *Operations with Series* (1924), p. 86.

When λ is even it is possible to get a result involving but $\frac{1}{2}\lambda(n-1)+1$ terms. To effect this simplification use is made of a simple algebraic indentity (see equation (6)).

In the remainder of the paper, an attempt is made to consider abstractly arrays of numbers analogous to the arrays mentioned above. Explicit expressions for the elements of a large class of arrays are obtained; these arrays include the generalized Stirling numbers.

2. *The Operator* $(x^\lambda D^\lambda)^n$. Writing D for d/dx, we put

$$(x^\lambda D^\lambda)^n = \sum_{s=1}^{(n-1)\lambda+1} \overset{(\lambda)}{a}_{n,s} x^{s+\lambda-1} D^{s+\lambda-1}; \tag{1}$$

and we seek a recursion formula for $a_{n,s}^{(\lambda)}$. Applying $(x^\lambda D^\lambda)$ to both sides of this equality,

$$(x^\lambda D^\lambda)^{n+1} = \sum_s \sum_{i=0}^{\lambda} \overset{(\lambda)}{a}_{n,s} \binom{\lambda}{i} (s+\lambda-1)!^i x^{s+2\lambda-1-i} D^{s+2\lambda-1-i},$$

where

$$m!^k = m(m-1)\cdots(m-k+1).$$

From this

$$\overset{(\lambda)}{a}_{n+1,s} = \sum_{i=0}^{\lambda} \binom{\lambda}{i} (s+i-1)!^i \overset{(\lambda)}{a}_{n,s-\lambda+i}. \tag{2}$$

(When $i=0$, $(s+i-1)!^i=1$.)

Equation (2) plus the "initial" conditions,

$$\begin{cases} a_{1,1} = 1 \quad a_{1,s} = 0 \quad (s > 1), \\ a_{n,s} = 0 \quad \textit{for} \quad s < 1, \end{cases} \tag{3}$$

define a triangular array of numbers. When $\lambda=1$ this is the array of Stirling numbers. When $\lambda=2$ we may easily calculate the first few elements, (n denotes the row, S the column.):

	1	2	3	4	5	6	7	8	9
1	1								
2	2	4	1						
3	4	32	38	12	1				
4	8	208	652	576	188	24	1		
5	16	1280	9080	16944	12052	3840	580	40	1

Let now $(x^\lambda D^\lambda)^n$ be applied to x^m. Then by (1),

$$(m!^\lambda)^n = \sum_{s=1}^{(n-1)\lambda+1} a_{n,s}^{(\lambda)} m!^{s+\lambda-1}. \tag{4}$$

But

$$\sum_{m=1}^{m} m!^k = \frac{(m+1)!^{k+1}}{k+1};$$

therefore, from (4),

$$\sum_{m=1}^{m} (m!^\lambda)^n = \sum_{s=1}^{(n-1)\lambda+1} a_{n,s}^{(\lambda)} \frac{(m+1)!^{s+\lambda}}{s+\lambda}. \tag{5}$$

3. *Summation of* (5) *for* λ *even.* We require the following identity:

$$m!^{2\lambda} = \sum_{i=0}^{\lambda} \Lambda_i (m+k+i)!^i (m-k-2\lambda)!^i \tag{6}$$

where

$$\Lambda_i = \Lambda_i(k) = \binom{\lambda}{i} (k+2\lambda)!^{\lambda-i} (k+\lambda)!^{\lambda-i}.$$

Proof: (6) may be proved by an induction on k. It obviously holds for $k=-\lambda$ and $1-\lambda$. Assuming then that (6) holds for k,

$$\begin{aligned} m!^{2\lambda} &= \sum_{i=0}^{\lambda} \Lambda_i(k)(m+k+i)!^i(m-k-2\lambda)!^i \\ &= \sum \Lambda_i(k)\cdot(m+k+1+i)!^i(m-k-1-2\lambda)!^i \\ &+ \sum i\Lambda_i(k)\cdot(2k+2\lambda+1+i)(m+k+i)!^{i-1}(m-k-1-2\lambda)!^{i-1} \\ &= \sum [\Lambda_i(k) + (i+1)\Lambda_{i+1}(k)\cdot(2k+2\lambda+2+i)] \\ &\qquad \cdot(m+k+i+1)!^i(m-k-1-2\lambda)!^i. \end{aligned}$$

But

$$\Lambda_i(k) + (i+1)\Lambda_{i+1}(k)\cdot(2k+2\lambda+2+i) = \Lambda_i(k+1),$$

as may be verified without difficulty. From this

$$m!^{2\lambda} = \sum \Lambda_i(k+1)(m+k+1+i)!^i(m-k-1-2\lambda)!^i,$$

completing the induction.

Let us now put

$$(m!^{2\lambda})^n = \sum_{s=1}^{(n-1)\lambda+1} b_{n,s}^{(\lambda)} (m+s-1)!^{2(\lambda+s-1)}. \tag{7}$$

The possibility of this expansion will appear incidentally in getting a recursion formula for $b_{n,s}^{(\lambda)}$.

Assuming (7) and using (6), we have

$$(m!^{2\lambda})^{n+1} = \sum_{s=1}^{(n-1)\lambda+1} b_{n,s}^{(\lambda)}(m + s - 1)!^{2(s+\lambda-1)} \cdot m!^{2\lambda}$$

$$= \sum_{i=0} b_{n,s}^{(\lambda)}(m + s - 1)!^{2(s+\lambda-1)} \cdot \sum_{i=0}^{\lambda} \Lambda_i(s-1)(m+s+i-1)!^i(m-s-2\lambda+1)!^i$$

$$\left(\Lambda_i(s-1) = \binom{\lambda}{i}(s + 2\lambda - 1)!^{\lambda-i}(s + \lambda - 1)!^{\lambda-i}\right)$$

$$= \sum_s \sum_i \binom{\lambda}{i}(s+2\lambda-1)!^{\lambda-i}(s+\lambda-1)!^{\lambda-i} b_{n,s}^{(\lambda)}(m+s+i-1)!^{2(\lambda+s+i-1)}$$

Collecting the coefficient of $(m+s-1)!^{2(\lambda+s-1)}$

$$b_{n+1,s}^{(\lambda)} = \sum_{i=0}^{\lambda} \binom{\lambda}{i}(s + i - 1)!^i(s + \lambda + i - 1)!^i b_{n,s-\lambda+i}^{(\lambda)}. \tag{8}$$

This formula together with (3) (writing b in place of a) defines for each value of λ an array similar to those in §1. Using these numbers we may replace (5) by the following expression:

$$\sum_{m=1}^{m}(m!^{2\lambda})^n = \sum_{s=1}^{(n-1)\lambda+1} b_{n,s}^{(\lambda)} \frac{(m + s)!^{2\lambda+s-1}}{2\lambda + 2s - 1}. \tag{9}$$

It is perhaps unnecessary to remark that (7) might have been obtained by expanding the operator $(x^{2\lambda}D^{2\lambda})^n$ in terms of $(x^{2\lambda+s-1}D^{2(\lambda+s-1)}x^{s-1})$:

$$(x^{2\lambda}D^{2\lambda})^n = \sum_s b_{n,s}^{(\lambda)}(x^{2\lambda+s-1}D^{2(\lambda+s-1)}x^{s-1}). \tag{10}$$

However, the method used above seems simpler.

4. *Arrays in general. The operator E.*

Definition: We define the operator[4] E by

$$Ef(s) = f(s - 1),$$

$f(s)$ being any function of the integral variable s.

Making use of E, some of our formulas may be put in much simpler shape. For example, (2) becomes

$$\begin{aligned} a_{n+1,s}^{(\lambda)} &= \sum_{i=0}^{\lambda} \binom{\lambda}{i}(s + i - 1)!^i E^{\lambda-i} \cdot a_{n,s}^{(\lambda)} \\ &= (E + s)(E + s + 1) \cdots (E + s + \lambda - 1) \cdot a_{n,s}^{(\lambda)}. \end{aligned} \tag{11}$$

Similarly, (8) may be written:

[4] E and its inverse are familiar operators in the calculus of finite differences.

$$b_{n+1,s}^{(\lambda)} = \sum_{i=0}^{\lambda} \binom{\lambda}{i} (s+i-1)!^i (s+\lambda+i-1)!^i E^{\lambda-i} \cdot b_{n,s}^{(\lambda)}$$

$$= [E + s(s+\lambda)] \cdots [E + (s+\lambda-1)(s+2\lambda-1)] \cdot b_{n,s}^{(\lambda)} \tag{12}$$

$$= [E + s(s+2\lambda-1)] \cdots [E + (s+\lambda-1)(s+\lambda)] \cdot b_{n,s}^{(\lambda)}. \tag{13}$$

Note that the operators in (11) are permutable; those in (13) also are permutable, but those in (12) are not.

We shall now consider arrays of numbers $c_{n,s}^{(\lambda)}$ defined by (3) and

$$c_{n+1,s}^{(\lambda)} = \sum_{i=0}^{\lambda} \alpha_i^{(\lambda)}(s) E^{\lambda-i} \cdot c_{n,s}^{(\lambda)}. \tag{14}$$

The properties of these arrays will depend on the operator

$$\sum_i \alpha_i^{(\lambda)}(s) E^{\lambda-i}.$$

Let us take the simplest case $\lambda = 1$; we have then the operator

$$\beta(s)E + \alpha(s);$$

we shall take $\beta(s) \equiv 1$. We inquire when two such operators are permutable. Now

$$[E + \alpha(s)][E + \beta(s)] = E^2 + [\alpha(s) + \beta(s-1)]E + \alpha(s)\beta(s),$$
$$[E + \beta(s)][E + \alpha(s)] = E^2 + [\beta(s) + \alpha(s-1)]E + \beta(s)\alpha(s).$$

But from

$$\alpha(s) + \beta(s-1) = \beta(s) + \alpha(s-1)$$

follows

$$\alpha(s) - \beta(s) = \alpha(s-1) - \beta(s-1) = \mu,$$

a quantity independent of s.

Therefore, *a necessary and sufficient condition that $E+\alpha(s)$ and $E+\beta(s)$ be permutable is that $\alpha(s)-\beta(s)$ be independent of s* (cf. (11) and (13)). This result will be of use in §5.

We next take $\lambda = 2$, and consider

$$E^2 + A(s)E + B(s). \tag{15}$$

Let us attempt to exhibit it as a product

$$[E + \beta(s)][E + \alpha(s)].$$

Since $\alpha(s-1) + \beta(s) = A(s)$ and $\alpha(s)\beta(s) = B(s)$,

$$\alpha(s) = \frac{B(s)}{A(s) - \alpha(s-1)},$$

and we obtain the following expression of $\alpha(s)$ as a continued fraction:

$$\alpha(s) = \frac{B(s)}{A(s) -}\ \frac{B(s-1)}{A(s-1) -}\ \cdots\ \frac{B(1)}{A(1) - \nu}, \tag{16}$$

where $\nu = \alpha(0)$ is arbitrary. As (12) and (13) indicate, the decomposition of (15) is by no means unique; and indeed this is evident from (16). As an example, take

$$E^2 + 2sE + s(s+1);$$

then by (16),

$$\alpha(s) = \frac{s(s+1)}{2s -}\ \frac{(s-1)s}{2(s-1) -}\ \cdots\ \frac{1.2}{2\cdot 1 - \nu};$$

$$\frac{\alpha(s)}{s+1} = \frac{1}{2-}\ \frac{1}{2-}\ \cdots\ \frac{1}{2-\nu}$$

$$= \frac{s - (s-1)\nu}{(s+1) - s\nu} = \frac{s(1-\nu)+\nu}{s(1-\nu)+1},$$

so that

$$\alpha(s) = \frac{s(1-\nu)+\nu}{s(1-\nu)+1}(s+1).$$

Since (15) can be expressed as a product in infinitely many ways it is natural to ask whether it is possible to pick ν in (16) so that the factors will coincide. In general this cannot be done; it is easy to show that a ν can be found only when

$$A(s) = [B(s)]^{1/2} + [B(s-1)]^{1/2}.$$

More generally, (15) can be expressed as a product of permutable operators only when

$$A(s) = [\mu + B(s)]^{1/2} + [\mu + B(s-1)]^{1/2}$$

for some constant μ.

5. *Explicit expressions for the elements of certain arrays.* We go back to the simple case $\lambda = 1$ of (14):

$$c_{n+1,s} = [E + \alpha(s)]c_{n,s},$$

Evidently

$$c_{n+1,s} = [E + \alpha(s)]^n c_{1,s}. \tag{17}$$

We shall now expand the right member of (17). If we put

$$(E + \alpha(s))^n = \sum_{i=0}^{n} A_i^{(n)}(s)E^i,$$

it is easily seen that

$$A_i^{(n)}(s) = \sum_{j=0}^{i} B_{ij}(s)\alpha^n(s-j). \tag{18}$$

Let i be fixed; from the definition, $A_i^{(i)}(s)=1$,

$$A_i^{(n)}(s) = 0 \quad \text{for} \quad 0 \leqq n \leqq i-1.$$

Solving the system (18) by Cramer's Rule,

$$B_{ij}(s) = \frac{(-1)^j}{\prod_{h=0}^{j-1}[\alpha(s-h)-\alpha(s-j)]\prod_{k=j+1}^{i}[\alpha(s-j)-\alpha(s-k)]}$$

Therefore, using (18),

$$(19) \qquad E+\alpha(s))^n = \sum_{i=0}^{n}\left\{\sum_{j=0}^{i}\frac{(-1)^j\alpha^n(s-j)}{\prod_{h=0}^{j-1}[\alpha(s-h)-\alpha(s-j)]\prod_{k=j+1}^{i}[\alpha(s-j)-\alpha(s-k)]}\right\}E^i.$$

By (17) and (3),

$$(20) \qquad \overset{1}{c}_{n+1,s} = \sum_{j=1}^{s}\frac{(-1)^{s-j}\alpha^n(j)}{\prod_{h=1}^{j-1}[\alpha(j)-\alpha(h)]\prod_{k=j+1}^{s}[\alpha(k)-\alpha(j)]},$$

from which all the remaining formulas of this section are derived. We next take the array defined by two permutable operators $E+\alpha(s)$ and $E+\beta(s)$; $\beta(s)$ is then $=\alpha(s)+\mu$. Hence, starting with

$$\begin{aligned}\overset{(2)}{c}_{n+1,s} &= (E+\alpha(s))(E+\alpha(s)+\mu)\overset{(2)}{c}_{n,s} \\ &= \{(E+\alpha(s))(E+\alpha(s)+\mu)\}^n \overset{(2)}{c}_{1,s} \\ &= (E+\alpha(s))^n(E+\alpha(s)+\mu)^n \overset{(2)}{c}_{1,s} \\ &= \sum_{i=0}^{n}\binom{n}{i}\mu^{n-i}(E+\alpha(s))^{n+i}\overset{(2)}{c}_{1,s} \\ &= \sum_{i=0}^{n}\binom{n}{i}\mu^{n-i}\sum_{j=1}^{s}\frac{(-1)^{s-j}\alpha^{n+1}(j)}{\prod(\alpha(j)-\alpha(h))\prod(\alpha(k)-\alpha(j))},\end{aligned}$$

we find that

$$(21) \qquad \overset{(2)}{c}_{n+1,s} = \sum_{j=1}^{s}\frac{(-1)^{s-j}\alpha^n(j)(\alpha(j)+\mu)^n}{\prod(\alpha(j)-\alpha(h))\prod(\alpha(k)-\alpha(j))}.$$

The close analogy between (20) and (21) indicates clearly the form of $\overset{(\lambda)}{c}_{n+1,s}$ for

$$(E+\alpha(s))(E+\alpha(s)+\mu)\cdots(E+\alpha(s)+\pi).$$

We shall now apply (20) and (21) to (11) and (13). For (11), $\lambda=1$, and we find

$$a_{n,s}^{(1)} = \frac{1}{(s-1)!} \sum_{i=1}^{s} \binom{s-1}{s-i} (-1)^{s-i} i^{n-1}. \tag{22}$$

This may be put in the form

$$\frac{1}{s!} \sum_{i=1}^{s} (-1)^{s-i} \binom{s}{i} i^{n},$$

which is the expression given, among other places, in Schwatt's[5] book. For general λ, (11) yields

$$a_{n,s}^{(\lambda)} = \frac{1}{(s-1)!} \sum_{i=1}^{s} (-1)^{s-i} \binom{s-1}{i-1} [i(i+1)\cdots(i+\lambda-1)]^{n-1}. \tag{23}$$

The treatment of (13) is somewhat more detailed; we find for $\lambda = 1$

$$b_{n,s}^{(1)} = \frac{1}{(2s+1)!} \sum_{k=1}^{s} (-1)^{s-k} \binom{2s+1}{s-k} (2k+1)[k(k+1)]^{n}. \tag{24}$$

For $\lambda = 2$, we take $\alpha(s) = s(s+3)$, $\mu = 2$.

Then

$$b_{n,s}^{(2)} = \frac{1}{(2s+3)!} \sum_{k=1}^{s} (-1)^{s-k} \binom{2s+3}{s-k} (2k+3)[k(k+1)(k+2)(k+3)]^{n}.$$

For general λ we find

$$b_{n,s}^{(\lambda)} = \frac{1}{(2s+2\lambda-1)!} \sum_{k=1}^{s} (-1)^{s-k} \binom{2s+2\lambda-1}{s-k} (2k+2\lambda-1)[k\cdots(k+2\lambda-1)]^{n}. \tag{25}$$

It should be noticed that (21) can be obtained by the method of finite differences; this is apparently not the case with (25). At any rate the method used above enables one to easily obtain an explicit expression for the elements of any array corresponding to operators composed of permutable linear operators.

6. *A property of permutable operators and their associated arrays.*

Let $F = \alpha_0(s)E^i + \cdots + \alpha_i(s)$ and $G = \beta_0(s)E^j + \cdots + \beta_j(s)$ be any two permutable operators. To F and G correspond the arrays $f_{n,s}$, $g_{n,s}$; to the product $H = FG$ corresponds the array $h_{n,s}$. Then the $(2n-1)$st row of the array that begins as $f_{n,s}$ for the first n rows and continues as $g_{n,s}$ is the n-th row of $h_{n,s}$.

Proof: Let η_s be the element in the $(2n-1)$st row and s-th column of the compound array described above. Evidently

$$\eta_s = G^{n-1}F^{n-1}c_{1,s},$$

where of course $c_{1,s}$ is zero for $s \neq 1$ and one for $s = 1$. Now

$$h_{n,s} = H^{n-1}h_{1,s} = H^{n-1}c_{1,s}.$$

But, since $FG = GF$,

$$H^{n-1} = G^{n-1}F^{n-1},$$

which completes the proof.

[5] Loc. cit., p. 86. For other references, see Nielsen, *Nombres de Bernoulli*, p. 26.

A PROBLEM IN DIFFERENTIAL GEOMETRY

By B. F. KIMBALL, University of New Hampshire

One approves of using vectors in the study of differential geometry, but at times their seeming simplicity hides an essentially complicated state of affairs. Take, for instance, the vector equation:

$$\boldsymbol{r} = \boldsymbol{a} \cos t + \boldsymbol{b} \sin t + \boldsymbol{c}t, \tag{1}$$

where $\boldsymbol{r}$, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are position vectors. In a certain text, along with several simple problems, one is asked to prove that the above equation represents an elliptical helix. The above vector form at once shows that the curve lies on an elliptical cylinder, and that it winds around the cylinder, the various windings being at equal distances apart as measured along elements of the cylinder. But it is not, strictly speaking, a helix unless $\boldsymbol{c}$ has special directions, as determined by the relative lengths of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ and in *that* case the curve does not make a constant angle with the *elements of the cylinder*. Perhaps the author of the text is using the term "helix" in a rather general sense.

It is interesting to note what the facts are concerning this curve. We use a dot over a letter to denote differentiation with respect to the parameter t. Let τ denote the torsion, k the curvature and s the length of the curve. The vector product is denoted by an $\times$ and the scalar product by a dot placed between the letters denoting the vectors. A parenthesis containing three vectors indicates the determinant of their vector components. A familiar formula gives:

$$\frac{\tau}{k} = \frac{(\dot{\boldsymbol{r}}\ddot{\boldsymbol{r}}\dddot{\boldsymbol{r}})}{k^3(\dot{\boldsymbol{r}}^2)^3}. \tag{2}$$

Now for the above curve

$$(\dot{\boldsymbol{r}}\ddot{\boldsymbol{r}}\dddot{\boldsymbol{r}}) = (\boldsymbol{abc}). \tag{3}$$

Thus a necessary condition that τ/k be constant is that the quantity $k^2(\dot{\boldsymbol{r}}^2)^2$ be constant. One finds that

$$k^2(\dot{\boldsymbol{r}}^2)^2 = \frac{[(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{c} \times \ddot{\boldsymbol{r}})]^2}{\dot{\boldsymbol{r}}^2} \tag{4}$$

We wish to find conditions on $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ which make this a constant.
We write

$$W[(\boldsymbol{a} \times \boldsymbol{b}) + (\boldsymbol{c} \times \ddot{\boldsymbol{r}})]^2 = \dot{\boldsymbol{r}}^2, \tag{5}$$

where W is constant. On expanding $\dot{\boldsymbol{r}}^2$ and $[(\boldsymbol{a}\times\boldsymbol{b})+(\boldsymbol{c}\times\ddot{\boldsymbol{r}})]^2$ one finds that each is equal to an expression in t of the form

$$A \cos 2t + B \sin 2t + C \cos t + D \sin t + E. \tag{6}$$

Thus in order that (5) be an identity in t, corresponding coefficients of (6) must be equal and one obtains the five conditions:

$$
\begin{aligned}
\frac{b^2 - a^2}{2} &= \frac{(\mathbf{c} \times \boldsymbol{a})^2 - (\mathbf{c} \times \boldsymbol{b})^2}{2} W; \\
- \boldsymbol{a} \cdot \boldsymbol{b} &= (\mathbf{c} \times \boldsymbol{a}) \cdot (\mathbf{c} \times \boldsymbol{b}) W; \\
- 2(\mathbf{c} \cdot \boldsymbol{a}) &= - 2(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\mathbf{c} \times \boldsymbol{b}) W; \\
+ 2(\mathbf{c} \cdot \boldsymbol{b}) &= - 2(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\mathbf{c} \times \boldsymbol{a}) W; \\
c^2 + \frac{a^2 + b^2}{2} &= \left[(\boldsymbol{a} \times \boldsymbol{b})^2 + \frac{(\mathbf{c} \times \boldsymbol{a})^2 + (\mathbf{c} \times \boldsymbol{b})^2}{2} \right] W.
\end{aligned} \tag{7}
$$

We define three angles α, β, and γ to satisfy the relations

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab \cos \alpha, \quad \boldsymbol{a} \cdot \mathbf{c} = ac \cos \beta, \quad \boldsymbol{b} \cdot \mathbf{c} = bc \cos \gamma,$$

where the italic letters denote the lengths of the vectors. Thus solving the 2nd, 3rd, and 4th equations for the cosines of these angles one obtains;

$$\cos^2 \alpha = \frac{(Wa^2 - 1)(Wb^2 - 1)}{W^2 a^2 b^2}, \qquad \cos^2 \beta = \frac{(Wa^2 - 1)(Wc^2 + 1)}{W^2 a^2 c^2}, \tag{8}$$

$$\cos^2 \gamma = \frac{(Wb^2 - 1)(Wc^2 + 1)}{W^2 b^2 c^2}.$$

Now using the values of the sines obtained from the above relations it is found that the first and last of equations (7) are identically satisfied. Thus the equations (8) are necessary conditions that the curve (1) be a helix. Without loss of generality in the discussion, one can take $\boldsymbol{a}$ and $\boldsymbol{b}$ as giving the major and minor axes of the ellipse defined by the first two terms of (1). This allows two possible choices of W. If one takes $W = 1/a^2$ one finds that $\cos^2 \gamma$ will be negative. This leads to the conditions

$$W = 1/b^2, \quad \cos^2 \alpha = 0 \tag{9}$$

and

$$\cos^2 \gamma = 0, \quad \cos^2 \beta = \frac{(a^2 - b^2)(b^2 + c^2)}{a^2 c^2}, \quad \sin^2 \beta = \frac{b^2(b^2 + c^2 - a^2)}{a^2 c^2}, \tag{10}$$

which show that in any case $\mathbf{c}$ must be perpendicular to the minor axis of the directrix ellipse.

Under (9) and (10) we have

$$k = \frac{b}{\dot{\boldsymbol{r}}^2}, \quad \tau = \frac{(\boldsymbol{abc})}{b^2 \dot{\boldsymbol{r}}^2}, \qquad (\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}})^2 = b^2 \dot{\boldsymbol{r}}^2. \tag{11}$$

Let $\boldsymbol{d}$ denote the fixed direction, if such exists, with which the helix makes a constant angle. From the theory of the helix we have

$$\boldsymbol{d} \text{ parallel to } \tau \boldsymbol{t} + k\boldsymbol{b}$$

or

$$\boldsymbol{d} \text{ parallel to } \frac{(\boldsymbol{abc})}{b^2\dot{\boldsymbol{r}}^2}\frac{\dot{\boldsymbol{r}}}{\sqrt{\dot{\boldsymbol{r}}^2}} + \frac{b}{\dot{\boldsymbol{r}}^2}\frac{\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}|}. \tag{12}$$

Using (11) one can write this vector as

$$\frac{1}{b^2}\frac{1}{\dot{\boldsymbol{r}}^2}\frac{1}{\sqrt{\dot{\boldsymbol{r}}^2}}[(\boldsymbol{abc})\dot{\boldsymbol{r}} + b^2(\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}})].$$

We set

$$\boldsymbol{d} = (\boldsymbol{abc})\dot{\boldsymbol{r}} + b^2(\dot{\boldsymbol{r}} \times \ddot{\boldsymbol{r}}),$$

and expanding in terms of t and noting that

$$b^2(\boldsymbol{c} \times \boldsymbol{a}) = (\boldsymbol{abc})\boldsymbol{b}$$

because of the conditions (10), we find that

$$\boldsymbol{d} = [(\boldsymbol{abc})\boldsymbol{c} + b^2(\boldsymbol{a} \times \boldsymbol{b})] + [b^2(\boldsymbol{b} \times \boldsymbol{c}) - (\boldsymbol{abc})\boldsymbol{a}] \sin t.$$

Now the two brackets can be shown to represent parallel vectors under the choice of β given above.

Thus under the special conditions (10) the curve is a helix which makes a constant angle with the vector

$$(\boldsymbol{abc})\boldsymbol{c} + b^2(\boldsymbol{a} \times \boldsymbol{b}).$$

The cotangent of that angle will be $(\boldsymbol{abc})/\boldsymbol{b}^3$.

EXTENSION OF THE CONCEPT OF GROUP OF ISOMORPHISMS

By G. A. MILLER, University of Illinois

It is well known that the group of isomorphisms, or the group of automorphisms, of a given group G is one of the most important concepts of group theory. Hence it may be desirable to note here a certain extension of this concept since this may throw some new light on its real nature. The group of isomorphisms of G may be constructed as follows: Write G in the form of a regular substitution group and construct the group formed by all the substitutions on the letters of G which satisfy the conditions that they transform G into itself and also omit a fixed letter of G. The substitution group thus formed is simply isomorphic with the group of isomorphisms of G when G is regarded as an abstract

group. When G is regarded as a regular substitution group the given substitution group may be regarded as the actual group of isomorphisms of G. The substitutions and the subgroups of G which are transformed into themselves under this group of isomorphisms are called characteristic substitutions and characteristic subgroups respectively of G. The symmetric group on the letters of G may contain many regular substitution groups which are simply isomorphic with G. In abstract group theory it is customary to regard all simply isomorphic groups as the same group and this is also done in the enumeration of the regular substitution groups. It is obvious that the symmetric group on all except one of the letters of G transforms G into all of its conjugates on its own letters and involves no two substitutions which transform all the substitutions of G in the same way. Hence such a symmetric group may be regarded as the *symmetric group of isomorphisms of* G, and to distinguish it from the group of isomorphisms of G noted in the preceding paragraph this may be called the *holomorph group of isomorphisms of* G. It may be noted that there are exactly g substittions in the symmetric group on the letters of G which transform G according to the same substitution in each of these groups of isomorphisms, g being the order of G, and that at least one of these is of the same order as the isomorphism in question.

It is evident that under the symmetric group of isomorphisms of G there are no characteristic substitutions and no characteristic subgroups of G besides the identity except in the trivial case when the order of G is 2, and that these two definitions of the group of isomorphisms coincide when and only when G is one of the following four groups: The identity, the group of order 2, the group of order 3, and the non-cyclic group of order 4. All the groups of the same order have the same symmetric group of isomorphisms but they usually have different holomorph groups of isomorphisms. The quotient obtained by dividing the order of the symmetric group of isomorphisms by the order of the holomorph group of isomorphisms is equal to the number of the conjugates of G under the symmetric group on its letters. Characteristic substitutions and characteristic subgroups correspond to such substitutions and to such subgroups of the corresponding group under the symmetric group of isomorphisms but they do not necessarily correspond to themselves under the isomorphisms of this group.

Under the holomorph definition of the group of isomorphisms only the possible interchanges of the operators in the possible automorphisms of a group are considered while according to the extended definition isomorphisms with simply isomorphic groups are also considered. Since all simply isomorphic groups are frequently regarded as the same group and all such groups may be supposed to be represented as regular substitution groups on the same set of letters whose number is equal to their common order, it seems desirable to note the effect of such an extension of the notion of a given group on the concept of its group of isomorphisms. It should however be observed that the ordinary, or holomorph, definition of the group of isomorphisms seems to be the more useful, and should

always be understood unless the contrary is explicitly stated. In closing it may be added that the present note has close contact with the article published in this Monthly, volume 29(1922), page 319, and may be regarded as an extension thereof along certain lines.

POSSIBLE TYPES OF MULTIPLICATION OF SERIES

By E. T. BELL, California Institute of Technology

1. Throughout the theory of numbers there is a sharp and generally impassable break between additive and multiplicative properties. For example, in order to devise a definition of addition for ideals, it seems to be necessary to revise the existing theory, which is purely multiplicative, ab initio. Again, in rational arithmetic, some of the most intractable questions are those combining both additive and multiplicative properties, for example Goldbach's hypothesis that every even integer is the sum of two primes. For additive rational arithmetic the natural analytical tool is the power series; for multiplicative, the Dirichlet. It is conceivable but not probable that other types of arithmetic may exist; namely, sections of arithmetic in which the laws of algebra are preserved in so far as they are relevant, which are neither multiplicative nor additive only. In trying to disprove such existence, I was led to certain simple functional equations in two variables, one of which is restricted to be an integer $\geqq 0$; to settle the question indicated above it is sufficient to prove that the obvious solutions of these equations are the general solutions. Possibly these equations are familiar to some reader of this Monthly who will supply the desired proof.

2. Let x be a parameter, n an integer $\geqq 0$, and let a_n be independent of x. Let a_n, $f(x,n)$ be defined and single-valued for all values of x, n considered. Further let $a_n, f(x,n)(n=0,1, \ldots)$ be such that

2.1 $$A(x) \equiv \sum_{n=0}^{\infty} a_n f(x, n)$$

is uniquely defined and has a meaning such that the product of two functions of the same kind as $A(x)$ is a function of the same kind:

2.2 $$A(x) \cdot B(x) = C(x),$$

where

$$B(x) \equiv \sum_{n=0}^{\infty} b_n f(x, n), \quad C(x) \equiv \sum_{n=0}^{\infty} c_n f(x, n).$$

Further suppose that the terms in the product on the left of 2.2 can be rearranged in any way after multiplication without affecting the meaning of $A(x)B(x)$.

The multiplication in 2.2 will be possible if $f(x,n)$ is such that

2.3 $$f(x, n_1)f(x, n_2) = f(x, \phi(n_1, n_2)),$$

where $\phi(n_1, n_2)$ is an integer $\geqq 0$ whenever n_1, n_2 are integers $\geqq 0$. From the hypotheses it follows that $\phi(n_1, n_2)$ is a symmetric function of n_1, n_2,

2.4 $$\phi(n_1, n_2) = \phi(n_2, n_1).$$

It follows also from the hypotheses that the multiplication defined by 2.2 is associative. Hence, and by 2.4,

2.5 $$\phi(\phi(n_1, n_2), n_3) = \phi(\phi(n_3, n_2), n_1)$$

for every triple of integers $n_1,n_2,n_3 \geqq 0$. If for the moment we drop the restriction that n_1, n_2, n_3 be integers, and consider only polynomials $\phi(n_1, n_2)$, it follows by a simple argument that ϕ is of the first degree in n_1, n_2 separately, and that there are two essentially distinct solutions of 2.4, 2.5; namely,

2.6 $$\phi(n_1, n_2) = n_1 + n_2 + c; \ \phi(n_1, n_2) = cn_1n_2,$$

in which, if $\phi(n_1,n_2)$ is to be a positive integer when n_1,n_2 are positive integers, c is an arbitrary constant positive integer (independent of n_1,n_2).

3. The functional equations to be solved are therefore, from 2.6, 2.3,

3.1 $$f(x, n_1)f(x, n_2) = f(x, n_1 + n_2 + c),$$

3.2 $$f(x, n_1)f(x, n_2) = f(x, cn_1n_2).$$

A solution of 3.1 is $f(x, n) = [F(x)]^{n+c}$; a solution of 3.2 is $f(x, n) = (c \cdot n)^{F(x)}$; where, in either, $F(x)$ is an arbitrary function of x, arbitrary to the extent demanded by the definitions concerning 2.1.

Hence, taking a new variable $y = F(x)$, we see that 3.1, 3.2 define respectively a power series in y and a Dirichlet series in y, and that 2.2 is in the respective cases Cauchy multiplication or Dirichlet multiplication of series (for $c=0$, $c=1$ respectively).

What are the general solutions of 3.1, 3.2? It need not be assumed that $f(x,n)$ is a continuous function of x.

I have since found a much more general solution of 3.2 which, however, does not seem to settle the question of complete generality.

SOLUTION OF EQUATIONS BY ADDITION-SUBTRACTION LOGARITHMS

By P. H. NYGAARD, North Central High School, Spokane, Wash.

By the use of addition-subtraction logarithms (also called Gaussian logarithms) it is possible to find the logarithm of the sum or the difference of two numbers directly from their logarithms. The tables used for this purpose contain what are known as A, B, and C logarithms. All cases of addition can be done by means of the A and B logarithms. Substraction problems require either the A and B or the B and C combination depending upon how large a difference exists between the logarithms of the two numbers. The following discussion is based upon the addition-subtraction logarithms as found in Jones's *Logarithmic Tables.*[1]

The purpose of this paper is to present a method of approximating the values of the roots of two types of equations by means of addition-subtraction logarithms.[2] In the first part equations involving powers of x will be considered, while the second part will deal with exponential equations.

Formulas similar to those of the first part have been stated by Lüroth in the Rendiconti del Circolo Matematico di Palermo, vol. 27 (1909), pp. 393–401. Lüroth's formulas, however, are not adapted to simple tables such as those of Jones. As far as the writer knows, the formulas of the second part are new.

1. *Trinomial Equations.*

Let $gx^m+hx^n=k$ represent the equation under consideration. The method to be explained will give all of its real solutions provided each of the five constants is real. Any or all of the constants may be negative, fractional, or irrational. The only limitation is that there must be only two terms containing the unknown.

Case 1. $gx^m+hx^n=k$, *and* g, h, k, *and* x *are all positive.* The value of x to be found should be roughly estimated. The equation should be arranged so that, for this value of x, $gx^m>hx^n$. In deriving the formulas the terms gx^m and hx^n are considered the numbers to be added and k their sum. Then from the definition of the quantities A and B, as used in the table of addition-subtraction logarithms,

$$A = \log h - \log g - (m - n) \log x,$$
$$B = \log k - \log g - m \log x.$$

[1] George W. Jones, *Logarithmic Tables*, The Macmillan Co. (Tenth Edition, 1905).

[2] *Editor's Note*: For an account of Gauss's method of solving certain algebraic equations by means of addition-subtraction logarithms, see his *Beiträge zur Theorie der algebraische Gleichungen* (Zweite Abteilung) in his Werke, vol. 3, p. 85; see also the paragraph entitled *Die Gauss'sche Methode der Auflösung der trinomischer Gleichungen* in Weber's *Lehrbuch der Algebra* (Zweite Auflage, 1898), pp. 393–397.

Elimination of log x between these two equations gives

$$(1)\qquad A - m^{-1}(m - n)B = \log h - \log g - m^{-1}(m - n)(\log k - \log g).$$

Solving the first of them for log x gives

$$(2)\qquad \log x = (\log h - \log g - A)/(m - n).$$

The value of each of the five given constants should be substituted in formula (1) and the formula simplified. A and B will be unknown. We find from the table of addition-subtraction logarithms the value of A which, with its corresponding value of B, satisfies formula (1). After A has been thus determined, we find log x from (2), and then find x from its logarithm.

Illustrative Example. Solve the equation $x^4+31.8x=28.72$. We shall proceed to find the positive root whose value is approximately .9. Since for this root $31.8x>x^4$, we arrange the equation in the form, $31.8x+x^4=28.72$. Then $g=31.8$, $h=1$, $m=1$, $n=4$, and $k=28.72$. From (1),

$$A-1^{-1}(1-4)\ B=\log 1-\log 31.8-1^{-1}(1-4)\ (\log 28.72-\log 31.8)$$

or

$$A + 3B = 0 - 1.502427 + 3(1.458184 - 1.502427).$$

This gives $A+3B=8.364844-10$. From the table of addition-subtraction logarithms we find that if $A=8.336-10$,

$$B = .009314, \text{ and } A + 3B = 8.363942 - 10;$$

if and $A=8.337-10$,

$$B = .009335, \text{ and } A + 3B = 8.365005 - 10.$$

Since $A+3B$ is supposed to equal $8.364844-10$, we find by interpolating that[3] $A=8.33685-10$. From (2) we get

$$\log x = (\log 1 - \log 31.8 + 10 - 8.33685)/(- 3) = 9.94643 - 10.$$

Therefore $x=.88395$.

Case 2. $gx^m-hx^n=k$ and g, h, k, and x are all positive. The value of x to be found should first be approximated and a rough calculation should be made to see whether, for this value of x, $\log gx^m-\log hx^n$ is less than or greater than .4.

[3] Finding A to 5 decimal places from three-place tabulated values is justifiable, because the function $A+kB$, for any k, is almost a linear function of A for nearby values. The successive differences are large enough to determine two interpolated figures correctly—often 3. Thus, when

$$\begin{aligned}
&A=8.335,\ A+3B=8.362876 \\
&\qquad\qquad\qquad\qquad \to\text{Diff.}=1066 \\
&A=8.336,\ A+3B=8.363942 \\
&\qquad\qquad\qquad\qquad \to\text{Diff.}=1063 \\
&A=8.337,\ A+3B=8.365005 \\
&\qquad\qquad\qquad\qquad \to\text{Diff.}=1063 \\
&A=8.338,\ A+3B=8.366068
\end{aligned}$$

Case 2a. When log $gx^m - \log hx^n < .4$. By making use of the definitions of A and B logarithms it may be shown as in case 1 that

$$A + n(m-n)^{-1}B = \log k - \log h + n(m - n)^{-1}(\log g - \log h), \tag{3}$$

and

$$\log x = (\log k - \log h - A)/n. \tag{4}$$

We find from the table of addition-subtraction logarithms the value of A which, with its corresponding value of B, satisfies formula (3). We determine log x from (4), and then find x from its logarithm.

Illustrative Example: Solve the equation $x^3 - 7x^{\frac{1}{2}} = 2$. We shall find the root between 2 and 3. For this root, $\log x^3 - \log 7x^{\frac{1}{2}} < .4$, so the equation comes under *Case* 2*a*. Here $g=1$, $h=7$, $k=2$, $m=3$, and $n=\frac{1}{2}$. Substituting in (3),

$$A + \tfrac{1}{2}(3 - \tfrac{1}{2})^{-1}\cdot B = \log 2 - \log 7 + \tfrac{1}{2}(3 - \tfrac{1}{2})^{-1}(\log 1 - \log 7),$$

or

$$A + \tfrac{1}{5}B = .301030 - .845098 + \tfrac{1}{5}(0 - .845098).$$

This gives $A + \frac{1}{5}B = 9.286912 - 10$. From the tables find a value of A which, when one fifth of the corresponding value of B is added, gives $9.286912 - 10$. We find $A = 9.27202 - 10$. From (4),

$$\log x = (\log 2 - \log 7 + 10 - 9.27202)/\tfrac{1}{2} = .36782.$$

Hence $x = 2.3325$.

Case 2b. When $\log gx^m - \log hx^n > .4$. For this case the B and C logarithms must be used. The formulas, found as in case 1, are:

$$B + m^{-1}(m - n)\cdot C = \log g - \log h + m^{-1}(m - n)(\log k - \log g), \tag{5}$$

and

$$\log x = (B - \log g + \log h)/(m - n). \tag{6}$$

We find from the table of addition-subtraction logarithms the value of B which, with its corresponding value of C, satisfies formula (5). We then find log x from (6), and x from its logarithm.

Illustrative Example: Solve the equation $(\sqrt{845})x^{\pi} - 93.26x^{-\sqrt{17}} = 1000$. This equation has purposely been made difficult to show the general utility of the method. There is a root between 3 and 4 which will be determined. Since for this root $\log[(\sqrt{845})x^{\pi}] - \log[93.26x^{-\sqrt{17}}] > .4$, the equation comes under *Case* 2*b*. Here $g = (\sqrt{845})$, $h = 93.26$, $k = 1000$, $m = \pi$, and $n = -\sqrt{17}$. From (5),

$$B + \pi^{-1}(\pi + \sqrt{17})C = \log\sqrt{845} - \log 93.26 + \pi^{-1}(\pi + \sqrt{17})(\log 1000 - \log\sqrt{845})$$

which gives $B + 2.3124C = 3.046902$. We locate a value of B so that the sum of

this B and 2.3124 times its corresponding C is 3.046902. That value of B is 3.0478. From (6) we get

$$\log x = (3.0478 - 1.463428 + 1.969695)/7.2647 = .4892.$$

Hence $x=3.085$.

If a root is to be found under *Case* 1, it may save time in determining A to change formula (2) into

$$A = \log h - \log g - (m - n) \log x.$$

By substituting in this formula a rough approximation to x it is possible to get an idea as to just about what value of A to consider. Formulas of similar usefulness may be obtained for *Case* 2a by changing (4) into

$$A = \log k - \log h - n \log x,$$

and for *Case* 2b by changing (6) into

$$B = \log g - \log h + (m - n) \log x.$$

These supplementary formulas are especially helpful when it is desired to find two roots so close together that they both come under the same case. Thus, $7x-x^2=12$ comes under *Case* 2a for both $x=3$ and $x=4$. Using the formula suggested in this paragraph for *Case* 2a, we find that for the first root $A=.12494$ and for the second root $A=9.87596-10$. These values both satisfy formula (3), and when substituted in formula (4) give, respectively, $x=3.000$ and $x=4.0000$.

Negative roots can be obtained, as is ordinarily done, by finding the corresponding positive roots of $f(-x)$. Hence x may be confined to positive values. If k is negative, the equation should be multiplied through by -1. Then either both the coefficients will be positive as in *Case* 1, or one will be positive and the other negative as in *Case* 2.

Limiting the number of terms containing powers of x to two does not greatly impair the usefulness of the method. Any cubic equation with integral exponents can be handled by making the proper substitution so as to eliminate the second degree term. Equations of higher degree often arise through the process of eliminating radicals or fractions. Since the method here proposed does not require exponents and coefficients to be integral, many of these higher degree equations can be avoided. Thus, $x^2-x^{\frac{1}{2}}=5$ would, upon changing to the form $x^2-5=x^{\frac{1}{2}}$ and squaring each side, give a fourth degree equation with three terms containing x to which the logarithmic method of this paper does not apply. However, by leaving the equation in the form, $x^2-x^{\frac{1}{2}}=5$, the method applies directly.

The advantages of this logarithmic method over the more usual methods of solving equations containing powers of x may be summarized as follows: (1) It eliminates much of the tedious arithmetical work, by substituting for it the use of tables. (2) It is applicable to a large variety of such equations which

are otherwise solvable only by the process of successive guesses. (3) Answers can be found to 4 or 5 significant figures in much less time than by other methods. The writer's experience shows that it requires on the average about one half as much time as Horner's method.

2. *Exponential Equations.*

The logarithmic method is applicable also to exponential equations. Here we shall confine ourselves to only one special form, which occurs however in many different connections. It is

$$ge^{cx}+he^{-cx}=\mathrm{k},$$

where g, h, c, and k are any positive real numbers, $e=2.71828\cdots$, and $ge^{cx}>he^{-cx}$ for the value of x to be found. For this equation the following two formulas may be derived in much the same way as before:

$$A-2B=\log g+\log h-2\log k, \tag{7}$$

and

$$X=(\log h-\log g-A)/.86859c \tag{8}$$

The values of A and B are found by reference to the table of addition-subtraction logarithms. We determine the value of A which, with its corresponding value of B, satisfies formula (7). We find x by substituting this value for A in (8).

Two illustrative problems dealing with exponential functions will be solved.

Suppose it is desired to find the angle whose hyperbolic cosine is 3/2. Since $\cosh x=\frac{1}{2}(e^x+e^{-x})$, we have

$$.5e^x+.5e^{-x}=1.5.$$

Here $g=.5$, $h=.5$, $c=1$, and $k=1.5$. From (7),

$$A-2B=2\log .5-2\log 1.5=9.045757-10.$$

From the table of addition-subtraction logarithms we find that if $A=9.164-10$, $B=.059140$, and $A-2B=9.045720-10$. If $A=9.165-10$, $B=.059267$, and $A-2B=9.046466-10$. Since $A-2B$ is supposed to be $9.045757-10$, we find by interpolating that $A=9.16405-10$. From (8), $x=-A/.86859=.96242$.

The equation commonly given for the catenary is $y=\frac{1}{2}a(e^{x/a}+e^{-x/a})$. Suppose we know that $a=60$, and want to determine x so that $y=183.7$. Here we have

$$30e^{x/60}+30e^{-x/60}=183.7,$$

for which $g=30$, $h=30$, $c={}^{1/60}$, and $k=183.7$. From (7),

$$A-2B=2\log 30-2\log 183.7=8.426025-10.$$

The value of A which, when 2 times the value of its corresponding B is subtracted from it, gives $8.426025-10$ is $A=8.45018-10$. From (8),

$$x = -A/.86859c = 1.54982/.014476 = 107.06.$$

Conclusion

No attempt has been made to give an exhaustive survey of the logarithmic method which has been explained. Enough has been included to show the general utility of the method in the solution of equations of many different types by the use of the same set of tables and the same procedure. Only one type of exponential equations has been solved, but the method can be readily extended to others.

QUESTIONS AND DISCUSSIONS

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A NOTE ON A PAPER CONCERNING A BESSEL FUNCTION

By S. A. SCHELKUNOFF, Bell Telephone Laboratories, Inc.

In a recent issue of this Monthly, Mr. C. C. Furnas presented an empirical formula for the modified Bessel function.[1] The method by means of which he arrived at this approximation is an interesting one, but his final formula and table are not so accurate as he believes them to be, partly because of the inherent limitations of the method itself and partly because of inaccuracies of mechanical integration. The purpose of this note is to call attention to a more accurate formula, and to correct the table for the benefit of those who are interested in practical applications of Bessel functions. Our method is less ingenious but more straightforward than his.

While the method used by Mr. Furnas leads to a series defining log I_0, it appears that his interest is really centered in the values of $I_0(x)$ itself, since he tabulates the values of this latter function. If this is the case it would have been a fairly simple matter to make use of the well-known asymptotic expansion

$$I_0(x) = \frac{e^x}{\sqrt{(2\pi x)}}\left[1 + \frac{1}{8x} + \frac{9}{128x^2} + \frac{75}{1024x^3} + \cdots\right], \tag{1}$$

which may be found, for example, in Watson's *Theory of Bessel's Functions*, p. 203, equation (2).

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[1] C. C. Furnas, *Evaluation of the modified Bessel function of the first kind and zeroth order*, this Monthly, vol. 37 (1930), pp. 282–287.

If, on the other hand, there is some special reason for desiring a series for $\log I_0(x)$, we can easily obtain one by the simple expedient of taking the logarithm of (1). Thus we are led to

$$(2)\ \log I_0(x) = x - \frac{1}{2}\log x - \frac{1}{2}\log 2\pi + \log\left[1 + \frac{1}{8x} + \frac{9}{128x^2} + \frac{75}{1024x^3} + \cdots\right]$$

or, expanding the last term in a series by means of the usual formula for $\log(1+\epsilon)$ when ϵ is small, to

$$(3)\quad \log I_0(x) = x - \frac{1}{2}\log x - \frac{1}{2}\log 2\pi + \frac{1}{8x} + \frac{1}{16x^2} + \frac{100}{1536x^3} + \cdots.$$

This process could readily be justified by introducing the remainder term in (2) and treating the resulting expression in the customary way.

Of course the series for $\log_{10} I_0$ may be obtained immediately by multiplying (3) throughout by $\log_{10} e$. For the purpose of comparison with Mr. Furnas' equation (16), it may be interesting to write down the first few terms of this series. They are:

$$(4)\qquad \log_{10} I_0(x) = 0.43429\left[x + \frac{1}{8x} + \cdots\right] - \frac{1}{2}\log_{10} x - 0.39909.$$

The accompanying table was computed by the use of formula (1) and its accuracy checked by using (3) independently as a control. It is believed to be accurate to within half a unit in the last place.

Comparing our table with that given by Mr. Furnas, we observe that the differences between his values for $I_0(x)$ and ours range between -0.149 and $+0.111$ times the corresponding powers of 10. Considering the inherent limitations of the method used by Mr. Furnas it is not at all surprising that such inaccuracies should have crept in.

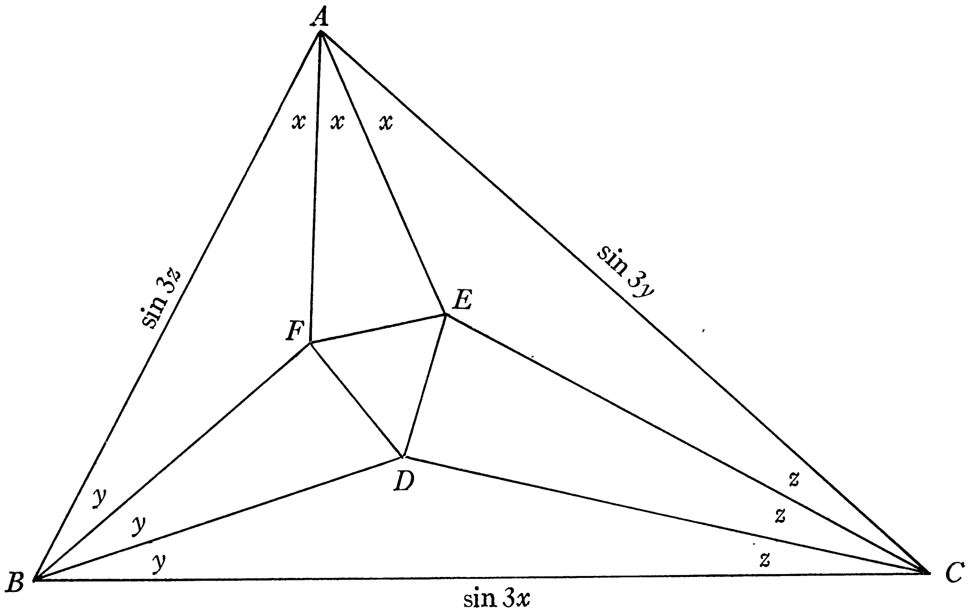
Values of the Modified Bessel Function $I_0(x)$

x	$I_0(x)$	x	$I_0(x)$
12	1.8949×10^{4}	90	5.1392×10^{37}
14	1.2942×10^{5}	100	1.0738×10^{42}
16	8.9345×10^{5}	200	2.0397×10^{85}
18	6.2184×10^{6}	300	4.4758×10^{128}
20	4.3558×10^{7}	400	1.0419×10^{172}
30	7.8167×10^{11}	500	2.5048×10^{215}
40	1.4895×10^{16}	600	6.1463×10^{258}
50	2.9326×10^{20}	700	1.5296×10^{302}
60	5.8941×10^{24}	800	3.8461×10^{345}
70	1.2016×10^{29}	900	9.7473×10^{388}
80	2.4752×10^{33}	1000	2.4857×10^{432}

A Simple Proof of the Theorem of Morley

By Jacob O. Engelhardt, Brooklyn, N. Y.

Theorem[1]: *If the three angles of a triangle be trisected, the triangle whose vertices are each the intersection of a pair of trisectors adjacent to a side, is equilateral.*



$2R=1$, where R is the radius of the circumscribed circle.

Lemma 1: $\sin 3a = 4\sin a \sin(60°+a)\sin(60°-a)$.

Lemma 2: If $a+b+c=180°$, $\cos^2 a+\cos^2 b+\cos^2 c = 1-2\cos(a)\cos(b)\cos(c)$.

Lemma 3: $x+y+z=60°$.

Proof:

$$BD = \frac{\sin 3x \sin z}{\sin(y+z)} = 4 \sin x \sin z \sin(60+x).$$

Similarly $BF = 4 \sin x \sin z \sin(60+z)$.
Therefore

$$(FD)^2 = 16 \sin^2 x \sin^2 z[\sin^2(60+x) + \sin^2(60+z) - 2\sin(60+x)\sin(60+z)\cos y] = 16\sin^2 x \sin^2 z[\cos^2(30-x) + \cos^2(30-z) - 2\cos(30-x)\cos(30-z)\cos y]$$

[1] For other proofs of this theorem see R. A. Johnson, *Modern Geometry*, p. 253; and Philip Franklin, in Contributions of the Mathematics Department of the Massachusetts Institute of Technology, Second Series, No. 117 (Nov., 1926), p. 57.

But, by Lemma 2,

$$\cos^2(30-x)+\cos^2(30-z) = 1-\cos^2(180-y)-2\cos(30-x)\cos(30-z)\cos(180-y)$$
$$= \sin^2(180-y)+2\cos(30-x)\cos(30-z)\cos y.$$

Therefore $(FD)^2=16\sin^2 x\sin^2 z\sin^2(180-y)$ and $FD=4\sin x\sin y\sin z$. Similarly $DE=4\sin x\sin y\sin z$, $EF=4\sin x\sin y\sin z$, and therefore $FD=DE=EF$.

Construction of A Table of Hyperbolic Sines and Cosines

By S. A. Corey, Des Moines, Iowa

In the May, 1930 issue of this Monthly, Professor J. P. Ballantine[1] has given a method of computing a table of sines and cosines, which by making a slight modification may also be used in computing a table of hyperbolic sines and cosines.

Employing the identities,

$$\begin{aligned} \sinh(n+1)x &= \sinh(n-1)x+2\cosh nx\sinh x,\\ \cosh(n+1)x &= \cosh(n-1)x+2\sinh nx\sinh x, \end{aligned} \tag{1}$$

and taking

$$2\sinh x = (.1)^k, \tag{2}$$

where k may for convenience be 1, 2, or 3 we get by substituting in (1):

$$\begin{aligned} \sinh(n+1)x &= \sinh(n-1)x+(.1)^k\cosh nx,\\ \cosh(n+1)x &= \cosh(n-1)x+(.1)^k\sinh nx. \end{aligned} \tag{3}$$

Taking this value of x and tabulating $\cosh nx$ for even values of n and $\sinh nx$ for odd values of n, we get by taking $k=1$:

$$\begin{aligned} 1.\quad &= \cosh 0x\\ .05\quad &= \sinh 1x \quad \text{(From Equation (2))}\\ 1.005\quad &= \cosh 2x = \cosh 0x + .1\sinh 1x\\ .1505\quad &= \sinh 3x = \sinh 1x + .1\cosh 2x\\ 1.02005\quad &= \cosh 4x = \cosh 2x + .1\sinh 3x\\ .25250\quad &= \sinh 5x = \sinh 3x + .1\cosh 4x, \end{aligned}$$

and so on.

Each new entry is obtained by adding one-tenth of the last entry obtained to the entry next preceding. By taking k larger, our unit of measurement, $2x$, may be made as small as desired. As the unit of measurement decreases the tabular accuracy increases. Professor Ballantine's equation (4) would indicate that 20-place accuracy could be obtained by taking $k=7$.

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[1] Vol. 37 (1930), pp. 248–250.

An Elementary Upper Bound to the Roots of Equations

By J. M. Feld, Columbia University

1. *Introduction.* Cauchy has given a rule for the determination of an upper bound to the roots of algebraic equations, viz., the positive roots of

$$x^n + a_1x^{n-1} + \cdots + a_n = 0$$

cannot exceed the greatest of the quantities

$$(ka_r)^{1/r},\ (ka_s)^{1/s}, \cdots,\ (ka_t)^{1/t},$$

where a_n, $a_s \cdots$, a_t are the negative coefficients and k is the number of negative coefficients. The purpose of this note is to extend this result to equations

$$\sum_0^m a_i e^{\alpha_i x} = 0.$$

Two cases will be considered. First an upper bound to the positive roots will be found, assuming the a's to be real; and secondly we shall determine an upper bound to the real part of the roots of exponential equations, allowing the a's to be complex. In both cases the α's are real.

2. Let

$$e^{\alpha_n x} + a_{n-1}e^{\alpha_{n-1}x} + \cdots + a_0e^{\alpha_0 x} = 0, \tag{1}$$

where the α's and a's are real numbers and let $\alpha_n > \alpha_{n-1} > \cdots > \alpha$. In order that (1) have a positive root at least one of the a's must be negative. Transposing the terms with negative a's to the right we get

$$e^{\alpha_n x} + a_pe^{\alpha_p x} + \cdots + a_{p+i}e^{\alpha_{p+i}x} = a_qe^{\alpha_q x} + \cdots + a_{q+k-1}e^{\alpha_{q+k-1}x} \tag{2}$$

Since

$$e^{\alpha_n x}/k \geqq a_{q+j}e^{\alpha_{q+j}x}$$

if

$$x \geqq \frac{\log (ka_{q+j})}{\alpha_n - \alpha_{q+j}},$$

we see that

$$e^{\alpha_n x} \geqq a_qe^{\alpha_q x} + \cdots + a_{q+k-1}e^{\alpha_{q+k-1}x}$$

provided x is greater than or equal to the greatest of the quantities

$$\frac{\log (ka_{q+j})}{\alpha_n - \alpha_{q+j}} \qquad (j = 0, 1, \cdots, k-1). \tag{3}$$

Therefore (1) can have no positive root greater than the greatest of the quantities (3).

If the α's in (1) are integers such that $\alpha_k = k$, letting $e^{kx} = y^k$ (1) becomes

$$y^n + a_{n-1}y^{n-1} + \cdots + a_0 = 0; \tag{4}$$

and we have the result that (4) can have no positive root greater than the greatest of the quantities,

$$(ka_{q+j})^{1/(n-q-j)} \qquad (j = 0, 1, \cdots, k-1). \tag{5}$$

In the equation

$$e^{\alpha_n x} - ce^{\alpha_{n-1}x} - \sum_{i=2}^{n} n^{\beta_i} c^{\gamma_i} e^{\alpha_{n-i}x} = 0, \tag{6}$$

where c is positive and

$$\beta_i = \frac{\alpha_{n-1} - \alpha_{n-i}}{\alpha_n - \alpha_{n-1}}, \qquad \gamma_i = \frac{\alpha_n - \alpha_{n-i}}{\alpha_n - \alpha_{n-1}}$$

the quantities (3) all equal

$$\frac{\log (nc)}{\alpha_n - \alpha_{n-1}},$$

which is a root of (6).

3. Let the a's in (1) be complex. Then for any x that satisfies (1) we have

$$|e^{\alpha_n x}| \leqq |a_{n-1}e^{\alpha_{n-1}x}| + \cdots + |a_0 e^{\alpha_0 x}| . \tag{7}$$

If k is the number of terms in the right member of (7) we have

$$|e^{\alpha_n x}| > |a_{n-1}e^{\alpha_{n-1}x}| + \cdots + |a_0 e^{\alpha_0 x}|$$

for any x such that

$$|e^{\alpha_n x}| > k|a_{n-p}e^{\alpha_{n-p}x}| \qquad (p = 1, 2, \cdots, n); \tag{8}$$

and consequently such values of x cannot satisfy (1). From (8) we obtain

$$|e^{(\alpha_n - \alpha_{n-p})x}| > k|a_{n-p}|$$

or

$$R(x) > \frac{\log k|a_{n-p}|}{\alpha_n - \alpha_{n-p}} \qquad (p = 1, 2, \cdots, n), \tag{9}$$

where $R(x)$ represents the real part of x. Thus (1) can have no roots in the half plane

$$R(x) > \text{the greatest } \frac{\log k|a_{n-p}|}{\alpha_n - \alpha_{n-p}} \qquad (p = 1, 2, \cdots, n).$$

If in (4) the a's are complex this result becomes: the absolute value of the roots of (4) cannot exceed the greatest of the numbers,

$$(k|a_{n-p}|)^{1/p} \qquad (p = 1, 2, \cdots, n).$$

On Collateral Reading in Mathematics

By J. H. Kusner, University of Florida

In the May, 1928 issue of this Monthly[1] there appeared a report of a committee of the Mathematical Association of America, presenting a list of books for use as collateral reading by freshmen and sophomores, together with a list of assignments and a proposed method of administration. These suggestions struck a responsive chord in the mind of the present writer and he has for the past two years been experimenting with collateral reading, and has, he believes, learned one or two things worthy of communication.

Probably the main objection which teachers of college mathematics have to such innovations as collateral reading is the combined one of the difficulty of working the reading in as an integral part of the course and also the objection to taking time away from the study *of* mathematics for the purpose of studying *about* mathematics and mathematicians. The committee proposed that the instructor devote a small portion of time, say one half hour per week, to class discussion of the material of the reading, which not only has the objection of taking class time, but also necessitates a multiplicity of copies of reserve books since all members of the class must read the same assignment each week. Also, it involves making the collateral reading required work, which many teachers who are in sympathy with the general idea are not ready to do.

The present writer has developed a method of administration which, although undoubtedly capable of much improvement, has little in it that is objectionable on the grounds mentioned above.

First of all, the reading has been placed on a voluntary basis. At the beginning of each semester, the student is offered the opportunity of entering upon the course of reading *in addition* to the regular work of the course and entirely separate from it, and if he elects it he must read one assignment each week. Naturally, so long as there exists the necessity of assigning grades, it is only just that the satisfactory completion of the collateral reading have some effect on the grade awarded. The present writer has adopted the scheme of raising the semester grade one letter if the reading for the entire semester has been completed satisfactorily and if the student has already earned a passing grade in the course. Thus a student earning D, C, B, or A in the regular work of the course receives C, B, A, or A respectively if he does the reading. In this way, no student receives credit for the regular course as a result of collateral reading alone, and yet some recognition is awarded, except for the A student, who generally is motivated strongly enough by intellectual curiosity to need no recognition.

The weekly discussion in class suggested by the committee has been replaced by a five to ten minute discussion between the student and instructor on the completion of an assignment, these discussions taking place during a regular

[1] Vol. 35 (1928), pp. 221–228.

office hour, the student appearing whenever he has a report to make. Many teachers will, of course, object to the great burden of so many personal conferences, and the writer can do little but agree that it is a great burden if the instructor has very many students doing the reading. Capacities, however, vary widely, and if the instructor is not overburdened he will find these collateral reading conferences very interesting and enjoyable. Indeed, the present writer was most pleasantly surprised when he first undertook this work and found that even the duller students have their interest and appreciation awakened as a result of the readings and conferences. In fact he has come to believe that some of his best work has been done by means of the close and effective personal contact which this makes possible. Rare indeed are the students in whom there is no response to the compelling appeal of the story of mathematics and its creators to be found in the course of collateral reading, and the opportunity for vital guidance on the part of the instructor in the report conference is greater than it seems to be before it has been experienced. The present writer feels that the report conference is perhaps the most valuable element of the collateral reading course, and he recommends it strongly.

Further, the assignments themselves, as indicated in the appended copy of the latest edition of the list used by the writer, have been modified somewhat, and for two reasons. First, there is too much duplication in the books of the original list. Second, it has been found to be more satisfactory to have in the first few assignments a survey of the history of mathematical science so that the student may soon receive a perspective picture into which may fit the more detailed historical and biographical readings which are placed later in the list. This seems to develop an earlier hold on the interest of the all too prevalent type of student who feels—"what good is mathematics anyhow?"

It is only these first few assignments which need be read in a definite order. The later historical readings fit into the picture already obtained and the philosophical and miscellaneous readings are satisfactory anywhere. There is the additional advantage of this plan that fewer reserve copies of the books are needed. The writer recommends one copy per six students of the books of assignments 1 to 6, one copy per twelve students of the books of assignments $A7$ and $B7$, and one copy per seventy-five students of each of the others.

It is obvious that a number of books suitable for this purpose are missing from the list. This is due solely to library limitations and delays involved in securing out-of-print books. The list will, of course, grow as time passes inasmuch as books satisfactory for this purpose are appearing with ever increasing frequency.

In conclusion, it should be remarked that some of the reading list is usable even among secondary school students. Some years ago the present writer experimented with collateral reading with boys and girls fourteen and fifteen years old, and found that if done judiciously, good results could be obtained.

The following is a copy of the reading list which has been issued to students taking this work.

University of Florida—Collateral Readings in Mathematics

Mathematics is, par excellence, the science of exact thought. As such, it is the natural study of all persons possessing the attribute of intellectual curiosity, those seeking entrance to the life of the mind. Furthermore, as the language of the deductive sciences, its role in modern life is a very important one.

However, the place of mathematics in a liberal education rests not only upon these considerations, but also upon those of another sort. In the words of a great philosopher and mathematician, "Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry." (Bertrand Russell in "Mysticism and Logic.")

Unfortunately, in most cases this exaltation comes but gradually—and even then, generally only after rather extensive study. However, a *thoughtful and critical* consideration of certain aspects of mathematics and mathematicians does make it possible to obtain more quickly a glimpse of what Wordsworth describes as "that independent world created out of pure intelligence." It is these aspects which are considered in the readings listed below.

One assignment is to be read each week and a five or ten minute oral discussion with the instructor will be necessary upon the completion of each assignment. All the books listed are on reserve in the main reading room of the Library.

The first ten assignments must be read in the order in which they appear in the list. The remaining ones may be read in any order after the completion of the first ten.

1. *History of Elementary Mathematics*—F. Cajori. Pp. 1–92.
2. *Mathematics* (*Our debt to Greece and Rome*)—D. E. Smith. Pp. 1–89 and 160–164.
3. *History of Elementary Mathematics*—F. Cajori. Pp. 93–138.
4. *History of Mathematics in Europe*—J. W. N. Sullivan. All.
5. *Early Mathematics in North and South America*—F. Cajori. Pp. 11–86.
6. *Early Mathematics in North and South America*—F. Cajori. Pp. 90–149.

There are two alternatives for 7, 8, 9, 10.

A. 7. *Pioneers of Science*—Sir Oliver Lodge. Pp. 5–107.
   8. *Pioneers of Science*—Sir Oliver Lodge. Pp. 108–202.
   9. *Pioneers of Science*—Sir Oliver Lodge. Pp. 206–303.
   10. *Pioneers of Science*—Sir Oliver Lodge. Pp. 305–397.

B. 7. *Makers of Science*—I. B. Hart. Pp. 19–102.
   8. *Makers of Science*—I. B. Hart. Pp. 103–209.
   9. *Makers of Science*—I. B. Hart. Pp. 210–314.

10. Read either 11 or 12 or 13 below.

These may be read in any order:

11. *Galileo*—W. W. Bryant. All.
12. *Kepler*—W. W. Bryant. All.
13. *Archimedes*—Sir T. L. Heath. All.
14. *Introduction to Mathematics*—A. N. Whitehead. Pp. 7–41.
15. *Introduction to Mathematics*—A. N. Whitehead Pp. 42–86.
16. *Philosophy of Mathematics*—J. B. Shaw. Pp. 1–30.
17. *Philosophy of Mathematics*—J. B. Shaw. Pp. 154–168 and 186–195.
18. *Science and the Modern World*—A. N. Whitehead. Pp. 1–54.
19. *Thinking about Thinking*—C. J. Keyser. All.
20. *The Fourth Dimension Simply Explained*—H. P. Manning. Pp. 7–51.
21. *Mysticism and Logic*—B. Russell. Pp. 58–73 and 33–45.
22. *The Human Worth of Rigorous Thinking*—C. J. Keyesr. Pp. 1–60.
23. *The Human Worth of Rigorous Thinking*—C. J. Keyser Pp. 271–314
24. *Mathematical Recreations and Essays*—W. W. R. Ball. Pp. 44–61 and 170–187.
25. *Canterbury Puzzles*—A. E. Dudeney. Pp. 11–57.
26. *Canterbury Puzzles*—A. E. Dudeney. Pp. 58–109.
27. *Canterbury Puzzles*—A. E. Dudeney. Pp. 110–161.
28. *The Beautiful Necessity*—Claude Bragdon. All.
29. *Projective Ornament*—Claude Bragdon. All.
30. *The Curves of Life*—T. A. Cook. Pp. 1–40.
31. *The Curves of Life*—T. A. Cook. Pp. 41–80.
32. *The Curves of Life*—T. A. Cook. Pp. 81–114.

RECENT PUBLICATIONS

Edited by Roger A. Johnson, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y. and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Number, the Language of Science. A Critical Survey written for the Cultured Non-Mathematician. By Tobias Dantzig. New York, The Macmillan Company, 1930. xiv+260 pages. $3.50.

La Gamme. Introduction a l'Etude de la Musique. By P. J. Richard. Preface by Marius Casadesus. Paris, Librairie Scientifique Hermann et Cie, 1930. viii+232 pages. Paper, 28 francs.

Applied Mechanics. By Norman C. Riggs. New York, The Macmillan Company, 1930. xii+328 pages. $3.75.

Mechanics for Students of Physics and Engineering. By Henry Crew and K. K. Smith. New York, The Macmillan Company, 1930. xviii+372 pages. $4.00.

Intermediate Mechanics: Dynamics. By D. Humphrey. London, Longmans Green and Co., 1930. xii+382 pages. $4.20.

A Course of Analysis. By E. G. Phillips. Cambridge, The University Press, 1930. viii+362 pages.

Cours d'Analyse, Professé a l'École Polytechnique. Par J. Hadamard. Tome II. Paris, Librairie Scientifique Hermann et Cie, 1930. vi+722 pages. Paper, 140 francs.

The Principles of Quantum Mechanics. By P. A. M. Dirac. Oxford, The Clarendon Press, 1930. x+258 pages. $6.00.

Infinite Series. By Tomlinson Fort. Oxford, The Clarendon Press, 1930. iv+254 pages.

Mathematical Introduction to Economics. By Griffith C. Evans. McGraw-Hill Book Company, Inc., 1930. New York, xii+178 pages. $3.00.

Eléments de Trigonométrie Sphérique. Par G. Papelier. Paris, Librairie Vuibert, 1930. 166 pages. Paper, 20 francs.

Vorlesungen über Grundlagen der Geometrie. Von Kurt Reidemeister. Berlin, Julius Springer, 1930. Grundlehren der mathematischen Wissenschaften, Band XXXII. x+148 pages. Paper 11 marks, bound 12.60 marks.

Curve Piane Speciali Algebriche e Trascendenti. Teoria e Storia. By Gino Loria. Vol. II.—Curve Trascendenti, Curve Dedotte da Altre. Primo Edizione Italiana. Milan, Ulrico Hoepli, 1930. xii+440 pages.

Darstellende Geometrie. I: Elemente, Ebenflächige Gebilde. By Robert Haussner. Sammlung Goschen, 142. Berlin, Walter de Gruyter & Co., 1929. 208 pages.

Fouriersche Reihen. By Werner Rogosinski. Sammlung Goschen, 1022. Berlin, Walter de Gruyter & Co., 1930. 136 pages.

Elementary Theory of Finite Groups. By L. C. Mathewson, under the editorship of J. W. Young. Boston, Houghton Mifflin Co., 1930. x+166 pages. $2.50.

Differentialgleichungen Reeller Funktionen. By E. Kamke. Leipzig, Akademische Verlagsgesellschaft M.B.H., 1930. xiv+436 pages.

REVIEWS

Automorphic Functions. By Lester R. Ford. New York, McGraw-Hill Book Company, 1929. xii+333 pages. Price $4.50.

This book is the first containing an extensive systematic treatment of the theory of automorphic functions in English. The author has succeeded in presenting this difficult subject in a manner which makes it accessible to those who are familiar with the fundamentals of the theory of functions of a complex variable.

The first chapter, entitled "Linear Transformations," and the second, "Groups of Linear Transformations," give an especially clear treatment of topics which are of great importance in later work. The isometric circle, a concept introduced at an early stage, proves to be a valuable tool.

The third chapter is devoted to "Fuchsian Groups," and the fourth to some of the fundamental properties of automorphic functions.

In the fifth chapter existence theorems are established by means of the Poincaré theta series, and some properties of the theta functions are proved.

The sixth chapter, "The Elementary Groups," contains also a treatment of inversion in a sphere and of stereographic projection, as well as an interesting discussion of the regular solids.

Chapter VII,"The Elliptic Modular Functions," contains the definitions and some of the properties of the functions $J(\tau)$ and $\lambda(\tau)$ and a discussion of their relation to each other.

Chapter VIII gives a thorough treatment of conformal mapping, including the mapping of a plane simply connected region on a circle (with particular attention to the behavior of the mapping function on the boundary), the mapping of limit regions, and of simply connected finite-sheeted regions. These results are applied in Chapters IX and X to problems in uniformization.

The text closes with a brief discussion of the relation between the theory of automorphic functions and certain parts of the theory of linear homogeneous differential equations of the second order.

Not the least valuable part of the book is an excellent bibliography containing more than three hundred titles. The arrangement is chronological, but it is easy to locate all the references to a particular author, if desired, since the Author Index refers to the bibliography as well as to the text itself.

FRED W. PERKINS

Einführung in die Nicht-Euklidische Geometrie. By Hans Mohrmann. Akademische Verlagsgesellschaft M.B.H. Leipzig, 1930. XII+126 pages.

This introduction to non-Euclidean geometry by Professor Mohrmann makes very pleasant and interesting reading on account of its aggressiveness and boldness of point of view. In the discussions of the philosophical implications of non-Euclidean and its relation to Euclidean geometry one in almost reminded of the late Study's vigorous style.

The first three chapters are given in the main to such meditations and historic accounts. Here, as usual with many German authors, the importance of Gauss in the development of non-Euclidean geometry is stressed beyond reason. Lobatchewsky and Bolyai are so to speak merely mentioned "en passant." Again it seems to the reviewer that altogether too much credit is given to the philosopher Cornelius for his alleged remarkable evaluation of the essence of Euclidean geometry. Mohrmann dedicates the book to this Cornelius and maintains with the latter that Euclidean geometry is the only geometry of reality (Wirklichkeit). Anybody that cannot see that is mistaken! This matter has of course been controversial for a long time, but Cornelius' dogmatic position is by no means secure.

On the other hand the criticism on axiomatic research and its importance for geometric development is well deserved. As Study, Poincaré and others have pointed out on various occasions the over-emphasis of axiomatics since Hilbert's "Foundations" has done as much harm as good to geometric advancement.

Mohrmann makes some very sarcastic comments on the extreme "axiomatists" or "Mathematomanen."

The chapters which deal with Euclidean, projective and non-Euclidean geometry proper, are very well done. The position that analysis furnishes the only rigorous foundation for geometry is well taken.

It has moreover the advantage of great simplicity in the establishment of the groundwork of geometry. The reader will find Mohrmann's method of exposition of the foundations very direct and simple and for this reason very interesting.

The mapping process of the projective plane upon the Steiner-surface and by topological deformation upon the Steinitz-octatredron is new to the reviewer.

Various models of non-Euclidean geometries are discussed and the Cayley-Klein metrics is given adequate treatment. All this is done in a clear and competent manner.

Mohrmann's book should thus prove a welcome addition to the existing treatises on non-Euclidean geometry. It is written in a refreshing and fearless style which will stimulate independent thinking and create a critical attitude toward orthodox and dogmatic statements even when issued by acknowledged authorities.

ARNOLD EMCH

Leçons sur les Nombres Transfinis, By Waclaw Sierpinski; Gauthier-Villars, Paris, 1928. 240 pages.

This volume appears as a unit in the Borel series of monographs on function theory. In the preface, signed by Professor Borel himself, it is very aptly remarked that Professor Sierpinski requires no introduction to students of the theory of aggregates. His contributions to that theory, appearing with especial frequency in the *Fundamenta Mathematica*, are impressive. As is well known, the Polish school, which has developed the *Fundamenta* so successfully within a mere ten or twelve years, is primarily interested in problems of analysis situs admitting resolution by the methods of the theory of aggregates. Since Professor Sierpinski is one of the outstanding members of the school, it is a little surprising that he chose the subject of transfinite numbers for his monograph. An account, from his pen, of principles and methods in analysis situs *a la Polonaise* could not fail to afford exceptionally instructive reading. It would also provide an admirable companion volume for the Lefschetz monograph.

No scholar, however, is more competent than Professor Sierpinski to produce a really scientific treatise on transfinite numbers. The present work is precisely that. Moreover, the exposition is well arranged, fluent, and pleasing. The reviewer has no hesitation in commending this book to the attention of all who wish to acquaint themselves, from the most recent point of view, with fundamentals of the theory treated.

Part 1, pages 1–138, is particularly interesting. Here the full cognizance

taken of recent researches, published principally in the *Fundamenta*, affects even the point of view and the entire course of the exposition. Thus we find the important Bernstein theorem based upon a new theorem due to Banach (*Fundamenta*, vol. 6, page 236), for which a proof is given by Sierpinski. In fact, the discussion of inequalities between cardinal numbers becomes, in several respects, quite noteworthy. In this connection, for example, we find not only the König theorem, but also a modification, and partial extension, which leads to interesting conclusions. Numerous other features of the six chapters of Part 1 result directly from the advanced scientific outlook of the author. A conspicuous achievement is the adroit handling of the vexatious Zermelo principle of selection (Allgemeines Auswahlprinzip). Without committing himself in this controversial matter, Professor Sierpinski directs attention to many theorems, some of them very surprising indeed, which can not be established unless the Zermelo postulate is accepted. He also shows the equivalence of this postulate and the affirmative solution of the problem of trichotomy.

Following the analysis of cardinal numbers in Part 1, we find, in Part 2, a systematic and readable exposition of the theory of ordinal numbers. The author begins with a discussion of types of simple order, and of the Dedekind-process extension of dense to continuous types. He then proceeds quickly to well-ordered types and the associated ordinal numbers. Having studied these numbers along familiar lines, he presents the system of classes of ordinals and their corresponding cardinals (Alephs). The existence of a well-ordering for an arbitrary aggregate is exhibited as equivalent to the affirmative solution of the problem of trichotomy and to the Zermelo principle of selection. An excellent unsolved problem, due to Souslin (*Fundamenta*, vol. 1, page 223) is re-recorded on page 153.

The typography is good, and there are apparently no serious errors.

LESTER S. HILL

Grundlagen der Analysis. Das Rechnen mit ganzen, rationalen, irrationalen, komplexen Zahlen. By Edmund Landau. Leipzig, Akademische Gesellschaft, 1930, xiv+134 pages.

Mastering the theory of the real number system is one of the severest tests to which students of higher mathematics are put. It is not easy for the teacher to convince his class that he is serious when he defines a rational number as an ordered pair of integers, or a real number as a class of rational numbers. Uprooting the student's haphazard notions of number, and replacing them by the theories of Dedekind, Cantor and Weierstrass is a maneuver which has its risks. The prognosis is favorable only when the patient has high vitality at the outset.

Professor Landau's little book looks as if it may have the qualities of a specific. Certainly no clearer treatment of the foundations of the number system

can be offered to a student. Starting with Peano's five axioms for the natural numbers, Landau develops, with the utmost elegance, precision and completeness, the theories of the natural numbers, fractions, real numbers and complex numbers. Never before has this subject been treated with such explicitness.

The book is written in the crystalline style for which Landau is so well known. There are two prefaces, one for students, one for authorities. Speaking to the student, whom he "duzes," Landau says: "Bitte vergisz was du auf der Schule gelernt hast, denn du hast es nicht gelernt."

One can only be thankful to the author for this fundamental piece of exposition, which is alive with his vitality and genius.

J. F. RITT

Projective Geometry. By John Wesley Young. The Carus Mathematical Monographs, number four. The Open Court Publishing Company, 1930. ix+185 pages.

It may be stated at the outset that this fourth volume of the Carus Mathematical Monographs will prove to be an excellent auxiliary text for students in a first course of projective geometry.

The decision of the author "to develop on an intuitive basis the concepts and the properties of projective space, without any admixture of metric ideas," is very fortunate. This idea is carried out in five chapters in which topics like perspective drawing, projection and section, correspondence, projective transformations, the principle of duality, Desargues' theorem, the fundamental theorem, Pascal's and Brianchon's theorems, etc., are being treated.

In chapter VI metric properties considered from a projective standpoint are discussed: parallel lines, midpoint, the classification of conics, perpendicular lines, the orthogonal involution, angle bisectors, axes of conics, foci, construction of an involution by means of circles, metric definitions of a conic. This is always particularly welcomed by the beginning student, because it forms for him a bridge between the old and the new soil.

The next three chapters bear a more theoretical aspect, but are of the utmost importance for a more advanced understanding of projective geometry. Thus chapter VII deals with groups of projective transformations, chapter VIII with the algebra of points and the introduction of analytic methods, and the concluding chapter with groups and geometries.

That the group-concept is here brought to the foreground is a proof for the modern standpoint which the author is taking in the teaching of even elementary projective geometry. In this manner the fact that projective geometry and its analytic formulation may be established without any Euclidean metric concepts whatsoever may be made clear by a relatively simple method.

This is done in an excellent fashion. But it seems to the reviewer that the exposition of this important fact could be simplified still more by stressing the important role of analysis in this demonstration. I have in mind the mechanism of projective coordinates $(A_1A_3E_2P_2)=x=x_1/x_3$, etc., based upon von Staudt's

harmonic quadrangle — construction + projectivity + continuity axiom, in which the Cartesian plane is implicitly contained by making a special assumption about the infinite region of the plane. The complete analytic equivalence only assures absolute rigor.

The last chapter gives the student a bright outlook on the geometries which are associated with various groups. The groups of collineations which leave given conics invariant offer the simplest and shortest approach to non-Euclidean geometry. This fact alone suffices to show the importance of the group-concept in geometry. Thus, the author concludes his monograph with a very stimulating subject which will urge the good student to penetrate farther into the beautiful realms of geometry.

The book is very clearly written and typographically well done. The reviewer has noticed very few errors. In the historic account perhaps too much emphasis is placed upon Poncelet with the exclusion of some equally important pioneers in this field.

But these criticisms are not important enough to detract from the value of the very creditable monograph which Professor Young has written.

ARNOLD EMCH

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3456. *Proposed by C. A. Rupp, Pennsylvania State College.*

Show that the equation

$$\sum_k \frac{a_{ij}}{a_{ii}a_{jj}} x_k = 0, \quad (a_{ij} = a_{ji}, \ ijk \text{ a permutation of } 123)$$

represents a Steiner-Plücker line of the Pascal hexagons whose vertices are the intersections of the conic of equation

$$\sum a_{ij}x_ix_j = 0 \quad (i, j = 1, 2, 3)$$

with the sides of the coordinate triangle.

3457. *Proposed by E. T. Bell, California Institute of Technology.*

If $\binom{m}{s}$ denotes the coefficient of x^s in $(1+x)^m$ when m is an integer >0, and $\binom{0}{0}$ denotes 1, show that if n is an integer >0,

$$2\sum_{j=0}^{n}2^{2n-2j}(-1)^j\binom{2j}{j}\binom{n}{j}\binom{n+j}{j} = [1+(-1)^n]\binom{n}{n/2}^2.$$

3458. *Proposed by J. Rosenbaum, Milford, Connecticut.*

In the triangle ABC, a tangent is drawn to the incircle cutting the lines BC and AC at X and Y respectively. Find the locus of the intersection of AX and BY.

3459. *Proposed by Norman Anning, University of Michigan.*

It is observed that $3003=\binom{15}{5}=\binom{14}{6}$. Solve in positive integers the equation

$$\binom{x+1}{y}=\binom{x}{y+1}$$

3460. *Proposed by B. F. Kimball, University of New Hampshire.*

Show that the integral:

$$V_n = \int_0^1\int_0^1\cdots\int_0^1 \frac{x_1{}^r + x_2{}^r + \cdots + x_n{}^r}{x_1+x_2+\cdots+x_n}dx_1dx_2\cdots dx_n$$

converges to the limit $2(r+1)^{-1}$ when n increases without limit, and that the product $n[V_n-2(r+1)^{-1}]$ remains bounded; r to be any real number greater than -1. A generalization of 3408[1930, 38].

3461. *Proposed by Boyd C. Patterson, Hamilton College.*

Given a triangle $A_1A_2A_3$, its circumcircle (O), and a line L in its plane. To prove the following construction of the point P on (O) whose Simson line with respect to $A_1A_2A_3$ is parallel to L: Through the circumcenter O draw a line OX parallel to L. If M is one of the points where OX cuts (O) then P may be located by the relation

$$\text{arc } MP \equiv \text{arc } MA_1 + \text{arc } MA_2 + \text{arc} MA_3 \qquad (\text{mod } 4\pi)$$

3462. *Proposed by R. C. Colwell and O. R. Ford, West Virginia University.*

A circular hoop, which is free to move on a smooth horizontal plane, has sliding on it a small ring of $1/n$th its mass, the coefficient of friction between the two being μ. Initially the hoop is at rest and the ring has an angular velocity ω round the hoop. Show that the ring comes to rest relative to the hoop after a time $(1+n)/\mu\omega$.

SOLUTIONS

3350. [1928, 494] *Proposed by Frank Irwin, University of California.*

Find the sum of all products of n factors each, the factors being chosen from the numbers 2, 3, 4, $\cdots$, $2+t$. Repetitions of factors are allowed; order is not considered; and two products $2\cdot 6$ and $3\cdot 4$ are regarded as distinct, though numerically equal.

Solution by Emma T. Lehmer, Brown University.

The problem is equivalent to finding the coefficient $E_n{}^m$ of x^{m-n} in the polynomial

$$P(x, m) = (x+2)(x+3)\cdots(x+m+1) = \sum_{k=0}^{m} E_{m-k}^{m} x^k, \quad E_0{}^m = 1,$$

where $t+1$ is replaced by m.
If we substitute $x+1$ for x in the above and multiply the result by $x+2$, we get

$$(x+2)P(x+1, m) = P(x, m+1),$$

or

$$(x+2)\sum_{k=0}^{m} E_{m-k}^{m}(x+1)^k = \sum_{k=0}^{m+1} E_{m+1-k}^{m+1} x^k.$$

Equating coefficients on both sides of this last equation, we have

$$(1)\quad E_n^{m+1} = E_n^m + \sum_{k=m+1-n}^{m}\left[2\binom{k}{m+1-n} + \binom{k}{m-n}\right]E_{m-k}^{m} (n = 1 \cdots m).$$

$$E_{m+1}^{m+1} = 2\sum_{k=0}^{m} E_{m-k}^{m}.$$

Since $(x+m+2)P(x, m) = P(x, m+1)$, we find by equating coefficients, the following relation:

$$(2)\qquad E_n^{m+1} = (m+2)E_{n-1}^{m} + E_n^m \; (n = 1 \cdots m)$$

$$E_{m+1}^{m+1} = (m+2)E_m^m.$$

From (1) and (2) we derive

$$(3)\; 2\binom{m}{n} + \binom{m}{n+1} = -\sum_{k=1}^{n-1}\left[2\binom{m-k}{n-k} + \binom{m-k}{n-k+1}\right]E_k^m + nE_n^m$$

$$(n = 1 \cdots m).$$

We now solve by determinants the set of m equations (3), of which the following are the first three:

$$2\binom{m}{1} + \binom{m}{2} = E_1{}^m,$$

$$2\binom{m}{2} + \binom{m}{3} = -\left[2\binom{m-1}{1} + \binom{m-2}{2}\right]E_1{}^m + 2E_2{}^m,$$

$$2\binom{m}{3} + \binom{m}{4} = -\left[2\binom{m-1}{2} + \binom{m-1}{3}\right]E_1{}^m$$

$$-\left[2\binom{m-2}{1} + \binom{m-2}{2}\right]E_2^m + 3E_3^m.$$

Hence

$$E_n^m=\frac{1}{n!}\begin{vmatrix} \left[2\binom{m}{1}+\binom{m}{2}\right] & -1 & 0 & \cdots & 0 \\ \left[2\binom{m}{2}+\binom{m}{3}\right] & \left[2\binom{m-1}{1}+\binom{m-1}{2}\right] & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left[2\binom{m}{n}+\binom{m}{n+1}\right] & \left[2\binom{m-1}{n-1}+\binom{m-1}{n}\right] & \left[2\binom{m-2}{n-2}+\binom{m-2}{n-1}\right] & \cdots & \left[2\binom{m-n+1}{1}+\binom{m-n+1}{2}\right] \end{vmatrix}$$

The first three of these functions of m are

$$E_1^m=\frac{1}{2}m(m+3),\quad E_2^m=\frac{m(m-1)(3m^2+17m+26)}{24},$$

$$E_3^m=\frac{m(m-1)(m-2)(m+3)(m^2+5m+8)}{48}.$$

3405 [1930, 37]. *Proposed by Paul Wernicke, Washington, D. C.*

Given in a plane three concurrent lines and a point P. Construct an equilateral triangle having its summits on the three lines and having the point P on one of its sides. Determine the number of solutions.

Solution by Lloyd C. Bagby, Linsly Institute of Technology.

Let l_1, l_2, l_3 be the three straight lines concurrent in O. On any one of the lines, say l_2, select a point Q, not at O, and construct the equilateral triangle QMN with the side MN lying upon l_1. Draw the straight lines NN', MM' so that they pass through Q', the symmetric of Q with respect to l_1. Then NN' and MM' are the loci of the third vertices of all equilateral triangles having one vertex at Q and another on l_1. Let NN' and MM' cut l_3 in N' and M', respectively. Then N' and M' determine two equilateral triangles with the sides QN' and QM', respectively, whose third vertices lie upon l_1. There are then three triangles homothetic to each of these triangles having a given point P on one of their sides or on a side extended. There are therefore six solutions.

Also solved by E. M. Berry, J. D. Leith, J. H. Neelley, A. Pelletier, J. A. Powers, O. J. Ramley, J. Rosenbaum, S. Kullback, and the Proposer.

3409 [1930, 94]. *Proposed by Norman Anning, University of Michigan.*

When $3b^2>8ac$, the curve, $y=ax^4+bx^3+cx^2+dx+e$, has two real inflectional tangents. Prove that they cut off equal areas.

Solution by Andrew G. Clark, Colorado Agricultural College.

If $F(x)$ is a polynomial, the area cut from $y=F(x)$ by the chord through the points of intersection of the two fixed lines, $x=x_1$ and $x=x_2$, with $y=F(x)$ is

invariant to the arbitrary character of the coefficients of the first and zero degree terms of $F(x)$.

This theorem holds providing the chord does not intersect $y=F(x)$ for $x_1<x<x_2$. The proof becomes evident when the expression,

$$\int_{x_1}^{x_2} F(x)dx - \tfrac{1}{2}(x_2 - x_1)[F(x_2) + F(x_1)],$$

for the area cut off by the chord is examined and found independent of the coefficients of the first and zero degree terms of $F(x)$.

The quartic to be considered loses nothing in generality if transformed into $y=f(x)=x^4+cx^2+dx+e$; it is clear that for the equation in this form, $c<0$ is the condition for two real points of inflection. Now, for convenience, replace c by $-6m^2$, whence

$$y = x^4 - 6m^2x^2 + dx + e;\ y' = 4x^3 - 12m^2x + d;\ \text{and}\ y'' = 12x^2 - 12m^2.$$

It is obvious, then, that the inflection points are $m, f(m)$ and $-m, f(-m)$, and that the equation of the tangent through the former is $y-f(m)=f'(m)\ (x-m)$. Now solving this equation with $y=f(x)$, we get the equation

$$f(x) - f(m) - f'(m)(x - m) = 0,\ \text{or}\ (x - m)^3(x + 3m) = 0.$$

The roots of this cubic are immediately seen to be independent of the coefficients d and e, in $f(x)$.

Therefore, if it can now be shown that for some particular values of d and e, the areas cut off by the two inflectional tangents are equal, an immediate application of our prestated theorem will complete the proof for the general quartic. This is easily done, since, for $d=0$, the quartic is symmetric to the y-axis, and the areas cut off are obviously equal.

Also solved by A. Pelletier, F. L. Wilmer, and Paul Wernicke.

3410. [1930, 94]. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Construct a triangle given the base, the difference of the base angles, and the bisector of the vertical angle.

Solution by Louise E. Wiederhold, William Smith College.

Let ABC be the required triangle, O its circumcircle, AU the bisector of angle BAC, intersecting BC at U, AD the altitude to BC from A, and AO the circumradius through A. Then the angle DAO equals angle ABC minus angle ACB. Since AU is the bisector of angle BAC, it is also the bisector of the angle DAO. Therefore angle DAU equals $\frac{1}{2}$(angle ABC minus angle ACB). Since angle ABC minus angle ACB is given, $\frac{1}{2}$(angle ABC minus angle ACB) is also known, which together with the length AU determines the right triangle DAU, which can be constructed. The points D, U then determine the line, l, on which BC is to lie. Since the point U on the line l is known, the point U' which is the intersection of the external bisector of angle BAC (perpendicular to AU at A) with the line l can be found. The circle K on UU' as diameter passes through

A. The points B, C are inverse points with respect to this circle since $(U'UBC)$ equals -1. A circle on BC as diameter would therefore be orthogonal to the circle K. But on l there is an infinity of pairs of points which are inverse with respect to K. However there is only one pair of points which are the length BC apart and which (considering AU as the internal bisector of angle BAC) fit the case. In order to find these points B, C it is necessary to construct a circle M orthogonal to K, of radius $\frac{1}{2}BC$. The center of this circle can be found in this manner: construct a right triangle with legs equal to $\frac{1}{2}UU'$ and $\frac{1}{2}BC$. The hypotenuse of this triangle will equal the distance of the center of M from the center of K on line l. (U must be located between K and M). The circle with M as center and $\frac{1}{2}BC$ as radius will cut the line UU' in the required points. B, C, with A determine the triangle.

Also solved by E. J. Buddenbaum, Solomon Kullbak, J. D. Leith, R. K. Morley, William Orange, A. Pelletier, W. A. Rees, Ralph Tocker, Paul Wernicke, and Frank L. Wilmer.

3411. [1930, 94] *Proposed by J. Rosenbaum, Milford, Conn.*

A particle within a closed plane curve is attracted by every point of the curve with a force proportional to the distance from the particle to the point. Find the position of equilibrium of the particle.

Solution by the Proposer.

This problem is a generalization of 3395, November, 1929. There the centers of attraction were finite in number, while here they are all the points of a continuum. Since there, the position of equilibrium is the center of mass of n equal masses placed at the n centers of attraction regardless of the value of n, of the magnitude of the masses, or of the manner of distribution of the centers of attraction, it follows that here, too, the position of equilibrium is the center of mass of the arc of the given curve.

It thus appears that the curve does not need to be a plane curve or closed. The centers of attraction may be even all the points of a given area or volume.

Also solved by A. Pelletier and Paul Wernicke.

NOTES AND NEWS

Professor George Birkhoff has been elected Corresponding member of the Royal Academy of Sciences of the Institute of Bologna.

Dr. M. A. Basoco, of the California Institute of Technology, has been appointed assistant professor of mathematics at the University of Nebraska.

Dr. Theodore Bennett has been appointed assistant professor of mathematics at the University of Wisconsin.

Dr. R. V. Churchill has been promoted to an assistant professorship of mathematics at the University of Michigan.

Assistant Professor A. J. Cook has been promoted to an associate professorship of mathematics at the University of Alberta.

Assistant Professor P. H. Daus, of the University of California at Los Angeles, has been promoted to an associate professorship of mathematics.

Dr. H. A. Davis, of West Virginia University, has been promoted to an assistant professorship of mathematics.

Assistant Professor W. H. Durfee, of Hobart College, has been promoted to a professorship of mathematics.

Dr. C. M. Erikson, of Michigan State Normal College, has been promoted to an assistant professorship of mathematics.

Mr. J. H. Fithian has been promoted to an assistant professorship of mathematics at the Newark College of Engineering.

Mr. H. J. Gay has been promoted to an assistant professorship of mathematics at the Worcester Polytechnic Institute.

Dr. Marion C. Gray has been appointed to a position in the department of development and research of the American Telephone and Telegraph Company.

Dr. E. K. Haviland has been appointed assistant professor of mathematics at Lehigh University.

Dr. G. A. Hedlund has been appointed associate in mathematics at Bryn Mawr College.

Dr. H. M. Hosford has been appointed to a professorship at the University of Arkansas.

Assistant Professor Emma Hyde has been promoted to an associate professorship of mathematics at Kansas State Agricultural College.

Miss Myra I. Johnson has been appointed head of the department of biological science at Whitney Women's College.

Associate Professor R. A. Johnson, of Hunter College, has been appointed associate professor of mathematics at Brooklyn College of the City of New York.

Assistant Professor V. F. Lenzen has been promoted to an associate professorship of physics at the University of California, Berkeley.

Mr. A. J. Lewis has been promoted to an assistant professorship of mathematics at the University of Denver.

Assistant Professor Mayme I. Logsdon has been promoted to an associate professorship of mathematics at the University of Chicago.

Assistant Professor H. F. MacNeish, of the College of the City of New York, has been promoted to an associate professorship of mathematics.

Assistant Professor E. B. Mode has been promoted to a professorship of mathematics at Boston University.

Assistant Professor C. H. Rawlins, of the United States Naval Academy, has been promoted to an associate professorship of mathematics.

Dr. Edward Saibel has been appointed assistant professor of mechanics at the Carnegie Institute of Technology.

Dr. J. A. Shohat has been appointed lecturer in mathematics at the University of Pennsylvania.

Assistant Professor C. A. Shook, of Yale University, has been appointed assistant professor of mathematics at Lehigh University.

Assistant Professor S. P. Shugert has been promoted to a professorship of mathematics at the University of Pennsylvania.

Assistant Professor E. B. Starke, of Rutgers University, has been promoted to an associate professorship of mathematics.

Associate Professor F. C. Touton, of the University of Southern California, has been promoted to be professor of educational research and vice-president of the University.

Dr. Morgan Ward has been promoted to an assistant professorship of mathematics at the California Institute of Technology.

Assistant Professor F. M. Weida, of Lehigh University, has been appointed associate professor of mathematics at George Washington University.

The following appointments to instructorships are announced:
Brooklyn College of the City of New York: Mr. A. W. Landers, Jr.
University of Buffalo: Dr. George C. Munro of the University of Michigan.
Case School of Applied Science: Dr. O. E. Brown.
Lehigh University: Mr. F. S. Beale and Dr. E. H. Cutler.
Los Angeles Junior College: Mr. E. J. Hills.
State Teachers College, Oshkosh: Miss Irene Price.
Princeton University: Mr. A. L. Foster, Mr. J. E. Merrill.
Purdue University: Mr. H. F. S. Jonah.
Radcliffe College: Dr. Mary Curtis Graustein (tutor).
Rutgers University: Mr. H. S. Grant.
Sweet Briar College: Dr. Ethel I. Moody.
South Dakota State College: Mr. Orlin L. Walder and Miss Irene Wente.
United States Naval Academy: Mr. Ernest Hawkins.
Yale University: Mr. Hassler Whitney.

Dr. James M. Taylor, professor emeritus of mathematics at Colgate University, died July 31, 1930 at the age of eighty-six.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TEASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.

INDIANA, Earlham College, May 2-3.

IOWA, Ames, Iowa, May 2-3.

KANSAS, February 15.

KENTUCKY, Lexington, Ky., April 5.

LOUISIANA-MISSISSIPPI, Cleveland, Miss., March 7-8.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, May 10.

MICHIGAN, Ann Arbor, Mich., March 22.

MINNESOTA, Carleton College, May 17.

MISSOURI, Columbia, Mo., November 28.

NEBRASKA, Peru, Neb., May 9.

OHIO, Columbus, Ohio, April 3.

PHILADELPHIA, Philadelphia, Pa., Nov. 29.

ROCKY MOUNTAIN, Denver, Colo., April 11-12.

SOUTHEASTERN, Atlanta, Ga., May 2-3.

SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, Calif., March 8.

TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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THE AMERICAN MATHEMATICAL MONTHLY

By H. S. EVERETT, The University of Chicago

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PROBLEMS—SOLUTIONS

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CORRIGENDA

Volume XXXVI, 1929:

P. 545, first line, for "K" read "k."
Pp. 532, 21st line, delete "s" in "New Yorks."
P. 558, under "Swezy," for "E.D." read "G.D."

Volume XXXVII, 1930:

P. 39, 4th line frim the bottom, for "S. Pelletier" read "A. Pelletier."
P. 162, 28th line, omit "e" in "Whiteman."
P. 203, 19th line, for "W.M." read "W. I."
P. 250, 11th line from bottom, for "cos" read "sin."
P. 386, 4th line from bottom, for "G" read "J."
P. 404, last line, for "μ^*" read "μ_*."
P. 486, 8th line, for "substraction" read "subtraction."
P. 504, 11th line, for "octatredron" read "octahedron."
P. 510, 7th line from bottom, for "Ramley" read "Ramler."

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE CUBIC AND BIQUADRATIC EQUATIONS VIETA'S TRANSFORMATION IN THE COMPLEX PLANE

By ARCHIBALD HENDERSON and A. W. HOBBS, University of North Carolina

1. In this paper, a number of methods, possessing a measure of novelty, are presented for the solution of the cubic and biquadratic equations. The transformation attributed to Vieta (1591) presents interesting points if treated as a problem in complex variables. The method employed by Heilermann[1] in the algebraic solution of the biquadratic affords an effective method of solving geometrically the biquadratic equation.

2a. *The Reduced Cubic.* If we replace w in the cubic

$$w^3 + pw + q = 0 \tag{1}$$

by $u+iv$ we obtain, after equating real and imaginary parts separately to zero:

$$u^3 - 3uv^2 + pu + q = 0, \tag{2}$$

$$3u^2v - v + pv = 0. \tag{3}$$

The real intersections of these two curves when considered as complex numbers are the roots of the original equation.

The transformation attributed to Vieta for the solution of the cubic is

$$w = z - (p/3z).$$

If $w = u + iv$ and $z = x + iy$, this transformation represents an inversion in the circle

$$x^2 + y^2 = - p/3,$$

followed by a reflection in the axis of reals, and a translation.

Putting $u+iv$ for w, and $x+iy$ for z we get:

$$u = \frac{x(x^2 + y^2 - \frac{1}{3}p)}{x^2 + y^2}, \quad v = \frac{y(x^2 + y^2 + \frac{1}{3}p)}{x^2 + y^2}.$$

Applying this transformation to equations (2) and (3) we obtain:

$$(x^3 - 3xy^2)(x^2 + y^2)^3 + q(x^2 + y^2)^3 - p^3(x^3 - 3xy^2)/27 = 0, \tag{4}$$

$$y(x^2 + y^2 + p/3)(x^4 + 2x^2y^2 + y^4 - px^2/3 - py^2/3 + p^2/9)(3x^2 - y^2) = 0. \tag{5}$$

The first of these equations, (4), represents a curve of the ninth degree having a triple point at the origin with distinct tangents:

$$x = 0, \quad x - y\sqrt{3} = 0, \quad x + y\sqrt{3} = 0.$$

These lines are also asymptotes of the curve. The only real parts of the second curve are:

$$y = 0, \quad x\sqrt{3} - y = 0, \quad x\sqrt{3} + y = 0,$$

[1] Zeitschrift für Mathematik und Physik, vol. 44 (1898).

and when p is negative, the circle $x^2+y^2+\frac{1}{3}p=0$. An interesting fact about the first curve is that it is its own inverse. In fact it is unaltered under inversion in the circle $x^2+y^2+\frac{1}{3}p=0$, also under reflection in the axis of reals. The six real intersections of the two curves are paired in such a way that if we consider one point of intersection with its reflected inverse and take their vector sum we get one of the values of w satisfying the original cubic.

Figure 1 is drawn for a specially favorable case, p being -3 and q being 52.

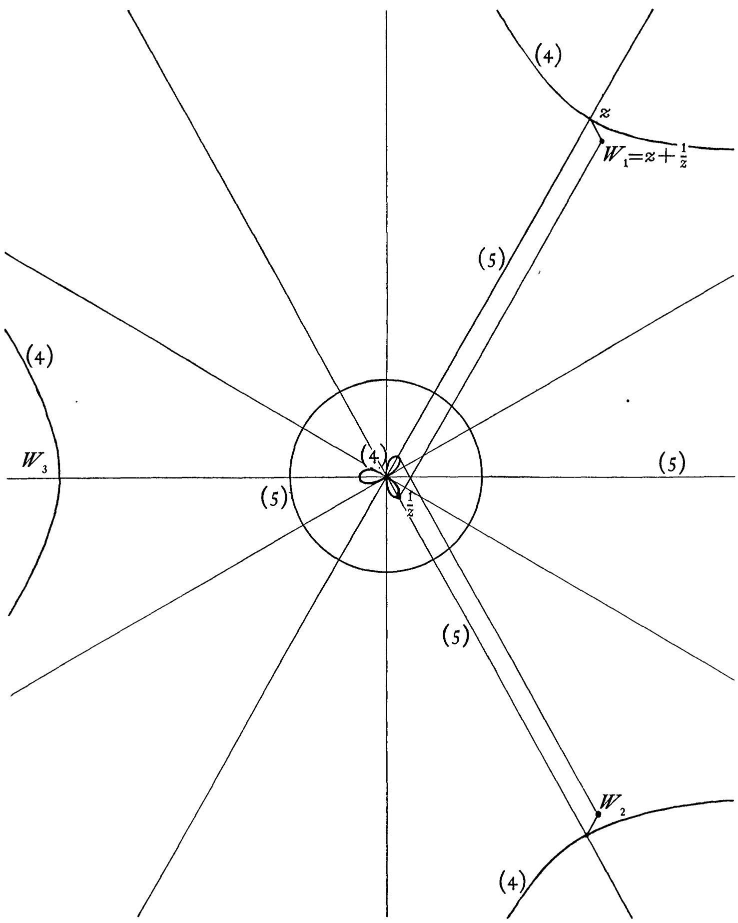

FIG. 1

2b. *The General Cubic.* Consider the general cubic

$$t^3 + 3at^2 + 3bt + c = 0 \quad (a, b, c \text{ real}). \tag{6}$$

Employing the substitution $t=s-a$, we obtain:

$$s^3 - 3(a^2 - b)s + (2a^3 - 3ab + c) = 0. \tag{7}$$

Further reducing this form, $s^3-3ls+m=0$, by the substitution $s=\nu W$, we have:

$$W^3 - \frac{3l}{\nu^2}W + \frac{m}{\nu^3} = 0.$$

Setting $l=\nu^2$, we have the form:

$$W^3 - 3W \pm K = 0, \quad \text{where} \quad K = m/l^{3/2}. \tag{8}$$

There are four cases. If $l=+\lambda$, $m=+\mu$; or $l=+\lambda$, $m=-\mu$, where λ and μ are both positive quantities, we have:

$$W^3 - 3W \pm \rho = 0, \quad \text{where} \quad \rho = \mu/\lambda^{3/2}. \tag{8A}$$

If however $l=-\lambda$, $m=+\mu$; or $l=-\lambda$, $m=-\mu$, where λ and μ are both positive quantities, we have

$$W^3 - 3W \pm \rho i = 0, \quad \text{where} \quad \rho = \mu/\lambda^{3/2}. \tag{8B}$$

Employing Vieta's transformation in the simplest form

$$W = z + (1/z) \quad \text{where} \quad W = u + iv, \quad z = x + iy \tag{9}$$

we have

$$u + iv = x + iy + (x + iy)^{-1}. \tag{10}$$

Solving we obtain:

$$u = \frac{x(x^2 + y^2 + 1)}{x^2 + y^2}, \quad v = \frac{y(x^2 + y^2 - 1)}{(x^2 + y^2)}. \tag{11}$$

Clearly

$$W = \frac{x(x^2 + y^2 + 1)}{x^2 + y^2} + \frac{y(x^2 + y^2 - 1)}{x^2 + y^2}i$$

is invariant under the substitution

$$x = \frac{x_1}{(x_1^2 + y_1^2)}, \quad y = \frac{-y_1}{(x_1^2 + y_1^2)}$$

as is otherwise obvious, by the substitution $z_1 = 1/z$ in the formula $W = z + (1/z)$.

To get the inverse transformation, we employ $z+(1/z)=u+iv$, which gives

$$z \equiv x + iy = \tfrac{1}{2}\left\{u + iv \pm [(u^2 - v^2 - 4) + 2uvi]^{1/2}\right\}. \tag{11A}$$

Setting

$$(\alpha + i\beta)^2 = (u^2 - v^2 - 4) + 2uvi$$

we obtain

$$\begin{aligned} \alpha^2 &= \tfrac{1}{2}\left\{(u^2 - v^2 - 4) + [u^4 + v^4 + 16 + 2u^2v^2 - 8u^2 + 8v^2]^{1/2}\right\}, \\ \beta^2 &= \tfrac{1}{2}\left\{-(u^2 - v^2 - 4) + [u^4 + v^4 + 16 + 2u^2v^2 - 8u^2 + 8v^2]^{1/2}\right\}. \end{aligned} \tag{12}$$

Hence from

$$x=\tfrac{1}{2}(u\pm\alpha),\qquad y=\tfrac{1}{2}(u\pm\beta),$$

with α and β from (12), we have

$$(13)\qquad \begin{aligned} x &= \frac{(\sqrt{2})u \pm \{[u^4+v^4+16+2u^2v^2-8u^2+8v^2]^{1/2}+(u^2+v^2-4)\}^{1/2}}{2\sqrt{2}},\\ y &= \frac{(\sqrt{2})v \pm \{[u^4+v^4+16+2u^2v^2-8u^2+8v^2]^{1/2}-(u^2+v^2-4)]^{1/2}}{2\sqrt{2}}. \end{aligned}$$

Making the substitution $W=u+iv$ in equation ($8A$), we have

$$(14)\qquad u^3-3uv^2-3u\pm\rho=0,\quad v(3u^2-v^2-3)=0.$$

Plotting, in either case, the paired loci, we obtain in general three points of intersection; (u_1, v_1), (u_2, v_2), and (u_3, v_3). These supply the three roots W_1, W_2, W_3 of the original equation. By this method we are spared the necessity of plotting nonic curves. The three values of (u, v), set in equations (13), furnish the required values of x and y, which likewise furnish the three values of z.

For the other case, $W^3-3W\pm\rho i=0$, we have

$$(15)\qquad u(u^2-3v^2-3)=0,\quad v^3-3u^2v+3v\pm\rho=0.$$

We then proceed as in the former case.

3. *Examples*:

A. Given $t^3+6t^2+9t+4=0$. Making the substitution $t=W-2$, we have $W^3-3W+2=0$. Using the substitution $W=u+iv$, we have

$$u^3-3uv^2-3u+2=0,\quad v(3u^2-v^2-3)=0.$$

The plotted or solved values for the intersections are $(1, 0)$, $(1, 0)$, $(-2, 0)$. Hence the values of W are $1, 1, -2$; and therefore the roots of the original equation are $-1, -1, -4$. The first curve above may be plotted by points from the equation,

$$v=\pm[(u^3-3u+2)/3u]^{1/2}.$$

The second curve consists of the straight line $v=0$ and the hyperbola, $u^2-\tfrac{1}{3}v^2=1$.

B. Given $t^3-6t^2+18t-40=0$. Making in order the substitutions $t=s+2$ and $s=+iW\sqrt{2}$, we have

$$s^3+6s-20=0 \quad\text{and}\quad W^3-3W-5i\sqrt{2}=0.$$

Equations (15) for this case are

$$u(u^2-3v^2-3)=0,\quad v^3+3(1-u^2)v+5\sqrt{2}=0.$$

Combining $u=0$ with the second equation, we have $v^3+3v+5\sqrt{2}=0$, giving one real root $v_1=-\sqrt{2}$. Hence $W_1=-i\sqrt{2}$. Combining $u^2-3v^2-3=0$ with the second equation, we have

$$8v^3+6v-5\sqrt{2}=0,$$

giving the single real value $\frac{1}{2}\sqrt{2}$. Since from the first equation $u = \pm 3\sqrt{2}/2$, the corresponding values of W are

$$W_2 = -\tfrac{1}{2}\sqrt{2}(3+i), \quad \text{and} \quad W_3 = -\tfrac{1}{2}\sqrt{2}(3-i).$$

The corresponding values of z, derived from equation (11*A*) are

$$\begin{aligned}
z_1 &= \tfrac{1}{2}\sqrt{2}(\sqrt{3}-1)i,\\
z_1' \equiv z_1^{-1} &= -\tfrac{1}{2}\sqrt{2}(\sqrt{3}+1)i,\\
z_2 &= \tfrac{1}{4}\sqrt{2}(\sqrt{3}+1)(\sqrt{3}+i),\\
z_2' \equiv z_2^{-1} &= \tfrac{1}{4}\sqrt{2}(\sqrt{3}-1)(\sqrt{3}-i);\\
z_3 &= -\tfrac{1}{4}\sqrt{2}(\sqrt{3}+1)(\sqrt{3}-i),\\
z_3' \equiv z_3^{-1} &= -\tfrac{1}{4}\sqrt{2}(\sqrt{3}-1)(\sqrt{3}+i).
\end{aligned}$$

From these six values of z, which are paired as indicated, the six pairs of values of x and y may be written down by inspection.

The geometrical aspects of the problem are clarified by figure 2. For instance, suppose we have z_1, z_2, and z_3, since only one set of three values is required. Choosing z_2, find its inverse point with reference to the unit circle, and then reflect this inverse point in the axis of reals. Complete the parallelogram formed by Oz_2 and Oz_2', giving the point W_2, as indicated by the formula

$$W_2 = z_2 + z_2^{-1} = z_2 + z_2'.$$

4. *The Biquadratic.* Employing Heilermann's method, we write the biquadratic equation,

$$t^4 + a_1t^3 + a_2t^2 + a_3t + a_4 = 0 \tag{16}$$

in the form

$$t^4 + a_1t^3 + \lambda t^2 + (a_2 - \lambda)t^2 + a_3t + a_4 = 0. \tag{16$'$}$$

Making the substitutions $x = t^2$, $y = t$, we have

$$F(x\ \ y) \equiv x^2 + a_1xy + a_2x + a_3y + a_4 + \lambda(y^2 - x) = 0. \tag{17}$$

We may now determine the parameter λ, so as to make the polynomial $F(x, y)$ break up into two linear factors. The required condition is

$$\begin{vmatrix} 2 & a_1 & a_2 - \lambda \\ a_1 & 2\lambda & a_3 \\ a_2 - \lambda & a_3 & 2a_4 \end{vmatrix} = 0$$

or

$$\lambda^3 - 2a_2\lambda^2 + (a_1a_3 + a_2^2 - 4a_4)\lambda + (a_1^2a_4 - a_1a_2a_3 + a_3^2) = 0.$$

The solution of this cubic in λ gives three values which, set in equation (17), give three quadratics, each of which breaks up into linear factors. Suppose, for example, that for one of the values of λ equation (17) takes the form

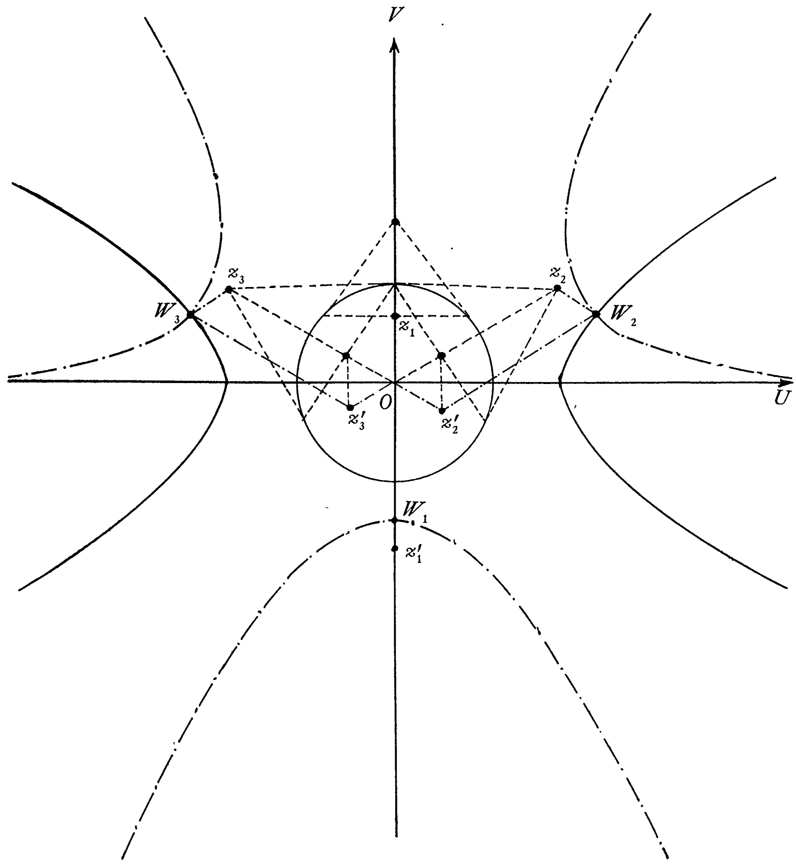

FIG. 2

$$(x + b_1y + c_1)(x + b_2y + c_2) = 0$$

or

$$(t^2 + b_1t + c_1)(t^2 + b_2t + c_2) = 0$$

or

$$\{(t - t_1)(t - t_2)\}\{(t - t_3)(t - t_4)\} = 0$$

For the three values of λ, the four roots t_1, t_2, t_3, t_4 are paired as follows:

$$t_1,\ t_2;\ t_3,\ t_4$$

$$t_1,\ t_3;\ t_2,\ t_4$$

$$t_1,\ t_4;\ t_2,\ t_3$$

5. *Geometrical Solution.* Consider the equation

$$(x^2 + a_1xy + a_2x + a_3y + a_4) + \lambda(y^2 - x) = 0.$$

This is a single parameter family of conics, passing through the points of intersection of the two conics

$$x^2 + a_1xy + a_2x + a_3y + a_4 = 0 \tag{18}$$

and

$$y^2 - x = 0. \tag{19}$$

A geometric solution without employing the cubic in λ is afforded directly, the four solutions of the original biquadratic in t being the ordinates of the four points of intersection of the plotted conics (18) and (19), the latter being a base parabola.

A second method is to find the intersections of a single pair of lines

$$x + b_1y + c_1 = 0, \quad x + b_2y + c_2 = 0,$$

found by the use of one solution of the cubic in λ, with the base parabola (19).

A third method is to find the intersections of two pairs of such lines, derived from two different solutions of the cubic in λ.

If the conics (18) and (19) intersect in four points, the three pairs of lines derived from the solutions of the cubic in λ form a complete quadrangle on the four points.[2]

Example. Given $t^4-2t^3-t^2+2t=0$. From the equation $\lambda^3+2\lambda^2-3\lambda=0$ we have $\lambda=0, -1, 3$. For $\lambda=0$,

$$x^2 - 2xy - x + 2y = 0 \quad \text{or} \quad (x-1)(x-2y) = 0.$$

Making this simultaneous with $y^2-x=0$, we find for the values of y, or t, $0, +1, -1, 2$.

TWO NOTES ON INSTRUCTION IN MECHANICS

By OLIVER D. KELLOGG, Harvard University

1. *On the "principle" that Newton's second law, together with the initial position and velocity, determine the motion of a particle.*

The following question appeared in a recent examination in mechanics: *Prove that the motion of a particle under a central force takes place in a fixed plane.* A number of answers were substantially as follows. "Consider the plane containing the given center, the initial position, and the initial velocity of the

[2] Consult Archibald Henderson, *Observations on simultaneous quadratic equations*, in this Monthly, vol. 35 (1928), pp. 337–346. A full discussion of the various cases arising from degeneration of the complete quadrangle is found in that paper.

particle. As the force lies in this plane, there is no normal acceleration, so that the particle remains in the plane."

The question arose as to whether this reasoning is valid. We are not here concerned with the failure of the students to notice that if the velocity is initially toward the center, the plane in question is not determined. Let us suppose the statement of the question precluded such an initial velocity. Since the theorem is true, it may appear a little difficult to assure oneself as to the propriety of the reasoning. This may, however, be done as follows. If we can construct a field of force (not, of course, central), to which the reasoning applies, and yet in which the motion is not necessarily plane, we shall know that the reasoning is not valid.

Now the solutions of the differential equations of motion are known to be unique in case the components of force on the coördinate axes satisfy a Lipschitz condition. And if the plane of the initial velocity be taken as the (x, y)-plane, and there is no component of force normal to that plane as long as the particle remains in the plane, then $z=0$ is part of *one* possible solution. We must therefore seek a field of force in which a Lipschitz condition is *not* satisfied. Such a one is given by the components

$$X = -x, \quad Y = -y, \quad Z = 12|z|^{1/2}.$$

If we take the mass of the particle as 1, and suppose that for $t=0$ it is at $(1, 0, 0)$, with the velocity $(0, 1, 0)$, the equations of motion and initial conditions will be found to be satisfied by

$$x = \cos t, \quad y = \sin t, \quad z = t^4,$$

and these equations do not represent a plane curve, in spite of the fact that there is no component of force normal to the plane $z=0$ which contains the particle and its velocity for $t=0$. Thus the students' "proof" is not a proof.

It is not contended that no credit should be given the indicated replies to the question. A discussion of the subject might be interesting, but this was not my intention in writing this paper.

As to the "principle" cited above, the example shows that it is a dangerous one. It may be urged that nature does not construct fields of force like the one given. But all our mathematical treatment of nature is based on idealizations, and there is no guarantee that an idealization might not lead to a field of force which does not have the Lipschitz property. Certainly, the "principle" as stated is not mathematically sound.

II. *On the Principle of Virtual Work.*

There are few topics in mechanics on which there is more lax writing and thinking than the principle of virtual work. We propose here, not so much to go into all the details of a precise formulation, as to point out an apparent failure of the principle.

Coördinates of a mechanical system must be defined. A *virtual displacement* of the system is then defined by a *variation* of the coördinates. A variation of the

coördinates is a system of *differentials*, satisfying the linear equations obtained by differentiating the equations of constraint (if the constraints are holonomic —otherwise satisfying the non-holonomic equations of constraint). Whether a virtual displacement is "actual," "real," or "possible," is utterly beside the point. If an equation of constraint is non-linear in the coördinates, and is not identically satisfied by them, then, in general, a virtual displacement does not correspond to a possible configuration of the system.

The constrains are assumed to be *smooth*, which means that the forces of constraint do no work for any virtual displacement. Calling all other forces applied, we may state the principle as follows: *A necessary and sufficient condition that the system be in equilibrium is that the virtual work of the applied forces vanishes for every virtual displacement.*

Some years ago, Professor W. F. Osgood called my attention to the following example. Imagine a weightless circular hoop, in a vertical plane, constrained to roll on the under side of an incline plane, without slipping. To a point of the hoop is attached a heavy particle. Study the possibility of equilibrium.

If we take the x-axis in the plane, and downward, and the y-axis perpendicular and also downward, it is natural to take as coördinate the angle through which the radius of the hoop through the particle has rolled from its position when the particle is in contact with the plane. The coördinates of the particle are then given by the usual equations of the cycloid:

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta),$$

and we find for the virtual work corresponding to a variation $\delta\theta$

$$\delta W = 2wa \sin \tfrac{1}{2}\theta \cos (\tfrac{1}{2}\theta - \alpha)\delta\theta,$$

where α is the inclination of the plane, and w the weight of the particle. The principle of virtual work tells us that a necessary and sufficient condition for equilibrium is the vanishing of the sine or the cosine in this expression. The vanishing of the cosine gives correct results: the particle is in equilibrium at every point of the path of minimum height. The vanishing of the sine, however, gives us the points of contact of the particle with the plane, which I think most of us would regard as spurious positions of equilibrium.

Perhaps an even more striking example is found in a bead, free to slide under gravity on a vertical wire. If z is the distance of the bead above an arbitrary point of the wire, and we take as the coördinate (and surely generalized coördinates are admissible) of the particle, q, defined by $z=q^3$, then the virtual work of the weight w of the bead is

$$\delta W = -\ 3wq^2\delta q,$$

and this vanishes at $q=0$ for every virtual displacement. We should thus be led to infer that *any given position of the bead is one of equilibrium!*

The second example will perhaps raise a suspicion as to where the difficulty lies, because of the artificiality of the coördinate used. Indeed this is just the point. It is not the principle that has failed us, it is the coördinates employed —in spite of the fact that in the first example θ is the most natural one to select. However, if we had used y instead, we should not have obtained the spurious positions of equilibrium.

The obvious way out is to lay proper restrictions on the choice of systems of coördinates. A sufficient restriction is the following. *A system of coördinates is not admissible unless to every variation of the coördinates* (other than that in which all differentials vanish) *there corresponds a change of position of at least one point of the system.* This may be formulated analytically by selecting a set of points of the system (assumed to have a finite number of degrees of freedom) such that the positions of these points fix those of all points of the system. Then the matrix of the partial derivatives of the cartesian coördinates of these points with respect to the chosen coördinates of the system must be of maximum rank.

Another way of avoiding the trouble lies in a change of definition of virtual displacement. We define *virtual velocities* as any system of velocities of the points of the system compatible with the constraints. A virtual displacement is then any system of displacements of the points of the system proportional to a system of virtual velocities. This procedure seems simpler. However, sooner or later the difficulties which coördinates may introduce must be faced.

THE HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

By MEYER SALKOVER, University of Cincinnati

This note deals with the solution of the differential equation in one dependent and one or two independent variables, linear, with constant coefficients, in the dependent variable and its derivatives, there being no term in the equation free from the latter quantities. By "homogeneous" is here meant the last requirement. The treatment is elementary, inductive, and characterized by the avoidance of artificial limiting processes, abstruse symbols (except in the last paragraph), and the customary auxiliary theorems. Moreover, the parallelism between the ordinary and the partial differential equation of the type studied will be clearly exhibited.

1. *The Ordinary Differential Equation*

The ordinary differential equation

$$(D - a)^m z = 0, \tag{1}$$

where D means the operator d/dx, the exponent m is a positive integer, any

power of D meaning the derivative of the corresponding order, and where a is a constant, may be solved as follows: We put

$$(D - a)z = R. \tag{2}$$

To find a serviceable expression for R, we might, in the usual way, apply the integrating factor e^{-ax}; or, equivalently, write (2) as

$$(d/dx)(\log z) - a = R/z$$

or

$$(d/dx)[\log (ze^{-ax})] = R/z;$$

whence

$$(D - a)z = R = z(d/dx)[\log (ze^{-ax})] = e^{ax}(d/dx)(ze^{-ax}). \tag{3}$$

By induction, we obtain

$$(D - a)^m z = e^{ax}(d^m/dx^m)(ze^{-ax}). \tag{4}$$

Hence (1) reduces to

$$(d^m/dx^m)(ze^{-ax}) = 0,$$

which, after repeated integration, yields

$$z = e^{ax}(C_0 + C_1 x + \cdots + C_{m-1}x^{m-1}), \tag{5}$$

where the C's are arbitrary constants.

Alternative forms for (3) and (4) are

$$(D - a)z = [(d/dx)(e^{C-ax}z)]_{C=ax} \tag{3'}$$

and

$$(D - a)^m z = [(d^m/dx^m)(e^{C-ax}z)]_{C=ax}. \tag{4'}$$

These identities will be cited later when (1) will be treated as a conditioned form of the partial differential equation of the corresponding type.

The general ordinary homogeneous linear differential equation with constant coefficients, which, by the theory of equations and the properties of the derivative, may be written in operationally factored form

$$(D - a_1)^{m_1}(D - a_2)^{m_2} \cdots (D - a_n)^{m_n} z = 0, \tag{6}$$

is easily solved by iteration of the above process. Let us put

$$v_1 = (D - a_2)^{m_2}(D - a_3)^{m_3} \cdots (D - a_n)^{m_n} z.$$

From (5),

$$v_1 = (C_{10} + C_{11}x + \cdots + C_{1m_1-1}x^{m_1-1})e^{a_1 x}. \tag{7}$$

We then introduce

$$v_2 = (D - a_3)^{m_3} \cdots (D - a_n)^{m_n} z,$$

so that

$$v_1 = (D - a_2)^{m_2} v_2;$$

we substitute in (7), use (4), divide by $e^{a_2 x}$ and integrate. Repeating this process, we ultimately find the solution of (6) to be

$$z = \sum_{i=1}^{n} (C_{i0} + C_{i1}x + \cdots + C_{im_i - 1} x^{m_i - 1}) e^{a_i x}. \tag{8}$$

The regular practice of obtaining (8) by adding the particular solutions arising from the separate factors of (6) is more elegant, though not essentially simpler. It requires, moreover, a digression to establish an auxiliary theorem.

The same procedure is applicable to the non-homogeneous equation, but here symbolic manipulation in some cases saves considerable time.

2. *The Partial Differential Equation*

In all elementary texts[1] to which the writer has referred, linear partial differential equations with constant coefficients are solved either symbolically by exponential operators suggested by the theory of the ordinary equation, or stepwise, without the aid of induction, or, in more complicated cases, by combining series of exponential functions with coefficients that are arbitrary constants into arbitrary functions. Here the subject is developed inductively along the lines of our treatment of the ordinary equation but without artificially and symbolically using previous results.

A. Binomial Factors of the Form $D_x - aD_y$. Let us consider first the equation

$$(D_x - aD_y)^m z(x, y) = 0, \tag{9}$$

where D_x and D_y are the operators $\partial/\partial x$ and $\partial/\partial y$, respectively. We put

$$(D_x - aD_y)z = R(x, y).$$

By the method of Lagrange,

$$\frac{dx}{1} = \frac{dy}{-a} = \frac{dz}{R};$$

whence

$$ax + y = C$$

and

$$z(x, C - ax) = \int^x R(\xi, C - a\xi) d\xi + f(C),$$

f being an arbitrary function of its argument. Thus we find

$$R(x, y) = (D_x - aD_y)z(x, y) = [\partial z(x, C - ax)/\partial x]_{C = ax + y}. \tag{10}$$

---

[1] e. g., Forsyth, *Differential Equations*, 4th edition, pp. 512, 516, 517; Cohen, *Differential Equations*, pp. 239–242, 246–247.

This is the analogue of (3) or (3′); in fact, (3′) may be derived from (10) by specialization as follows:

If $(D_x - aD_y)z$ reduces to $(D_x - a)z$,

$$D_y z = z;$$

and on integration

$$z = e^y f(x).$$

Substituting this in (10) gives us

$$R = \left[\partial(f(x)e^{C-ax})/\partial x\right]_{C=ax+y}. \tag{11}$$

The transition to the standpoint of the ordinary differential equation is easily made by considering y as an arbitrary constant. If we write K for the difference between y and C, the right side of (11) becomes

$$\left[d(ze^{K-ax})/dx\right]_{K=ax},$$

as in (3′).

By induction from (10) we find

$$(D_x - aD_y)^m z(x, y) = \left[\partial^m z(x, C - ax)/\partial x^m\right]_{C=ax+y}. \tag{12}$$

This is the analogue of (4) or (4′), which may be derived from (12) by specialization.

By (12), (9) becomes

$$\left[\partial^m z(x, C - ax)/\partial x^m\right]_{C=ax+y} = 0. \tag{13}$$

If in (9) the independent variable y had been replaced by $C - ax$, we would have obtained instead of (13),

$$\partial^m z(x, C - ax)/\partial x^m = 0,$$

whence, by repeated integration,

$$z(x, C - ax) = f_0(C) + xf_1(C) + \cdots + x^{m-1}f_{m-1}(C),$$

where the f's are arbitrary functions. Restoring $y = C - ax$, we obtain

$$z(x, y) = f_0(ax + y) + xf_1(ax + y) + \cdots + x^{m-1}f_{m-1}(ax + y) \tag{14}$$

as the solution of (9).

By the same sort of specialization as was applied to (10), (14) can be converted into (5), the solution of the corresponding ordinary differential equation (1).

The homogeneous linear partial differential equation with constant coefficients in which the derivatives are all of the same order and in which the derivative of highest order with respect to x is of zero order with respect to y may be written in the following general form:

$$(D_x - a_1D_y)^{m_1}(D_x - a_2D_y)^{m_2} \cdots (D_x - a_nD_y)^{m_n} z = 0. \tag{15}$$

It may be solved (see the indicated solution of (6)) by repeated use of (12), the operational binomial powers being unraveled one at a time, starting with $(D_x - aD_y)^{m_1}$. We thus find for the solution of (15) the sum of n expressions of the form (14), in the successive arguments

$$a_1 x + y,\ a_2 x + y, \cdots, a_n x + y.$$

B. Other Linear Factors. If operational linear factors such as are excluded by the restrictions of the previous section exist, they, too, can be brought within the scope of our method. Suppose that the repeated general linear factor $(D_x - aD_y - b)^m$ occurs. From the foregoing we may infer that its effect on the solution may be ascertained by solving the equation

$$(D_x - aD_y - b)^m z = 0. \tag{16}$$

Putting

$$(D_x - aD_y - b)z = R(x, y)$$

we find, from (10),

$$R(x, y) = [\partial z(x, C - ax)/\partial x]_{C=ax+y} - bz(x, y)$$

or

$$\begin{aligned} R(x, C - ax) &= \partial z(x, C - ax)/\partial x - bz(x, C - ax) \\ &= e^{bx}\partial[z(x, C - ax)e^{-bx}]/\partial x, \end{aligned}$$

(using (3)). By induction,

$$(D_x - aD_y - b)^m z = e^{bx}[\partial^m\{z(x, C - ax)e^{-bx}\}/\partial x^m]_{C=ax+y}.$$

Whence, returning to (16) and proceeding as with (13), we obtain the solution:

$$z(x, y) = e^{bx}[f_0(ax + y) + xf_1(ax + y) + \cdots + x^{m-1}f_{m-1}(ax + y)]. \tag{17}$$

It will be noticed that the arbitrary functions arise naturally instead of, as in the usual treatment, being built up from series of exponentials having as coefficients arbitrary constants.

For every factor of the type (16) in a homogeneous linear partial differential equation with constant coefficients, there will evidently be a term of the type (17) in the solution.

If repeated linear factors free from D_x, such as $(D_y - b)^m$, occur, the corresponding terms in the solution are readily seen from (16) and (17), interchanging x and y and putting $a = 0$, to have the form

$$e^{by}[f_0(x) + yf_1(x) + \cdots + y^{m-1}f_{m-1}(x)].$$

C. A Symbolic Solution. Our results so far obtained may be extended symbolically to more difficult cases. Take, for instance, the differential equation for the linear flow of heat (without imposing conditions on the function sought),

$$(D_x^2 - b^2 D_y)z = 0,$$

treated by Forsyth[2] by exponential operators. This may be written

$$(D_x - bD_y^{-1/2}D_y)(D_x + bD_y^{-1/2}D_y)z = 0.$$

Thus written, it formally comes under (15), with $a_1 = bD_y^{-\frac{1}{2}}$ and $a_2 = -bD_y^{-\frac{1}{2}}$. The symbolic solution is then

$$z = f_1(bD_y^{-1/2}x + y) + f_2(-bD_y^{-1/2}x + y).$$

If the arbitrary functions are expanded into Taylor's series in powers of the first term of each argument, the powers of $D_y^{-\frac{1}{2}}$ operating on the derivatives of $f_i(y)(i=1, 2)$, and then, as in Forsyth, the arbitrary functions are renamed to efface the obscure operator $D_y^{-1/2}$, the series for z in two arbitrary functions is obtained. The equivalent series in one arbitrary function is found by writing the equation in the form

$$(D_y - b^{-2}D_x \cdot D_x)z = 0,$$

which yields, symbolically,

$$z = f(b^{-2}D_x y + x);$$

and this is to be expanded into a Taylor's series in powers of the first term of the argument, the powers of D_x operating on the derivatives of $f(x)$.

SCALAR EXTENSIONS OF AN ORTHOGONAL ENNUPLE OF VECTORS

By A. D. MICHAL, California Institute of Technology

A general theory of scalar extensions of tensors has been developed by the author in another paper.[1] The aim of the present note is to single out an important special case and to give the simpler details independently of the general theory.

Let

$$g_{\alpha\beta}dx^\alpha dx^\beta; \quad g_{\alpha\beta} = g_{\beta\alpha} \tag{1}$$

be the fundamental metric differential form of an n-dimensional Riemann space. For the sake of simplicity we shall assume that the form (1) is positive definite.[2] Throughout our paper we shall adopt the convention of letting the repetition of an index in a term denote summation with respect to that index over all integral values $1, 2, \cdots, n$. Furthermore by x^i we shall understand the ith coordinate of a point $(x^1, x^2, \cdots x^i, \cdots, x^n)$ and not the ith power of a variable x.

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[2] Loc. cit., pp. 522–524.

[1] To be published in The Tôhoku Mathematical Journal.

[2] Obvious modifications will make our work valid for Eisenhart's more general metrics. See Eisenhart, *Riemannian Geometry* (1926), Chapt. III.

Let ${}_{(\alpha)}\xi^i(x^1, x^2, \cdots, x^n)$ denote the ith component of the αth contravariant vector in an orthogonal ennuple of contravariant vectors ${}_{(1)}\xi^i, {}_{(2)}\xi^i, \cdots, {}_{(n)}\xi^i$. The analytical expressions of these hypotheses are as follows:

(I) under an arbitrary analytic change of coordinates,

$$G : x^i = f^i(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^n), \quad \left|\frac{\partial x^i}{\partial \bar{x}^j}\right| \neq 0, \tag{2}$$

the components of the αth vector ${}_{(\alpha)}\xi^i$ undergo a transformation[3]

$${}_{(\alpha)}\xi^i(x^1, \cdots, x^n) = {}_{(\alpha)}\bar{\xi}^\sigma(\bar{x}^1, \cdots, \bar{x}^n)\frac{\partial x^i}{\partial \bar{x}^\sigma} \cdot \tag{3}$$

(II) the vectors ${}_{(\alpha)}\xi^i$ satisfy the invariant conditions

$$g_{\sigma\tau\,(\alpha)}\xi^\sigma{}_{(\beta)}\xi^\tau = \delta_{\alpha\beta}, \tag{4}$$

where

$$\delta_{\alpha\beta}\begin{cases} = 1 \text{ if } \alpha = \beta \\ = 0 \text{ if } \alpha \neq \beta. \end{cases}$$

Define ${}_{(\alpha)}\xi_i$ by

$${}_{(\alpha)}\xi_i = g_{ij\,(\alpha)}\xi^j. \tag{5}$$

Then it easily follows from (4) that ${}_{(\alpha)}\xi_i$ is the cofactor of ${}_{(i)}\xi^\alpha$ in the determinant of ${}_{(i)}\xi^\alpha$, divided by the determinant of ${}_{(i)}\xi^\alpha$.

Let $\Gamma^i_{\alpha\beta}$ stand for the Christoffel symbols of the second kind

$$\Gamma^i_{\alpha\beta} = \frac{1}{2}g^{i\sigma}\left(\frac{\partial g_{\sigma\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma}\right). \tag{6}$$

As is well known the law of transformation of the $\Gamma^i_{\alpha\beta}$ under an arbitrary transformation G is given by

$$\bar{\Gamma}^\sigma_{\alpha\beta}(\bar{x}^1, \cdots, \bar{x}^n)\frac{\partial x^i}{\partial \bar{x}^\sigma} = \Gamma^i_{\lambda\mu}\frac{\partial x^\lambda}{\partial \bar{x}^\alpha}\frac{\partial x^\mu}{\partial \bar{x}^\beta} + \frac{\partial^2 x^i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} \cdot \tag{7}$$

Let x^i be an arbitrarily given coordinate system and the point P an arbitrarily given point in our space with coordinates $x^i = q^i$. Define a coordinate system S_1, with coordinates y^i, implicitly by

$$x^i = q^i + [{}_{(\alpha)}\xi^i(x^1, \cdots, x^n)]_P y^\alpha - \frac{1}{2!}[\Gamma^i_{\lambda\mu\,(\alpha)}\xi^\lambda{}_{(\beta)}\xi^\mu]_P y^\alpha y^\beta. \tag{8}$$

Since the determinant of ${}_{(\alpha)}\xi^i$ does not vanish at the point P, it follows that (8) is a reversible transformation in the neighborhood of the point P. Let a star

[3] We shall always understand that any free index (not summed out) can take on any integral value $1, 2, \cdots, n$.

over a component denote the evaluation of that component in a coordinate system S_1. If we now consider the particular transformation G, given by (8), in the transformation laws (3) and (7), we obtain after some easy reductions the results

$$(9) \qquad (a) \quad [\overset{*}{_{(\alpha)}\xi}{}^i(y^1, \cdots, y^n)]_0 = \delta^i_\alpha; \qquad (b) \quad [{}^*\Gamma^i_{\alpha\beta}(y^1, \cdots, y^n)]_0 = 0,$$

where δ^i_α is unity or zero according as $i=\alpha$ or $i\neq\alpha$ and where the suffix 0 denotes evaluation for $y^i=0$. Now a coordinate system y^i for which (9) (b) is satisfied is known as a geodesic coordinate system. Hence S_1 is a special geodesic coordinate system.

Consider an arbitrary transformation G on the coordinates x^i. Then as before, the new coordinate system $\bar{x}^i$ and the *same* point P with new coordinates $\bar{x}^i=\bar{q}^i$ can define implicitly a coordinate system $\bar{y}^i$ with corresponding properties (9) by means of the transformation

$$(10) \qquad \bar{x}^i = \bar{q}^i + [{}_{(\alpha)}\bar{\xi}^i(\bar{x}^1, \cdots, \bar{x}^n)]_P\bar{y}^\alpha - \frac{1}{2!}[\bar{\Gamma}^i_{\lambda\mu}{}_{(\alpha)}\bar{\xi}^\lambda{}_{(\beta)}\bar{\xi}^\mu]_P\bar{y}^\alpha\bar{y}^\beta.$$

If we write (3) and (7) as transformation laws between components in the y^i coordinate system and the corresponding components in the $\bar{y}^i$ coordinate system we obtain by means of (9) and a similar set of formulas for the $\bar{y}^i$ coordinate system the results

$$(11) \qquad \left(\frac{\partial y^i}{\partial \bar{y}^\alpha}\right)_0 = \delta^i_\alpha; \qquad \left(\frac{\partial^2 y^i}{\partial \bar{y}^\alpha \partial \bar{y}^\beta}\right)_0 = 0.$$

We see from (5) that ${}_{(\alpha)}\xi_i$ is the ith component of the αth covariant vector of the ennuple of covariant vectors ${}_{(1)}\xi_i, \cdots, {}_{(n)}\xi_i$. Let y^i and $\bar{y}^i$ be geodesic coordinates of type S_1 with the same origin P. On differentiating the set of equations

$$(12) \qquad \overset{*}{_{(\alpha)}\bar{\xi}}_i(\bar{y}^1, \cdots, \bar{y}^n) = \overset{*}{_{(\alpha)}\xi}_\sigma(y^1, \cdots, y^n)\frac{\partial y^\sigma}{\partial \bar{y}^i}$$

and making use of (11) we obtain at the common origin of the coordinate systems y^i and $\bar{y}^i$

$$(13) \qquad \left(\frac{\partial \overset{*}{_{(\alpha)}\bar{\xi}}_i}{\partial \bar{y}^j}\right)_0 = \left(\frac{\partial \overset{*}{_{(\alpha)}\xi}_i}{\partial y^j}\right)_0.$$

Hence the functions $\gamma_{\alpha ij}(x^1, \cdots, x^n)$ defined by

$$(14) \qquad [\gamma_{\alpha ij}(x^1, \cdots, x^n)]_P = \left(\frac{\partial \overset{*}{_{(\alpha)}\xi}_i}{\partial y^j}\right)_0$$

are absolute scalar functions. This set of scalars will be called the *first scalar extension of the vectors* ${}_{(\alpha)}\xi_i$.

THEOREM I: *The scalars* γ_{ijk} *in the first scalar extension of the covariant vectors* ${}_{(i)}\xi_j$ *are identical with the Ricci coefficients of rotation of the orthogonal ennuple.*

To prove this interesting theorem, we write down the equations

$$
{}_{(i)}\overset{*}{\xi}_j(y^1, \cdots, y^n) = {}_{(i)}\xi_\lambda(x^1, \cdots, x^n)\frac{\partial x^\lambda}{\partial y^j}, \tag{15}
$$

where y^i is a geodesic coordinate system of type S_1, differentiate with respect to y^k and then evaluate the results at the origin of the coordinate system y^i. Finally by substituting for

$$
\left(\frac{\partial x^i}{\partial y^j}\right)_0 \text{ and } \left(\frac{\partial^2 x^i}{\partial y^j \partial y^k}\right)_0
$$

their expressions as given by (8), we obtain

$$
\gamma_{ijk} = \left(\frac{\partial\, {}_{(i)}\xi_\lambda}{\partial x^\mu} - {}_{(i)}\xi_\sigma \Gamma^\sigma_{\lambda\mu}\right){}_{(j)}\xi^\lambda\, {}_{(k)}\xi^\mu . \tag{16}
$$

Up to this point we have made use only of the existence and continuity of the first partial derivatives of the components of the vectors in the ennuple of vectors. Assume now the existence and continuity of all the second partial derivatives of the components of the vectors in the given orthogonal ennuple. On adding

$$
-\frac{1}{3!}\left[\Gamma_{\lambda\mu\nu}\,{}_{(\alpha)}\xi^\lambda\,{}_{(\beta)}\xi^\mu\,{}_{(\gamma)}\xi^\nu\right]_p y^\alpha y^\beta y^\gamma \tag{17}
$$

to the corresponding right hand sides of (8), the modified transformation of coordinates (8) defines implicitly another special type, say S_2, of geodesic coordinates y^i. In (17), the set of functions $\Gamma^i{}_{\lambda\mu\nu}$ are defined by

$$
\Gamma^i_{\lambda\mu\nu} = \frac{1}{3}P\left(\frac{\partial \Gamma^i_{\lambda\mu}}{\partial x^\nu} - \Gamma^i_{\sigma\mu}\Gamma^\sigma_{\lambda\nu} - \Gamma^i_{\lambda\sigma}\Gamma^\sigma{}_\nu\right),
$$

where P denotes the sum of the terms obtainable from the ones inside the parenthesis by permuting λ, μ, ν cyclically.[4]

The set of functions $g_{\alpha\beta\gamma\delta}(x^1, \cdots, x^n)$ defined by

$$
(g_{\alpha\beta\gamma\delta})_p = \left(\frac{\partial^2 \overset{*}{g}_{\alpha\beta}}{\partial y^\gamma \partial y^\delta}\right)_0
$$

with the aid of a geodesic coordinate system S_2 are absolute scalar functions. This set of scalars will be said to constitute the *second scalar extension of* $g_{\alpha\beta}$.

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[4] Their law of transformation is given by O. Veblen and T. Y. Thomas in the Transactions of American Mathematical Society, vol. 25 (1923), p. 577.

By an obvious extension of the method of proof of Theorem I, one can demonstrate the validity of the following theorem.

THEOREM II: *If g_{ijkl} are the scalar functions of the second scalar extension of the metric tensor g_{ij}, then the set of functions $g_{ikjl}-g_{iljk}$ is identical with the set of the Ricci four-index scalars,*[5] $\gamma_{ij,kl}$.

A NOTE ON THE NUMERICAL COMBINATION OF PROBABILITY SERIES

By GEORGE C. CAMPBELL, University of Iowa

The object of this note is to give a practical method for the numerical combination of probability series.

Consider the two series, P_r' and P_r'', derived from the American Men mortality table, giving the probability of exactly r deaths from two homogeneous groups of 100 lives, age 20 and 30, respectively.

r	P_r'	P_r''	P_r'''
0	0.675 184	0.639 545	0.431 811
1	0.265 714	0.286 515	0.363 386
2	0.051 762	0.063 537	0.152 134
3	0.006 654	0.009 298	0.042 247
4	0.000 635	0.001 010	0.008 754
5	0.000 048	0.000 087	0.001 444
6	0.000 003	0.000 006	0.000 197
7	0.000 000	0.000 000	0.000 023
8			0.000 002
9			0.000 000

We ask for the series P_r''', giving the probability of exactly r deaths from the heterogeneous group of 200 lives composed of the two sub-groups. It is fairly obvious that

$$
\begin{aligned}
P_0''' &= P_0' P_0'', \\
P_1''' &= P_0' P_1'' + P_1' P_0'', \\
P_2''' &= P_0' P_2'' + P_2' P_0'' + P_1' P_1'', \\
P_3'\ ' &= P_0' P_3'' + P_3' P_0'' + P_2' P_1'' + P_1' P_2'', \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

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[5] For the definition of $\gamma_{ij,kl}$ see T. Levi-Civita, *The Absolute Differential Calculus* (1927), p. 278. For more general metrics, see L. P. Eisenhart, *loc. cit.*

where P_r''' is the sum of the products of all pairs of probabilities whose subscripts add up to the sum r. It is clear that this direct approach leads to an enormous amount of tabulation, even for series of moderate length.

Let us tabulate the series P_r' on a strip of paper and the series P_r'' in reverse order on another strip. Fasten the two strips with paper clips to a strip of card-board so that they will slide, side by side. Set P_0' opposite P_0''. Their product gives P_0'''. Then we slide the strips so that two pairs of lines are opposite each other, P_1' opposite P_0'' and P_0' opposite P_1''. If we do both of these multiplications on the computing machine without clearing the first product, we have the sum of the products on the machine, giving P_1'''. We continue to slide the strips by one line each time, and at each setting we multiply together all pairs of numbers opposite each other. By leaving the products on the machine, we obtain directly the sum of products which is P_r'''.

	. . . . .
	. . . . .
	0.063 537
	0.286 515
0.675 184	0.639 545
0.265 714	
0.051 762	
. . . . .	
. . . . .	

We note that this simple device carries us directly from the component series to the resultant series without tabulating any intermediate steps.

QUESTIONS AND DISCUSSIONS

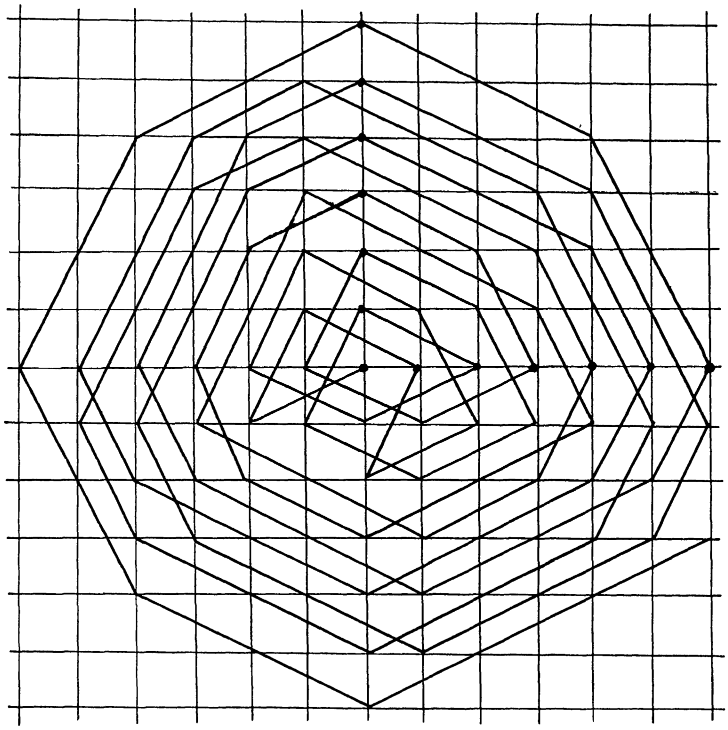
EDITED by R. E. GILMAN, Brown University, Providence, Rhode Island.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A RECREATION

By NORMAN ANNING, University of Michigan

The points of the plane which have integral coordinates form a countable set. Among the ways of counting them is that in which we proceed from each to the next by a knight's move. After a few irregular moves at the beginning the process can be systematized. The recreation consists in trying to find



a path which reduces to a minimum the number of irregular moves. Just where regularity sets in is open to question but please agree that the accompanying figure has 4 irregular moves. Similarly figures can be drawn for the half-plane and quarter-plane with irregularities as small as 8 and 13 respectively. It is not claimed that these numbers are the smallest of their classes; there's the puzzle.

A Note by the Editor

If one makes use of some of the well known configurations for the knight's tour in a square,[1] it is not difficult to show that the integral points in the whole

[1] See, for example, Dudeney's *Amusements in Mathematics*, pp. 102–103, 227–229.

plane can be counted in a sequence of knight's moves, and by an entirely systematic process involving no irregular moves whatever. (See figures 1, 2.) Indeed it is possible to do this in an infinite number of ways. The same statements

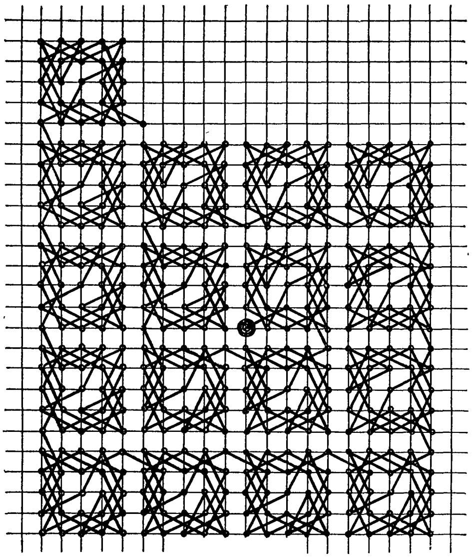


FIG. 1.

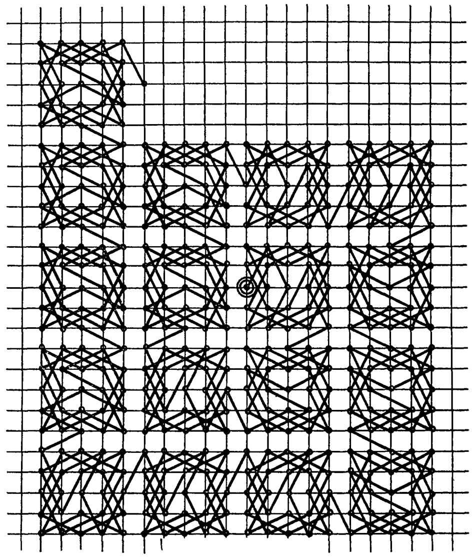


FIG. 2.

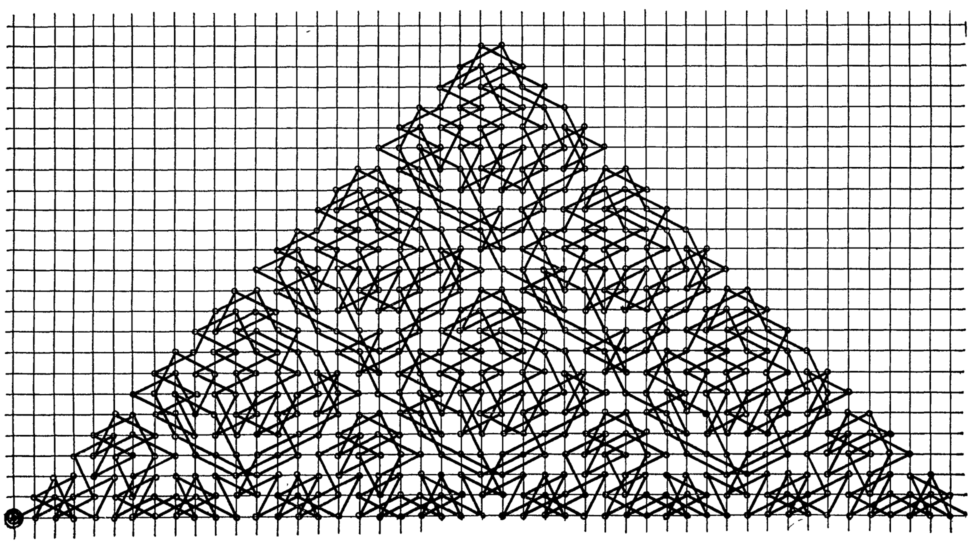


FIG. 3.

hold for half and quarter planes whose boundaries are parallel to the coordinate axes. It will be observed that the problem is very much changed by giving the boundaries of the half or quarter plane an orientation oblique to the coordinate

axes. This suggests the problem as to the possibility of counting the integer points in a sector of the plane. If the vertex of the sector is at the origin and the initial side coincides with the positive x-axis, the enumeration is impossible

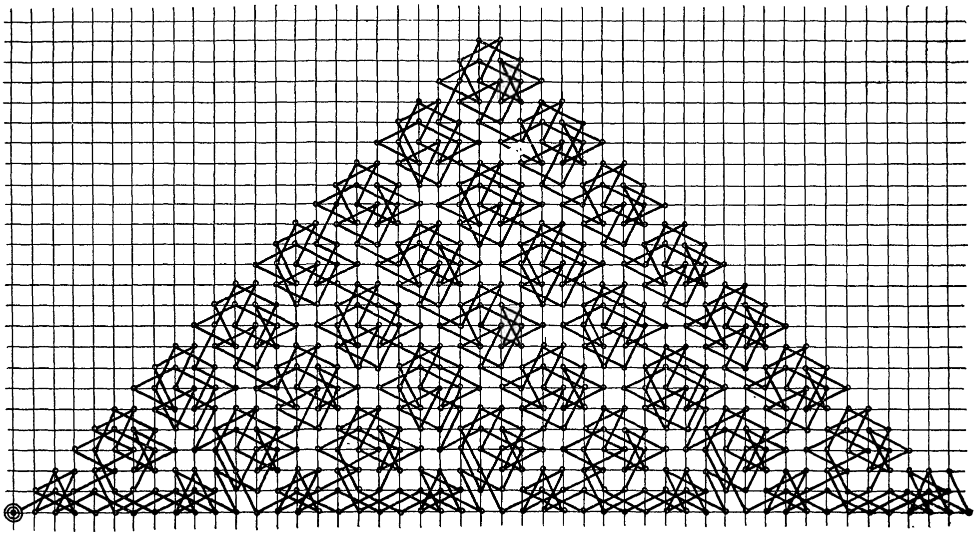


FIG. 4.

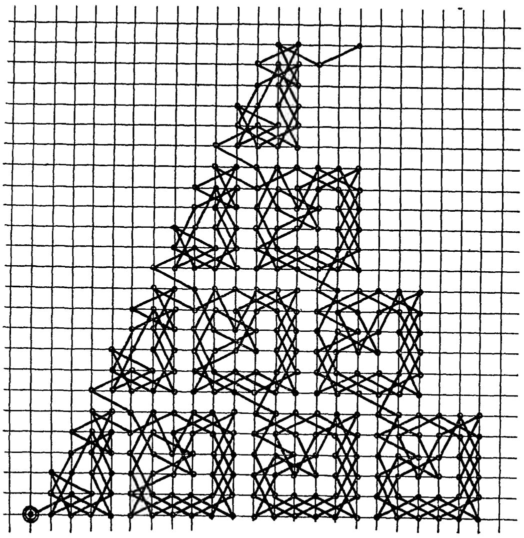


FIG. 5.

by any process systematic or otherwise, if the angle of the sector is less than 45°. For the 45° sector itself an infinite number of entirely systematic enumerations are possible (see figures 3, 4), and the same statement holds for the sector whose angle is $\tan^{-1} 2$ (see figure 5). More precisely, if as in figures 3 and 4 two

different designs are discovered each of which by suitable iteration effects the required enumeration in some sector, and if these designs, though different, are such that the areas occupied by the fundamental pattern and of each successive iteration in the one design are congruent respectively to the areas of the fundamental pattern and each successive iteration in the other, then arbitrary alternations from one pattern to the other are possible, and it can accordingly be asserted that there exists a non-denumerably infinite number of ways of counting the points in that sector, of which at least a denumerably infinite number are systematic. For sectors of 360°, 180°, 90°, 45°, $\tan^{-1} 2$, $\tan^{-1} 3/2$, and $\tan^{-1} 4/3$, such is actually the case.

It is probable that for any sector whose angle is not less than 45°, the points with integral coordinates, inside and on the boundary, can be enumerated in a sequence of knight's moves, and in a non-denumerably infinite number of ways, of which a denumerably infinite number are systematic.

This suggests a question.

QUESTION No. 59: Given a pair of compasses whose opening has been set equal to the distance between a pair of integral points (points having integral coordinates), it is proposed to straddle with this fixed opening of the compasses from one integral point to another, so as to enumerate consecutively, all such points inside and on the boundary of a certain sector of the plane. For what sectors and with what compass settings can this be done?

R.E.G.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Analytic Geometry. By D. R. Curtiss and E. J. Moulton. D. C. Heath and Company, New York, 1930. xiii+338+18 pages. $2.48.

The best way to begin a review of this excellent book is to sketch the plan of the book as the authors state it in the preface. (This preface, by the way, is certainly worth reading because it is so well written and has so many fine ideas concerning text-books). The authors plan to have enough material in the book for a year's course, yet so arranged as to permit selections of topics for shorter courses. They have included many starred sections and also other extra material for superior students. They strive for a clear and simple style and yet for accuracy of statement, for generality in their proofs, for clear-cut definitions. They have an introductory chapter containing formulas and tables from algebra and trigonometry. They use determinants freely, also the reduction and addition formulas for the trigonometric functions. They propose to tie their chapters together, to give more solved illustrative examples than is usual, to arrange the exercises so that the simpler problems precede the difficult.

The answers given at the end of the book are only for the odd-numbered problems. Many pairs of problems have the odd-numbered exercise similar to the even-numbered one. Polar coordinates are introduced at once and developed along with rectangular coordinates. Chapter I reviews elementary curve-plotting such as the student has already met with in algebra and trigonometry; a fuller treatment is then given in Chapter VIII. Chapter XIII, on curve-fitting, gives a practical treatment of the method of least squares. Chapters XIV–XVII contain a brief treatment of solid analytic geometry, including what is used in the calculus.

The authors have succeeded admirably in the plan they outline in their preface. Let us note now some other good points about this text. The book contains a wealth of material and numerous very good figures. It is easy to find any desired topics. No derivatives are used anywhere in the book. The treatments of loci and of symmetry and other parts of curve tracing are excellent. The book contains more advanced topics than many elementary texts, and this in spite of the fact that no space or words are spared in the effort to make everything clear. Many points that trouble the student and must usually be answered by the teacher are taken up in this book either in the text itself, or in the solved problems, or in the footnotes. This makes the book well adapted for home study.

The reader will be struck by some features of the text. For example, there are many ingenious proofs. Thus the derivation of the formulas for rotation of axes is made to depend on the expansion of cos $(\phi+\theta)$ and sin $(\phi+\theta)$. The famous nine point circle appears among the examples on page 109. Quite an exhaustive treatment is given of the reduction of conics to standard forms. Brief discussions are given of poles and polars with respect to conics, also of systems of conics, harmonic sets of points, and invariants.

There are some minor criticisms that might be made. For instance, why do the authors debar infinity from the book, and say that a line parallel to the y-axis has no slope? Why do not the authors discuss oblique coordinates? However, the good qualities of the book offset these few criticisms.

The book everywhere shows great care and precision. Thus each equation of a line is spoken of as an equation of the line and not as the equation. The chapter on parametric coordinates is very good and longer than usual. The chapter on curve-fitting will appeal to many teachers as very timely and useful. The treatment of the analytics of space is brief but quite satisfactory. The book has been painstakingly written and shows throughout its pages the earmarks of the long and successful teaching experience of the authors. In every way this is a splendid text, and it ought to appeal to a large clientele as very teachable and well adapted to their needs.

ALAN D. CAMPBELL

A Short History of Mathematics. By Vera Sanford. Houghton Mifflin Company, New York, 1930. xii+402 pages. $3.25

The need for a history of mathematics in one volume has increased during recent years with the growth of interest in this subject and of its importance as a background for the subject matter which is the equipment of secondary school teachers in mathematics. The books in the field twenty-five years ago were adequate for that period but a brief general work of real value will be welcome today as a contribution in supplementing or replacing these earlier works. This one is put out under the editorship of John Wesley Young with an introduction by David Eugene Smith. It will take its place among the standard histories of mathematics for its freshness of treatment, its wide range of subject matter and the sense of authority which it conveys.

The author has had access to and made generous use of the most recent studies and publications along many lines, notably *Sir Isaac Newton, 1727–1927*, edited by F. E. Brasch, *Source Book of Mathematics*, by David Eugene Smith, and *The Rhind Mathematical Papyrus*, by A. B. Chace. Use has been made of many excellent features which have already become familiar through the more extensive and elaborate two-volume work of the distinguished historian and author whose introduction has already been noted. These include maps, chronological details summarized in tables, facsimiles of early works, reproductions of photographs and pictures of instruments. The illustrations are profuse and several are made available for the first time. Such a one is: (p. 96) "Late Use of Roman Numerals in Commercial Accounts in England. From an account book of 1602 (44th year of Elizabeth) now in the collection of Professor David Eugene Smith."

An exceptionally good feature of the work is a summary at the end of each chapter. The one following the chapter on "Men Who Made Mathematics" is a particularly nice biographical characterization and is quoted in full to show the spirit of these summaries as well as the delightful style of the author.

"What has been given thus far is a picture of the development of mathematics, setting forth the type of people interested in the subject in different countries and at different periods: the priests of ancient Egypt, the merchants and astrologers of Babylonia, the philosophers of Greece, the practical Romans, the scholar-monks of the Dark Ages and of medieval times, the poet-minded writers of India and the Arabs who were as quick to assimilate the knowledge of other nations as they were quick to conquer their territory. Following these were the writers of practical textbooks in the century immediately after the invention of printing and, side by side with them, the speculative scholars who were frequently instructors in the universities and who were interested in the theoretical aspects of the subject. In the seventeenth century there is a group of notable figures, with Descartes, Pascal, Leibniz, and Newton as the most prominent. It is only by contrast with these exceptional figures that the men of the eighteenth century seem less important than their predecessors."

Various devices to aid in seeing the progress of a concept or a symbol are found throughout the book, such as (p. 153) "Methods of Representing Equality," and valuable classifications occur in connection with these. "The Representation of Unknown Quantities" will be cited; it has the sub-headings (p.

156): "By abbreviations and initial letters; By symbols; By exponents with no base given; By exponents with the base given." Such schemes as these help a reader in his first survey and help an instructor in presenting the subject through a text.

While the outstanding facts in connection with each chapter heading are well covered, especial attention may be called to the good general survey under "Commercial Mathematics," a chapter which is original in its treatment throughout and an original contribution from this standpoint, although much of the subject matter is found elsewhere. The space devoted to verbal problems is probably justified by the author in the initial sentence of the chapter: (p. 204) "The problem content of a textbook in arithmetic or in algebra is a measure of the mathematical interests and accomplishments of the group for whom it is written. To a somewhat less degree it is also an indication of the ability and interest of the author, . . . " The author speaks with special authority in this field. Her *History and Significance of Certain Standard Problems in Algebra* is a work which should be consulted generally in connection with the history of mathematics during the centuries which it covers.

The work shows insight and is well-expressed. One instance of this will suffice for many which might be given. (p. 85).

> "It is hard to judge the difficulty of computation with number systems other than our own, for we are apt to forget that the actual work is done mentally and that the figures are merely used to record the results. Thus lack of familiarity with the symbols hinders the rapid reading and writing of the numbers that are involved, and accordingly the numerals are branded as being too clumsy for use."

Speculation seems to be irresistible as to what would have happened if something else had not happened, all this from the standpoint of our present knowledge of development along one line. The author states first: (pp. 39–40) "His [Stifel's] *Arithmetica Integra* (1544) contained a comparison of geometric and arithmetic progressions which foreshadowed Napier's invention of logarithms nearly a century later." Later (p. 176) we find: "Stifel (1544) called the upper set of numbers *exponents* and extended both series to the left, thus including negative exponents . . . $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 & 512 \end{bmatrix}$ Close as this was to the idea of logarithms, Napier's invention was the result of a very different line of thought. It is an interesting speculation to try to gauge the influence of the recreational aspects of this subject in inhibiting the use of the idea in the earlier invention of logarithms. Had Napier's invention not come when it did, it is highly probable that logarithms would have come by way of Stifel's exponents." When will authors cease from these useless deductions! The sane attitude to take toward inventions is that found in the following: (p. 111) "The idea of the decimal fraction may indeed be traced to these isolated instances, but it is likely that the man who first developed laws for their use in ordinary computation was not influenced by these early and perhaps unwitting applications." This statement is found in a particularly good account of Stevin's work from the French edition of which the author has made a translation.

While in general no exception can be taken to the contents of the book, erroneous ideas are conveyed at times. (p. 14) " . . . thinker who saw that the two might be combined in a place-value system which would require only ten different characters to write any number, however large" might better read "only as many characters as there are units in the base,—in the common and generally accepted system ten." (p. 257) " . . . fifth century B.C. and from that time to the present the name 'circle-squarers' has been applied to men who have attempted to do things that seem to be impossible." Has the term not rather been applied to men who continue to assert that they can do these things after *science* has proved that they are impossible, making it an opprobrious term rather than a flattering one! (p. 265) "Hippocrates of Chios (c. 460 B.C.) had reduced the problem [duplication of the cube] of solving the equation $x^3=2a^3$ to finding two mean proportionals between the quantities a and $2a$." What Hippocrates did was to reduce the problem to one of plane geometry, namely, inserting two mean proportionals between the edge of the given cube and a line double that edge. The modern form may follow this if it is so stated explicitly. Is there any justification for: (p. 269) "In fact, it [Euclid's Elements] could aptly be called by the name 'General Mathematics'."? The publication of treatises covering different branches of mathematics under one cover does not fit into the blending of these branches, more or less successfully, which is understood by that designation. These instances may all be matters of differences of opinion with regard to an expression to use. The opinion of the reviewer has grown out of long experience in trying to make students in the classroom get the real meaning of just such expressions. Two footnotes relating to the same work should read the same, as they do not in references to a bulletin on pages 154 and 382, the title in the latter case being the correct one, although the middle name of the writer is misspelled.

The contents of the book are: Men Who Made Mathematics; Arithmetic; Commercial Mathematics; Algebra; Verbal Problems; Practical Geometry; Demonstrative Geometry; Trigonometry; Analytic Geometry; Calculus; Theory of Numbers; Calculating Devices; Weights and Measures; The Place of Mathematics in the School Curriculum. General Histories of Mathematics and a Chronological Outline conclude the work. Subtitles of one or two chapters will give a more comprehensive view of the ground covered. Those under "Arithmetic" are: The Name Arithmetic; Number Magic; Number Systems; Computation with Ancient Systems of Numerals; The Abacus; Hindu-Arabic Numerals; Computation with Hindu-Arabic Numerals; Fractions. The Chapter "Demonstrative Geometry" covers so wide a range of topics as: Early Greek Geometry; The Three Famous Problems; Euclid's "Elements"; Non-Euclidean Geometries; The Geometry of Conics; Modern Geometries.

The editor and the publisher are to be congratulated on this addition to the series of books relating to mathematics put out in recent years.

LAO G. SIMONS

Foundations of Potential Theory. (Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, XXXI). By O. D. Kellogg. Julius Springer, Berlin, 1929. ix+384 pages.

What are the desirable characteristics of a book on a classical field of applied mathematics? This question would probably receive somewhat different replies from different groups of scientific workers, such as the experimental physicists, the mathematical physicists, and the mathematicians. However, leaving aside the matter of individual or group preferences, let us consider the question from a broader standpoint. In order that a book of the type referred to may make the maximum contribution to the general development of scientific knowledge, what qualities should it possess? Without any expectation of securing universal agreement, the reviewer of the present volume ventures to formulate his own answer to this question.

First of all the fundamental hypotheses should be closely linked to current physical intuition. That is to say, they should be accurate mathematical formulations of what the general body of working physicists regard as the mechanisms of the physical phenomena involved. Second, the mathematical structure on these foundations should conform to present standards of mathematical rigor, and its generality should at least be adequate for any applications that have been made. If at all feasible, a margin of safety in the matter of generality should be allowed, as past history of the progress of physics shows clearly that the theoretical developments in this field demand an ever increasing degree of generality in the mathematical results that are used.

Professor Kellogg's book, which is one of the important Courant series and the first of American authorship, measures up well to the ideal standard described above. That it should agree with this standard in every particular is hardly to be expected, for even if the author's ideal were identical with the reviewer's, an occasional lapse is to be expected in any human performance. In this connection I am always reminded of one of the bits of dry humor with which Darboux was in the habit of enlivening his lectures. At the end of a rather lengthy and involved reduction he paused and looked quizzically at the final result. Then he closed that part of the discussion with the following peroration, " .... ce qui est exact, si je ne me trompe pas. Et pourquoi me tromperai-je? Parce que tout homme est faillible."

I do not mean to imply by the above anecdote that the details which I am about to criticize are necessarily instances of fallibility. They may merely illustrate a difference of opinion between the author and the reviewer. In any event they are what I should classify as minor defects, and I mention them primarily because they are exceptional instances in which the book departs from the ideal standard which I have formulated.

In the first place the reviewer does not approve of the terminology "continuously differentiable," originally introduced on page 97 and used subsequently throughout the book in connection with functions of one or several

variables. Since the word "continuously" modifies the word "differentiable," the implication of the phrase, taken by itself, is that there is something in the *process* of differentiation which is subject to continuous variation. However, the process of differentiation is obviously identical at all points; it is the *result* obtained from that process which exhibits the continuous variation in question. In the case of functions of a single variable, there is no gain in brevity by saying that a function is "continuously differentiable" in place of saying that it has a continuous derivative. In the case of functions of several variables, the statement that the function possesses a continuous directional derivative is somewhat longer but is free from the objection which has been raised.

One may contend that the above criticism has nothing to do with the essential requirements in my ideal standard, since the author accurately defines what he shall mean by functions "continuously differentiable." I should not agree with this contention, however; there is more than a mere matter of linguistic style involved. I prefer to interpret the term "mathematical rigor" in a broad manner. To conform to my idea of it, all definitions in words and all terminology in words should be so phrased as to furnish the maximum aid to clarity of thinking. I do not object to an abbreviated terminology which refers only to some central feature of the mathematical concept described. But I do object to a phrase whose internal structure implies in any way a property foreign to the concept in question.

On somewhat different but essentially similar grounds the reviewer does not approve of the term "piecewise differentiable" introduced on page 97, since the phrase implies that only the existence of a derivative is required, whereas the requirement of the continuity of that derivative is equally important. He objects to such phrases as "reasonably smooth functions" and "sufficiently smooth functions," found on pages 199 and 203 respectively, on the ground of their essential vagueness. In both cases a statement of adequate sufficient conditions would not require much more space. There is far too much shadowy formulation of mathematical requirements in the existing literature of mathematical physics. I always regret to see mathematicians condoning this tendency by even a slight leaning in this direction.

In the case of the discussion on page 203, referred to in the preceding paragraph, where the "sufficiently smooth function" is to be developed in a series of Bessel's functions, there is a further objection to the rather offhand statement as to the requirements to be made. It suggests that the conditions may be found in any standard treatise on the subject, whereas this is not the case. Even in the almost encyclopedic work of Watson, the uniform convergence of the series in question in an interval including the origin is not established, although the physical problem in which this uniform convergence enters goes back to Fourier. The only adequate mathematical discussion up to the present date is to be found in a paper by the present reviewer in the 1911 volume of the Transactions of the American Mathematical Society.

After formulating these criticisms of matters of detail, I do not wish to close

without reiterating my admiration for the general high standard of the book under discussion. It is a splendid piece of work, in which Professor Kellogg's colleagues and friends, as well as the author himself, can justly take pride.

CHARLES N. MOORE

Bernard Bolzano's Schriften. Band 1: Functionenlehre. Published by the Königlichen Böhmischen Gesellschaft der Wissenschaften, Prague, 1930. xx+184+24+vi pages.

In this book there appears in print for the first time the "Functionenlehre" of Bernard Bolzano, as it occurs in a manuscript which has been preserved in the National Library in Vienna.

The occasion for its publication is the nationalistic movement in Czechoslovakia, a movement intended to foster patriotic pride among the inhabitants of this ten-year-old republic which was created as a result of the war. As a part of this movement, efforts have been made to increase popular appreciation of the great artists and scholars which the Czechoslovakian nation has produced. Among these is Bernard Bolzano (1781–1848), famous as priest, philosopher, and mathematician.

Bolzano was born in Prague and spent his entire life in Bohemia. His father was an Italian who emigrated to Prague from northern Italy; his mother was a native of Prague. Appointed professor of the philosophy of religion at Prague at the age of 24, apparently Bolzano's teachings were not acceptable to the ruling powers of the state and consequently he was retired on pension after 15 years. Thus at the age of 40, with a sufficient income for the necessities of life, and forbidden to write on theological topics, he turned to writing on philosophy and mathematics.

Even before his appointment to the chair of theology Bolzano had been interested in mathematics. His first published paper appeared in 1804, a paper in which he attempted to prove Euclid's parallel axiom. He studied the works of Lagrange, Lacroix, and Cauchy, and until his death he continued writing on the foundations of philosophy and mathematics. To date comparatively little has been printed of the mathematical writings which he left in manuscript. The aim of the recently constituted Bolzano-Commission of the Königliche Böhmischen Gesellschaft is to publish the hitherto unprinted works of Bolzano and his scientific correspondence. "Functionenlehre" was chosen as the first of the series to be published by the commission, although it forms only the concluding part of a greater work on "Grössenlehre" which was planned by Bolzano. However, the complete manuscript of this larger work does not exist, and as "Functionenlehre" forms the most interesting part of the work and seems to be ready for publication, it has been published first.

The book begins with a general introduction by Professor K. Petr, describing the life and accomplishments of Bolzano and the purposes of this edition of his works. Then follows Bolzano's text, edited by Professor K. Rychlik with preface and notes. The text consists of three parts: An introduction (pp. 1–12)

on "Relations between two variables;" Chapter 1 (pp. 13–79) on "Continuous and discontinuous functions;" and Chapter 2 (pp. 80–183) on "Derived functions." The introduction is principally concerned with the form of the increment of $y=f(x)$ for certain particular types of functions, and with the limiting values of this increment as Δx approaches zero. Chapter 1 takes up continuity, and in particular left- and right-hand continuity of a function. In Chapter 2, the derivative of a function is defined in the usual way. Left- and right-hand derivatives are studied, but Bolzano decides not to introduce a new notation for them. The mean-value theorem, partial differentiation, integration, and Taylor's series are other topics which are taken up.

Probably the most interesting topic in the entire book is the construction of a function which is continuous in an interval but does not have a derivative at any point of its interval of definition. This example antedates by approximately fifteen years Weierstrass's famous example. The fact that Weierstrass and not Bolzano has received credit for this discovery only serves to emphasize how slight was Bolzano's influence on the development of mathematics. His printed papers on mathematics are few in number. He was not a teacher of mathematics; hence his ideas were not disseminated through the writings and teachings of his pupils. As his discoveries are now brought to light, we see that he was possessed of sound ideas on mathematical subjects, and often of ideas which were not generally known until much later. His criticisms of previously given definitions of continuity are for the most part valid. He points out an error of Galois (p. 96) and evaluates the work of others on Taylor's series. In regard to the function mentioned above, Bolzano proved only that it had no derivative at an everywhere dense set of points. Had he made this result more widely known, someone might have been able to point out the more general property of this function.

The text is printed practically as Bolzano wrote it. The spelling is for the most part as in the original manuscript. The chief difference from modern German seems to be in the use of "ey" for "ei" in words such as *bey*, *beyde*, *Beyspiel*, *frey*, *sey*, *seyn*, *zwey*. A vocabulary on page 1 of the "Notes" gives the modern equivalent of certain phrases used by Bolzano.

The editor has suppressed any desires to edit the text by eliminating the mistakes that Bolzano has made. (A few obvious errors, chiefly numerical errors in the examples, have been corrected.) In the "Notes," which form an exceedingly valuable portion of the book, Professor Rychlik points out which theorems are false and which are true but whose proof is either incorrect or incomplete. Occasionally an error in one theorem invalidates many others. The chief sources of error lie in Bolzano's ignorance of the idea of uniform continuity and his belief that a function of two variables which is continuous in each variable separately is necessarily continuous in both. Incidentally a reading of this text serves to emphasize the superiority of the modern "epsilon-delta" notation. When Bolzano's incorrect proofs are recast in modern notation, the errors in his reasoning are easily seen, whereas the proofs as originally given seem entirely plausible.

The last two pages of the book give a list of misprints. The reviewer noticed a dozen others, all trivial. On pages 22–4 of the "Notes" is a bibliography, which includes the titles of those books from Bolzano's private library which are now in the library of the University of Prague. The title of Young's book as given there does not agree with that given on page 170 of the text. Among the modern books, the third edition of Hobson's *Theory of Functions* might well have been mentioned.

This book will be of much interest to those studying the development of mathematics. It can not, of course, be evaluated as a text on function theory by comparing it with modern texts on that subject. Many things are done now in a better way; but taking into account the times in which he lived, Bolzano seems to have done a good job as a textbook writer. While his rigor is at times not up to modern standards, his manuscript might well serve as the framework of a modern treatise on the theory of functions.

Another interesting point is that Professor Rychlik states (p. 5 of the "Notes") that the proof of the theorem commonly known as the "Bolzano-Weierstrass Theorem" is not to be found either in Bolzano's manuscripts nor in his printed works.

HARRY MERRILL GEHMAN

Statistical Methods for Research Workers. By R. A. Fisher. Third Edition—Revised and Enlarged. Oliver and Boyd, London, 1930. xiii+283 pages+6 tables to be extended outside the text. 15s.

The original edition of this monograph, in a series of "Biological Monographs and Manuals," was greatly appreciated as a product of intimate co-operation with several biological research departments and of daily contact with statistical problems that arise amongst the most advanced of contemporaneous workers.

In the present edition, the section numbers have remained unchanged and the content of most of the sections is the same now as in the first edition. The chief aim of the two revisions seems to be to amplify the principles of statistical estimation, to arouse more interest in the "Method of maximum likelihood," pp. 14, 243 ff., and also to present the most recent developments in statistical method. Two new sections are added to chapter IX which replaced section 6 and example 1, of the first edition. They illustrate further the applicability of the method of maximum likelihood and the quantitative evaluation of information. In his preface, the author laments that, in spite of their practical importance, these new ideas have not met ready acceptance. The summary of principles on pages 270, 271 may show the reason as well as Professor Fisher's confidence in his method, which consists in multiplying the logarithm of the number expected in each class by the number observed, summing for all classes and finding the value of θ for which the sum is a maximum. To read, "In practice one need seldom do more than solve, at least to a good approximation, the

equation of maximum likelihood, and calculate the sampling variance of the estimate so obtained," brings to mind the admonition to beware of any magical process that works though roughly applied.

To consider the "number expected" implies that the type of the universe is known or assumed; to multiply the number observed by the logarithm of the number expected seems like weighting the observations in an unusual way and fitting the products to a selected type of distribution. The choice of the type of universe is far more difficult than curve-fitting, but this is taken as a matter of course, so the special curve-fitting is less impressive. We are told that when the theory of large samples no longer holds, efficient statistics, other than that obtained by the method of maximum likelihood, may fail, but the text has not yet enough in it to convince us of the trustworthiness of small samples without special, independent consideration in each case.

Whether fully warranted or not, some recent writers have attributed to the work of Dr. R. A. Fisher their inspiration for emphasizing the value of small samples to such a degree that we fear that the engineers are in danger of being beset by the small sample specialists somewhat as the psychologists were by the correlaters fifteen years ago.

Amongst the over enthusiastic advocates may be mentioned W. A. Shewhart and F. W. Winters,[1] whose results "so recently obtained" do not differ essentially from those obtained by Thiele,[2] without the questionable emphasis of small samples and without assuming a normal distribution.

As a matter of fact the groundwork and the principles of the socalled "new" work on samples was laid more than forty years ago by Thiele in his monumental work *Almindelig Iagttagelseslaere*,[2] a fact of which the small sample enthusiasts seem to be ignorant. It is therefore not at all surprising to find that Dr. John Wishart,[3] another of the small sample advocates, has published the following statement in a paper in "*The Proceedings of the Royal Society of Edinburgh*":—

"As an estimate of the semi-invariant, κ, Mr. R. A. Fisher has determined a series of symmetric functions k in such a way that the mean value of any k over all samples shall be equal to the corresponding semi-invariant, κ, in the sampled population. The simplest of his results are:

$$
\text{(5a)}\quad
\begin{aligned}
k_1 &= m_1' = N^{-1}\sum_1^N(x),\\
k_2 &= N(N-1)^{-1}m_2,\\
k_3 &= N^2(N-1)^{-1}(N-2)^{-1}m_3,\\
k_4 &= N^2(N-1)^{-1}(N-2)^{-1}(N-3)^{-1}\{(N+1)m_4 - 3(N-1)m_2^2\}.
\end{aligned}
$$

[1] Journal of the American Statistical Association, June (1928), p. 144 ff.

[2] Almindelig Iagttagelseslaere (Copenhagen, 1889). Since there is a nice idea wrapped up in Thiele's carefully chosen Danish title, it may be well to translate it, "The general doctrine of observations."

[3] Proceedings of the Royal Society of Edinburgh, Session 1928–29, vol. 49, part 1, No. 7

In the 1889 edition of Thiele's work we find the following formulas:[4]

$$
(46)\qquad
\begin{aligned}
\lambda_1 &= \mu_1 = m^{-1}\sum_1^m x,\\
\lambda_2 &= m(m-1)^{-1}\mu_2,\\
\lambda_3 &= m^2(m-1)^{-1}(m-2)^{-1}\mu_3,\\
\lambda_4 &= m^3(m-1)^{-1}(m^2-6m+6)^{-1}\{\mu_4+6(m-1)^{-1}\mu_2^2\}.
\end{aligned}
$$

The chief difference in the formulas seems to be that R. A. Fisher has replaced the letters λ, μ, and m in Thiele's notation by the letters k, m, and N. Another quotation from Dr. Wishart[3] may be of interest, viz.,

"Thus the well-known formulas for the moments of the distribution of the mean in samples are summed up, in the notation of this paper, by the formula

$$\kappa(1^r) = \kappa_r/N^{r-1},$$

or in words: The rth semi-invariant of the distribution of the mean is equal to the rth semi-invariant of the sampled population, divided by N^{r-1}."

On page 60 of Thiele's book[2] this result is written as

$$\lambda_r(\mu_1) = m^{1-r}\lambda_r.$$

One, of course, may be dense, but it is difficult to see in what essential way these older formulas differ from those attributed to the reputedly "new" work by the small sample experts. Moreover, Thiele published also in 1889 the error laws (expressed in terms of the presumptive semi-invariants and the number of observations, m) of the semi-invariants, μ, as observed in the sample, and gave besides a numerical table of the sum products of the power sums up to the eighth power so as to facilitate the calculation of the product moments of the semi-invariants.[5] For a concise presentation of the various symmetric functions as employed by the Scandinavian mathematicians in statistical work, interested readers may well look up a review by Mr. Arne Fisher of a book by R. Frisch.[6]

Another phase in which some of the "new" ideas of R. A. Fisher seem to have been anticipated, by several decades at least, by the Scandinavians is in the application he makes of orthogonal functions in fitting regression curves to polynomials. Work of a similar character was also done in 1927 by Birge and Shea in their article in the "University of California Publications in Mathematics"[7] on *A rapid method for calculating the least square solution of a polynomial of any degree.* This reputedly original and new method was, however, described by Gram in his 1879 Danish dissertation on "Raekkeudviklinger."

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[4] Loc. cit., p. 63.

[5] Loc. cit., pp. 61, 62.

[6] Journal of the American Statistical Association, No. 159, Sept. (1927), p. 402.

[7] Vol. 2, No. 5 (issued in March, 1927), pp. 67–118.

Further, Gram gave in the 1915 volume of the "Bulletin de l'Association des Actuaires Suisses," pp. 3–23, a very elegant demonstration of the method together with a numerical table of the generating polynomials up to the fourth order for uneven values of n from 7 to 21 inclusive. The values given in this table, except for an occasional constant multiplier, -1, are exactly the same as the corresponding values of the somewhat more extensive set of tables by Messrs. Birge and Shea, which also cover the even values of n. When they speak of "our new method" we are reminded that "there is nothing new under the sun." The polynomials themselves date back, as mentioned by Gram, even to the eminent Russian mathematician Tchebycheff. The same essential ideas of the Gram polynomials occur also in R. A. Fisher's work, but Dr. Fisher fails to mention either Tchebycheff or Gram.

Finally, we call attention to a paper by J. O. Irwin[8] wherein a comparison is made between Pearson's test for "goodness of fit" and another test devised by Irwin himself. This latter test, which appears superior to the Pearsonian test, is however a special case of Thiele's so-called "Fejlkritik" (error critique) which also is found in the 1889 edition of Thiele's work.

This review has been extended to this length for several reasons:

First, to point out that there is some ground for feeling that Dr. Fisher is too parsimonious in references to other workers outside the fellowship of his own countrymen, and has thus led his followers to remain ignorant of other rich fields.

Second, to call attention to that galaxy of astronomers and actuaries of the first magnitude who have persisted too long in being self-sufficient, publishing only in their native language. We note with appreciation that some of them have started to publish in English.[9]

Third, to serve to arouse some of our mathematicians, especially those who are linguistically inclined, to delve more into the Scandinavian languages.

CHARLES C. GROVE

Curve piane speciali algebriche e trascendenti. Teoria e storia. Volume II.—Curve trascendenti—Curve dedotte da altre con 58 figure illustrative intercalate nel testo. Prima edizione Italiana. By Gino Loria. Ulrico Hoepli, Milano, 1930. xi+439 pp.

The second volume of the first Italian edition of Loria's treatise has followed the first volume with unusual promptness. There are no essential departures from the well-known second German edition of 1910. This volume contains

[8] Journal of the Royal Statistical Society, vol. 92, part 2 (1929).

[9] T. N. Thiele, *Theory of Observations*, 1903 (an elementary extract of the 1889 Danish edition and a translation of a work in Danish issued in 1897). C. L. Charlier, *Researches on the Theory of Probability* (1905). J. F. Steffensen, *Interpolation*, 1927 (the first complete text to appear in English). *Skandinavisk Aktuarietidskrift*, started in 1914, has about two-thirds of its articles in English.

Book VI, consisting of 25 chapters on various transcendental curves, and Book VII with 12 chapters on curves derived from other curves. Some 200 new references, and 60 odd additions to the index of names bring the book thoroughly up to date.

We regret the lack of a subject index for this and for the first volume; and we regret the lack of care in proofreading which makes it highly unsafe to use references without checking them. Nevertheless we are convinced that this is the most useful reference book in this field.

B. H. BROWN

PROBLEMS AND SOLUTIONS

EDITED BY B. F FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

PROBLEMS FOR SOLUTION

N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

3463. *Proposed by D. L. Holl, Ames, Iowa.*

Sum

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3}\left[\operatorname{sech}\frac{1}{2}\pi(2n-1) + (-1)^{n+1}\tanh\frac{1}{2}\pi(2n-1)\right].$$

3464. *Proposed by I. Maizlish, Centenary College of Louisiana.*

A telegraph pole is in the form of a frustum of a right circular cone of altitude h, diameter of lower base D, and diameter of upper base d. A rope, of radius r, is wound spirally around this pole from the bottom to the top, covering the whole pole. A device is constructed which unwinds the rope—beginning at the top. If the angular velocity, w, with which the rope is being unwound is a function of the length of the rope already unwound, find the time it will take the device to unwind the whole rope.

3465. *Proposed by J. P. Dalton, University of Witwatersrand, Johannesburg, South Africa.*

Prove that for all positive integral values of n

$$\sum_{i=n}^{2n}(-1)^{2n-i}\frac{(2i+2)!}{i!(2n-i)!(2i-2n)!} = 2^{n-1}(n+1)(2n+1)(2n^2+7n+4).$$

3466. *Proposed by Leonard M. Blumenthal, Rice Institute.*

The lineo-linear expression, which we may exhibit in the matrix form

$$\begin{array}{c|cccc}
 & x_1 & x_3 & x_5 & x_{2n-1} \\
\hline
\bar{x}_2 & n-1 & -1 & -1 \cdots & -1 \\
\bar{x}_4 & -1 & n-1 & -1 \cdots & -1 \\
\bar{x}_6 & -1 & -1 & n-1 \cdots & -1 \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\bar{x}_{2n} & -1 & -1 & -1 & n-1
\end{array}$$

where the coefficient of $x_{2k-1}\bar{x}_{2j}$ is found at the intersection of the kth column and the jth row, and $\bar{x}_{2j}$ is the conjugate of x_{2j}, is invariant under translations applied to either set x_{2i-1}, x_{2i} of the variables, and also under the rotations $y_i = tx_i$, $t = e^{i\theta}$ $(i = 1, 2, \cdots 2n)$.

Considering the complex numbers x_{2i-1}, x_{2i} $(i = 1, 2, \cdots, n)$ as defining the vertices of two ordered plane n-gons, find a geometric interpretation of this invariant.

3467. *Proposed by J. Rosenbaum, Melford, Conn.*

Solve in positive integers: $2x^2+2x+1=y^2$.

3468. *Proposed by Charles A. Rupp, Pennsylvania State College.*

Show that the determinant of n^2 elements in the upper left corner of the Pascal triangle

$$\begin{array}{cccccc}
1 & 1 & 1 & 1 & \cdot & \cdot \\
1 & 2 & 3 & \cdot & \cdot & \\
1 & 3 & \cdot & \cdot & & \\
1 & \cdot & \cdot & & & \\
\cdot & \cdot & & & & \\
\cdot & & & & &
\end{array}$$

has the value unity.

SOLUTIONS

2856. [1920, 377]. *Proposed by O. S. Adams, U. S. Coast and Geodetic Survey.*

Show that for the real domain defined by $+1 > x > -1$, s a positive integer,

$$\frac{1}{(1-x^s)^{1/s}} \int_0^x \frac{dx}{(1-x^s)^{(s-1)/s}} = x + \sum_{n=1}^{n=\infty} \frac{2(s+2)(2s+2)\cdots(ns-s+2)}{(s+1)(2s+1)\cdots(ns+1)} x^{ns+1}$$

and

$$\frac{1}{(1-x^s)^{(s-1)/s}}\int_0^x \frac{dx}{(1-x^s)^{1/s}} = x + \sum_{n=1}^{n=\infty} \frac{n!s^n}{(s+1)(2s+1)\cdots(ns+1)}x^{ns+1}.$$

Solution by Morgan Ward, California Institute of Technology.

Let

$$\int_0^x \frac{dx}{(1-x^s)^{(s-1)/s}} = x(1-x^s)^{1/s}F(u), \tag{1}$$

$$\int_0^x \frac{dx}{(1-x^s)^{1/s}} = x(1-x^s)^{(s-1)/s}G(u), \quad u = x^s.$$

The solution of the problem then reduces to showing that

$$F(u) = 1 + \sum_{n=1}^{\infty} \frac{2(s+2)(2s+2)\cdots(ns-s+2)}{(s+1)(2s+1)\cdots(ns+1)}u^n, \tag{2}$$

$$G(u) = 1 + \sum_{n=1}^{\infty} \frac{n!s^n}{(s+1)(2s+1)\cdots(ns+1)}u^n. \tag{3}$$

Assume that $+1 > x > -1$. Then we may differentiate both sides of (1) with respect to x, obtaining

$$\frac{1}{(1-x^s)^{(s-1)/s}} = [(1-x^s)^{1/s} - x^s(1-x^s)^{(1-s)/s}]F + sx^s(1-x^s)^{1/s}\frac{dF}{du},$$

or

$$su(1-u)(dF/du) + (1-2u)F = 1. \tag{4}$$

If equation (4) has a solution expressible as a power series,

$$F(u) = \sum_{n=0}^{\infty} a_n u^n,$$

it is evident that $a_0 = 1$. By substituting this expression in equation (4) and equating coefficients of u^n, we obtain the recurrence formula

$$a_n = a_{n-1}(ns-s+2)/(ns+1) \qquad (n = 1, 2, \cdots),$$

from which (2) follows immediately. Then, since (2) is absolutely convergent for $|u| < 1$, it is a solution of (4).

In a similar manner, we find that $G(u)$ satisfies the differential equation

$$su(1-u)(dG/du) + (1-su)G = 1. \tag{5}$$

As before, we find that the only solution expressible as a power series in u is

$$G(u) = \sum_{n=0}^{\infty} b_n u^n,$$

where

$$b_0 = 1,\ b_n = b_{n-1}ns/(ns+1)\ \ (n = 1, 2, \cdots\ ;$$

from this (3) follows at once. Again, since (3) is absolutely convergent for $|u|<1$, it is a solution of (5). It is evidently unnecessary to restrict s to be an integer, but it is necessary that s be positive in order that $|x|<1$ may imply that $|u|<1$.

3124. [1925, 138]. *Proposed by M. B. Porter, University of Texas.*

$f(x)$ and $\phi(x)$ are polynomials and all the zeros of $\phi(x)$ are real. Let $P(x)=\phi(x)[f^{(i)}(x)]^r$ where $f^{(i)}$ stands for the ith derivative of $f(x)$.

Prove that $[P'(0)]^2<P''(0)P(0)$ is a sufficient condition that $f(x)$ has imaginary zeros. Newton's test[1], $C_i^2 \leqq C_{i-1}C_{i+1}$, where C_i is the coefficient of x^i in $f(x)$, is a special case.

I. Solution by Otto Dunkel, Washington University.

All the polynomials considered here are required to have real coefficients. The theorems of the problem may be deduced in various ways from well known and easily proved theorems. One of these theorems is as follows:

If the derivative $P'(x)$ of the polynomial $P(x)$ has $2i$ imaginary roots, then $P(x)$ has at least $2i$ imaginary roots.

We proceed to apply this lemma. Replace x in $P(x)$ by $h+y^{-1}$, where h is a real quantity. The equation $P(x)=0$ is replaced by

$$(1)\qquad P(h)y^p + P'(h)y^{p-1} + \tfrac{1}{2}P''(h)y^{p-2} + \cdots = 0,$$

where p is the degree of $P(x)$, $p \geqq 2$. The $(p-2)$th derived equation is

$$(2)\qquad p(p-1)P(h)y^2 + 2(p-1)P'(h)y + P''(h) = 0.$$

If now

$$(3)\qquad [P'(h)]^2 < p(p-1)^{-1}P(h)P''(h),$$

the equation (2) has imaginary roots; and, by the repeated use of the lemma, we see that (1) has at least a pair of imaginary roots. It then follows that $P(x)=0$ has at least a pair of imaginary roots.

If then h is real and

$$(4)\qquad [P'(h)]^2 < P(h)P''(h), \quad \text{or} \quad = P(h)P''(h) \neq 0,$$

it is obvious that (3) must be true. For the same value of h the test (3) is better than (4), since it does not require as much as (4). If then (3) or (4) is true $P(x)=0$ has at least one pair of imaginary roots. If $P(x)\equiv\phi(x)[f^{(i)}(x)]^r$, $r>0$, where $\phi(x)$ has no imaginary roots, then $f^{(i)}(x)$ must have some imaginary roots, and it then follows that $f(x)$ must have at least a pair of imaginary roots.

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[1] Cf. Netto: *Vorlesungen über Algebra*, vol. I, p. 234, edition of 1896.

If we take $P(x) = f^{(i-1)}(x)$, where

$$f(x) = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n, \quad C_0C_n \neq 0, \tag{5}$$

and set $h=0$, then (3) becomes

$$C_i^2 < \frac{n-i+1}{n-i}\,\frac{i+1}{i}\,C_{i-1}C_{i+1}. \tag{6}$$

If (6) is satisfied for any one of the $n-1$ values of i, then the equation (5) has at least one pair of imaginary roots. Also (6) is obviously satisfied if

$$C_i^2 < C_{i-1}C_{i+1}, \text{ or if } C_i^2 = C_{i-1}C_{i+1} \neq 0. \tag{7}$$

If two consecutive coefficients of (5) are zero, it is well known that there must be at least one pair of imaginary roots. This fact may also be proved by the above reasoning. A similar remark applies to (1).

Another method for obtaining (4) directly from (1) is as follows: If $r_1, r_2, \cdots, r_p$ are the roots of (1), $-\sum r_i = P'(h)/P(h)$, $\sum r_ir_j = P''(h)/2P(h)$, and

$$\left[\sum r_i\right]^2 - 2\sum r_ir_j = \sum r_i^2. \tag{8}$$

The rest of the proof follows easily.

A third method of obtaining (4), without the use of (1), is to employ the identity

$$\frac{P'(h)}{P(h)} = \sum_{i=1}^{i=p} \frac{1}{h-r_i}. \tag{9}$$

By taking the derivatives of both sides and making a slight transformation in the result, we find

$$[P'(h)]^2 - P(h)P''(h) = \sum{}'[(h-r_1)(h-r_2)\cdots(h-r_p)]^2, \tag{10}$$

where the accent mark means that in each of the p products a linear factor is omitted.

A fourth form of proof may be obtained by multiplying $P(x)$ by a linear factor $x-a$, where a is real, and then choosing for a a value which makes one of the intermediate coefficients of the product zero. Then apply Descartes' rule of signs in the usual way to detect imaginary roots. See the article by Oliver D. Kellogg, *A necessary condition that all the roots of an algebraic equation be real*, Annals of Mathematics, ser. 2, vol. 9, 1908, p. 97.

The tests (6) and (7) may be called quadratic tests. By a slight variation of the first method a variety of other tests may be obtained.

II. Solution by Paul Wernicke, Washington, D. C.

Sylvester's extension of Newton's rule, which is capable of some simplification *if from the start* we assume Netto's[1] Case I, leads to $a_{n-1}^2-2a_{n-2}a_n>0$ if $f(z)$ is to have real roots only. This is $[f'(0)]^2-f''(0)f(0)>0$. With r a positive integer, writing $[f^{(i)}(z)]^r=F$, we have

$$(P')^2 - P''P = [(\phi')^2 - \phi''\phi]F^2 + [(F')^2 - F''F]\phi^2$$

the first term of which cannot be negative for $z=0$ by the condition on ϕ. Therefore the bracket of the second term is negative and $F(z)=0$ or $[f^{(i)}z]^r=0$; hence $f^{(i)}(z)=0$ has complex roots. By the last paragraph on p. 208 (Netto, loc. cit., III) this is a sufficient condition for all preceding derivatives and $f(z)$ itself to have complex roots.

Since it is always possible to choose three coefficients of ϕ so that (for $r=1$) $(P')^2-P''P$ for any i becomes equal to one of the expressions (15) referred to, Newton's "test" is a special case of the above.

3412. [1930, 157]. *Proposed by P. R. Rider, Washington University.*

The vertex of a right circular cone of semi-vertical angle α is the center of a unit cube, the axis of the cone coinciding with a diagonal of the cube. Find the area of that part of the conical surface which is inside the cube.

Partial Solution by V. F. Ivanoff, San Francisco, Cal.

Let the center O of the cube be the origin of rectangular coordinates; let the positive z-axis lie upon the sem-diagonal OC, where C is a vertex of the cube; and let the plane zOx be perpendicular to a face through C. The equation of this face is

$$z = -2^{1/2}x + 3^{1/2}2^{-1}; \tag{1}$$

and the equations of the cone are

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \quad z = \rho\cot\alpha. \tag{2}$$

From (1) and (2) we have

$$\rho = \frac{3^{1/2}}{2(\cot\alpha + 2^{1/2}\cos\theta)} \cdot \tag{3}$$

The differential of the conical surface inside the cube may be written

$$dP = \frac{1}{2}\cdot\rho d\theta\cdot\rho\csc\alpha = \frac{3d\theta}{8\sin\alpha(\cot\alpha + 2^{1/2}\cos\theta)^2} \cdot \tag{4}$$

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[1] Loc. cit.; p. 231, bottom; p. 234, bottom; p. 234, expressions (15).

Hence

$$P = \frac{9}{4 \sin \alpha} \int_0^{\pi/3} \frac{d\theta}{(\cot \alpha + 2^{1/2} \cos \theta)^2}, \tag{5}$$

where P is the surface of one sheet of the cone within the cube. This integral may be transformed into a rational integral by known methods of the calculus.

The result (5) is true only when $\tan \alpha \leqq 2^{3/2}$. If $\tan \alpha > 2^{3/2}$ the conical surface of one sheet intersects all faces of the cube and the problem becomes more complicated.

3413. [1930, 157]. *Proposed by P. R. Rider, Washington University.*

The axis of a right circular cylinder of radius r coincides with a diagonal of a unit cube. Find the area of that part of the cylindrical surface which is inside the cube.

Partial Solution by V. F. Ivanoff, San Francisco, Cal.

With the same disposition of the axes as in the solution of 3412, we have again the equation

$$z = -2^{1/2}x + 3^{1/2}2^{-1}, \tag{1}$$

and the equations of the cylinder,

$$x = r \cos \theta, \quad y = r \sin \theta. \tag{2}$$

Hence for the intersection of the cylinder with the face (1) we have

$$z = -2^{1/2}r \cos \theta + 3^{1/2}2^{-1}. \tag{3}$$

The differential of the cylindrical surface between the xOy-plane and the face is

$$dP = rd\theta \cdot z = r(3^{1/2}2^{-1} - 2^{1/2}r \cos \theta)d\theta; \tag{4}$$

and the total surface within the cube is

$$\begin{aligned} P &= 6r \int_0^{\pi/3} (3^{1/2} - 2^{3/2}r \cos \theta) d\theta \\ &= 2 \cdot 3^{1/2} r(\pi - 3 \cdot 2^{1/2} r). \end{aligned} \tag{5}$$

The result (5) is true only when all the elements of the cylinder lie within the cube, i.e. when $r \leqq 2^{-\frac{1}{2}}$. In the cases where $2^{-1/2} < r < 2^{\frac{1}{2}} \cdot 3^{-\frac{1}{2}}$ the problem is more complicated. If $r \geqq 2^{\frac{1}{2}} \cdot 3^{-\frac{1}{2}}$ the cube is inside the cylinder.

Also solved by A. Pelletier.

Note by the Editors: The proposer of these two problems (3412, 3413) states that they are special cases of problems which arise in certain statistical studies. See the article by Paul R. Rider, *On the distribution of the ratio of mean to standard deviation in small samples from non-normal universes*, Biometrika, vol. XXI, Dec., 1929, pp. 139–141, and the references cited therein. See also a similar

problem in an article by G. A. Baker, *Random sampling from non-homogeneous populations*, Metron, vol. 8, no. 3, Feb., 1930, pp. 74–76.

3414. [1930, 157]. *Proposed by B. C. Wong, Berkeley, California.*

Prove or disprove:

$$\sum_{i=0}^{t}\left[\frac{(r-1)!(r-2i)}{(r-i)!i!}\right]^2=\frac{(2r-2)!}{r!(r-1)!},$$

where $t=(r-2)/2$ if r is even and $t=(r-1)/2$ if r is odd.

Solution by Emma T. Lehmer, Brown University.

Using binomial coefficient notation we are to prove that

$$f(r)\equiv\sum_{i=0}^{[(r-1)/2]}\left[\binom{r}{i}\frac{r-2i}{r}\right]^2=\frac{1}{r}\binom{2r-2}{r-1}. \tag{1}$$

Changing i to $r-i$ in (1) and adding the result to (1), we get

$$f(r)=\frac{1}{2}\sum_{i=0}^{r}\left[\binom{r}{i}\frac{r-2i}{r}\right]^2$$

or

$$f(r)=\frac{1}{2}\sum_{i=0}^{t}\binom{r}{i}^2-2\sum_{i=0}^{r-1}\binom{r}{i+1}\binom{r-1}{i}+2\sum_{i=0}^{r-1}\binom{r-1}{i}^2. \tag{2}$$

To evaluate these sums, consider the identity

$$(1+x)^r(1+x)^q\equiv(1+x)^{r+q}.$$

Equating coefficients of x^{r-s} on both sides we obtain

$$\sum_{i=0}^{q}\binom{r}{s+i}\binom{q}{i}=\binom{r+q}{q+s}. \tag{3}$$

As special cases of (3) we have

$$\sum_{i=0}^{r}\binom{r}{i}^2=\binom{2r}{r}\quad\text{and}\quad\sum_{i=0}^{r-1}\binom{r}{i+1}\binom{r-1}{i}=\binom{2r-1}{r}.$$

Substituting these results in (2) we obtain

$$\begin{aligned}f(r)&=\frac{1}{2}\binom{2r}{r}-2\binom{2r-1}{r}+2\binom{2r-2}{r-1}\\&=\binom{2r-2}{r-1}\left[\frac{2r-1}{r}-2\frac{2r-1}{r}+2\right]=\frac{1}{r}\binom{2r-2}{r-1}.\end{aligned}$$

Also solved by W. I. Miller.

3415. [1930, 157]. *Proposed by L. S. Johnston, University of Detroit.*

Let A, C, O, C', A' be collinear points, with $OA=OA'$, $OC=OC'$; let l_1, l_2, and l_3 be lines perpendicular to AA' through A, O, and A' respectively. With any point P on l_2 as center and PC as radius a circle is drawn. Let d_1 and d_2 be the diameters determined by the intersections of this circle with l_1 and l_3. Find the envelope of d_1 and d_2.

I. Solution by A. Pelletier, Montreal, Canada.

Taking AA' for the x-axis, l_2 for y-axis, and letting $(0, m)$ be the coördinates of any point, P, on l_2, $OA=OA'=a$, $OC=OC'=c$, we have for the equation of the circle having center P and passing through C, $x^2+(y-m)^2=m^2+c^2$, or $x^2+y^2-2my-c^2=0$.

Solving this equation simultaneously with the equations of the lines $x=\pm a$ for points of intersection and then writing the equations of the lines passing through the two pairs of points symmetric with respect to the center, $(0, m)$, we have the equations of the two diameters d_1 and d_2:

$$y - m = \pm (m^2 + c^2 - a^2)^{1/2}a^{-1}x.$$

Eliminating m, the variable parameter, between these two equations and their first derivative equation with respect to m, we obtain for the equation of the envelopes of d_1 and d_2,

$$a^{-2}x^2 - (c^2 - a^2)^{-1}y^2 = 1,$$

which is the equation of an hyperbola when $c>a$, and an ellipse when $c<a$.

II. Solution by Otto Dunkel, Washington University.

Set $OA=a$, $C'O=OC=c<a$, and consider one of the variable lines d_1 cutting l_2 and l_1 in P and Q, respectively, where $PQ=PC$. Let the foot of the perpendicular from C to d_1 be N, and produce CN to D so that $CN=ND$. Draw the line $C'D$ cutting d_1 in T, and then draw CT. It will be shown that d_1 is tangent at T to the ellipse having foci C', C, and vertices A', A. For the triangles CPQ and AON are similar and isosceles, as easily follows from the fact that C, O, P, N are concyclic, and similary for Q, A, C, N. Hence $ON=OA=a$. Moreover, $C'T+TC=2ON=2a$, and TC' and TC make equal angles with d_1. From symmetry the same reasoning applies to d_2.

If $c>a$, the same kind of reasoning shows that the envelope is an hyperbola.

Also solved by Rufus Crane, P. S. Dwyer, J. H. Neelley, and the Proposer.

3416 [1930, 157]. *Proposed by William Hoover, Columbus, Ohio.*

If a circumscribed triangle to a given circle have two sides fixed and the third variable, to determine the envelope of its circumcircle.

Solution by William I. Miller, University of Pittsburgh.

Using oblique Cartesian coordinates, place the triangle with the fixed vertex at the origin, and the fixed sides along the positive ends of the axes. Let the

fixed angle of the triangle be 2ϕ. Let the center of the circle be the point (h, h), and its radius $r=h \sin 2\phi$. Let 2α be the angle exterior to the triangle which the variable side makes with the X-axis, and let the X and Y intercepts of this variable line be a and b, respectively. By trigonometry,

$$a = h + h \cos 2\phi + r \tan \alpha = 2h \cos \phi \cos (\alpha - \phi) \sec \alpha,$$

$$b = h + h \cos 2\phi + r \cot (\alpha - \phi) = 2h \cos \phi \sin \alpha \csc (\alpha - \phi).$$

The equation of the circle through $(0, 0)$ $(a, 0)$, and $(0, b)$ is

$$f(x, y) \equiv x^2 + y^2 + 2xy \cos 2\phi - ax - by = 0.$$

To obtain the equation of the envelope, substitute the above values for a and b and eliminate α between $f=0$ and $\partial f/\partial\alpha=0$. The latter equation reduces to

$$\cos \alpha \csc (\alpha - \phi) = (x/y)^{1/2}.$$

From this,

$$a = 2h[1 + (y/x)^{1/2} \sin \phi], \quad b = 2h[1 + (x/y)^{1/2} \sin \phi].$$

Only the positive values of the radicals are used, since $0<\phi<\alpha<90°$ and $a>2h$, $b>2h$. Substituting these values for a and b in the equation of the circle, we obtain an equation which is factorable:

$$(x + y + 2(xy)^{1/2} \sin \phi)(x + y - 2(xy)^{1/2} \sin \phi - 2h) = 0.$$

The first factor, if rationalized, represents two conjugate imaginary lines, whose real point of intersection is the origin. The second factor, if rationalized, represents a circle tangent to the coordinate axes with center at $(h \sec^2\phi, h \sec^2\phi)$ and radius $2h \tan \phi$. But as it stands, this factor represents only the larger of the two arcs determined by the points of contact with the axes; the smaller arc is obtained if the given circle be escribed to rather than circumscribed about the given triangle.

To construct the envelope circle geometrically, let $OABC$ be the rhombus circumscribing the given circle, with OA and OC the fixed lines. At A erect a perpendicular to OA, intersecting the diagonal OB at D; then D is the center and AD the radius of the required circle. The larger of the two arcs AC, together with the point O, is the required envelope.

Also solved by Rufus Crane, A. Pelletier, O. J. Ramler, Mabel M. Young, and the Proposer.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

Associate Professor Karl Menger, of the University of Vienna, who is lecturing at Harvard University during the present semester, will lecture at Rice Institute during the second semester this year. At Rice Institute he will give an undergraduate course on "Metrical Geometry" and a graduate course on "Theory of Dimension and Curve Theory."

Professor O. D. Kellogg, of Harvard University, has been appointed exchange professor to a group of three middle-western colleges for the second half of the present academic year. He will spend about five weeks lecturing at each. The three colleges are: Carleton College, Northfield, Minnesota; Knox College, Galesburg, Illinois; and Colorado College, Colorado Springs, Colorado.

Princeton University has conferred an honorary doctorate on President K. T. Compton of the Massachusetts Institute of Technology.

The following persons have been appointed National Research Fellows in Mathematics for the academic year 1930–31 (This list includes reappointments, and may not be complete): R. P. Agnew, A. C. Berry, Leonard Carlitz, L. W. Cohen, C. C. Craig, H. T. Engstrom, D. H. Lehmer, S. B. Littauer, N. H. McCoy, E. J. McShane, Gordon Pall, W. T. Reid, N. W. Rutt, W. J. Trjitzinsky, Jacob Yerushalmy.

Mr. O. J. Farrell has been appointed Rockefeller International Research Fellow in Mathematics.

John V. Atanasoff, of the University of Wisconsin, has been appointed assistant professor of mathematics at the Iowa State College.

Professor Ella E. Bernstorf, of Friends University, Wichita, Kansas, has been appointed dean of women at the State Teachers College, California, Pa.

Dr. Nat Edmonson, Jr., of Rice Institute, has been appointed assistant professor of mathematics at the Texas Technological College.

Assistant Professor J. D. Eshleman, of the University of Pennsylvania, has been appointed acting professor of mathematics at Heidelberg College, Tiffin, Ohio.

Mr. Ivan L. Hebel, of the Colorado School of Mines, has been promoted to an assistant professorship of mathematics.

Mr. E. R. Heineman has been appointed assistant professor of mathematics at the Texas Technological College.

Dr. A. W. Hobbs, professor of applied mathematics at the University of North Carolina, has been appointed dean of the College of Liberal Arts of that University.

Professor Alfred Hume, of the University of Mississippi, has been appointed professor of mathematics at Southwestern College, Memphis, Tenn.

Dr. P. W. Ketchum, of the University of Illinois, has been promoted to be an associate in mathematics.

Dr. F. W. Kokomoor, of the University of Florida, has been promoted to an associate professorship of mathematics.

Assistant Professor H. I. Lane, of the University of South Dakota, has been promoted to an associate professorship of mathematics.

Assistant Professor J. D. Leith, of the University of North Dakota, has been promoted to an associate professorship of mathematics.

Assistant Professor Elizabeth LeStourgeon, of the University of Kentucky, has been promoted to an associate professorship of mathematics.

Professor Gertrude I. McCain, of Westminster College, has been appointed professor of mathematics at State College, Radford, Va.

Dr. Wilhelm Maier, on leave from the University of Frankfort, has been appointed visiting professor of mathematics at Purdue University for the year 1930–31.

Dr. Morris Marden has been appointed assistant professor of mathematics in the Extension Division of the University of Wisconsin, at Milwaukee.

Dr. C. G. Phipps has been promoted to an associate professorship of mathematics at the University of Florida.

Dr. P. G. Robinson has been promoted to an associate professorship of mathematics at Iowa State College.

Assistant Professor H. M. Sheffer has been promoted to an associate professorship of philosophy at Harvard University.

Professor J. A. G. Shirk, of Kansas State Teachers College, who has been granted a year's leave of absence, is studying mathematical statistics at Stanford University. Professor W. H. Hill is acting head of the department during his absence.

Dr. R. G. Smith, of the University of Kansas, has been elected associate professor of mathematics at Kansas State Teachers College.

Associate Professor F. W. Sparks has been promoted to a professorship of mathematics at the Texas Technological College.

Dr. C. W. Strom has been promoted to a professorship of mathematics at Luther College.

Assistant Professor D. J. Struik has been promoted to an associate professorship of mathematics at the Massachusetts Institute of Technology.

One Hundred Per Cent Membership

The membership in the Association is now about 2200, including 137 institutional members. These figures indicate one hundred percent increase over the total charter membership in 1916. Presumably this also indicates that among the mathematics faculties in many institutions the Association membership has doubled during these fourteen years. However, in some institutions, probably in very many, *all* members of the mathematics staff and the institution itself have belonged to the Association from the outset. Such a one hundred percent membership has been maintained, for instance, by the University of Chicago. It is urgently desired by the membership committee to ascertain all of the institutions of which this is now true, and to this end the committee on membership requests the cooperation of each mathematical staff. Will the secretary or some representative of each such department transmit to the Secretary of the Association at Oberlin, Ohio, the following information:

(1) Has your department at present a one hundred percent individual membership in the Association?

(2) Has your institution an institutional membership in the Association?

(3) Will you cordially invite any non-members in your group to join the Association?

(4) Will you present to your institution the desirability of becoming an institutional member of the Association?

Membership in the Association is a mark of professional standing and a contribution to the promotion of mathematical interests in America. Members are entitled to receive all publications of the Association at cost, including the Carus Monographs and the Rhind Mathematical Papyrus. The Association now has eighteen sections distributed over the country so that any member may attend a meeting within reasonable distance; and all meetings, sectional and national, are fully reported in the American Mathematical Monthly.

The near approach of the Annual Meeting in Cleveland, Ohio, is an auspicious time to secure new members for 1931. The Secretary will supply application blanks on request, and they should be returned to him not later than December 27. Why not adopt the slogan—"One Hundred Percent Membership."

THE CHAUVENET PRIZE

In the year 1925, the Mathematical Association of America established a prize of one hundred dollars for the best expository paper published in English during successive periods of five years by a member of the Association.

The purpose of the prize is to stimulate expository contributions in mathematical journals. The award does not apply to books, although the Carus Monographs are expository in character and on this score might be included. They carry their own reward in the form of a cash honorarium to each author.

The prize will be awarded hereafter every three years. The last award was in December, 1929, to Professor T. H. Hildebrandt. The next award will be in December, 1932, for the period of 1929-1931.

The Carus Mathematical Monographs

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THE CARUS MONOGRAPH COMMITTEE is pleased to announce that the first edition of Number Four is well advanced in sales and that each of the others has gone into a second edition; also that a German Edition of number One is being brought out by the firm of Teubner in Munich. The titles of the monographs are: (1) "Calculus of Variations" by Professor GILBERT A. BLISS; (2) "Analytic Functions of a Complex Variable" by Professor DAVID R. CURTISS; (3) "Mathematics of Statistics" by Professor HENRY L. RIETZ; "Projective Geometry," by Professor JOHN W. YOUNG.

The price of these Monographs is $1.25 to institutional and individual members of the Association when ordered directly through the Secretary, one copy to each member; this is the bare cost of production. The price to all non-members of the Association and for all quantity orders for class use is $2.00 per copy, obtained only through the Open Court Publishing Company, 337 East Chicago Avenue, Chicago, Illinois, distributors to the general public of Association publications.

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DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Fourteenth Summer Meeting of the Association, Providence, Rhode Island, Sept. 8-9, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1930.

ILLINOIS, Lake Forest, Ill., May 2-3.

INDIANA, Earlham College, May 2-3.

IOWA, Ames, Iowa, May 2-3.

KANSAS, February 15.

KENTUCKY, Lexington, Ky., April 5.

LOUISIANA-MISSISSIPPI, Cleveland, Miss., March 7-8.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, May 10.

MICHIGAN, Ann Arbor, Mich., March 22.

MINNESOTA, Carleton College, May 17.

MISSOURI, Columbia, Mo., November 28.

NEBRASKA, Peru, Neb., May 9.

OHIO, Columbus, Ohio, April 3.

PHILADELPHIA, Philadelphia, Pa., Nov. 29.

ROCKY MOUNTAIN, Denver, Colo., April 11-12.

SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, Calif., March 8.

TEXAS.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.